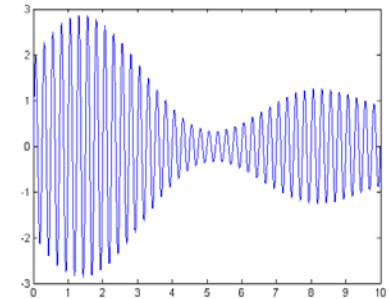


ECE3614

Introduction to Communications Systems

Fall 2007

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Lecture #8: The Energy Spectral
Density and Power Spectral
Density



Overview

- The Fourier Transform provides frequency domain information for both energy signals and power signals.
- The energy spectral density (ESD) provides information of the energy of the signal at each frequency
 - Applies only to energy signals
- The power spectral density (PSD) provides information of the power of a signal at each frequency
 - Applies only to power signals
 - Can be applied to random as well as deterministic signals
- Today we will review the ESD and PSD and their relationship to the autocorrelation function
- Reading
 - 2.8 – 2.9

Developing Energy Spectral Density

- Recall the Convolution/Multiplication property

$$x_1(t) * x_2(t) \iff X_1(f) X_2(f)$$

- Writing this as an equality

$$\int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau = \int_{-\infty}^{\infty} X_1(f) X_2(f) e^{j2\pi ft} df$$

- Reversing time and conjugating we have

$$\int_{-\infty}^{\infty} x_1(\tau) x_2^*(\tau - t) d\tau = \int_{-\infty}^{\infty} X_1(f) X_2^*(f) e^{j2\pi ft} df$$

- Letting $t = 0$ and setting $x_2(t) = x_1(t)$

$$\int_{-\infty}^{\infty} x_1(\tau) x_1^*(\tau) d\tau = \int_{-\infty}^{\infty} X_1(f) X_1^*(f) df$$

ESD – cont.

$$\int_{-\infty}^{\infty} x_1(\tau) x_1^*(\tau) d\tau = \int_{-\infty}^{\infty} X_1(f) X_1^*(f) df$$

□ Rewriting

$$\underbrace{\int_{-\infty}^{\infty} |x_1(\tau)|^2 d\tau}_{\text{Energy}} = \int_{-\infty}^{\infty} |X_1(f)|^2 df$$

- Since the integral of $|X_1(f)|^2$ over f provides the total energy, we can call it the energy spectral density

$$\psi_x(f) = |X_1(f)|^2$$

Autocorrelation

- The autocorrelation function provides a measure of the similarity of a signal with a delayed version of itself

$$R_x(\tau) = \int_{-\infty}^{\infty} x_1(t) x_1^*(t - \tau) dt$$

- Setting the delay $\tau = 0$ we arrive at the energy

$$R_x(0) = \int_{-\infty}^{\infty} x_1(t) x_1^*(t) dt = \int_{-\infty}^{\infty} |x_1(t)|^2 dt$$

Autocorrelation – cont.

- Starting from the correlation relationship we described earlier

$$\int_{-\infty}^{\infty} x_1(t) x_1^*(t - \tau) dt = \int_{-\infty}^{\infty} X_1(f) X_1^*(f) e^{j2\pi f\tau} df$$

- However, the first integral is simply the autocorrelation function

$$\begin{aligned} R_x(\tau) &= \int_{-\infty}^{\infty} X_1(f) X_1^*(f) e^{j2\pi f\tau} df \\ &= \int_{-\infty}^{\infty} |X_1(f)|^2 e^{j2\pi f\tau} df \\ R_x(\tau) &= \int_{-\infty}^{\infty} \psi_x(f) e^{j2\pi f\tau} df \end{aligned}$$

- Thus, the autocorrelation function and the energy spectral density are Fourier Transform pair

Example 8.1

- Determine the autocorrelation function of a sinc pulse
- Solution: We could use the definition directly, but it doesn't appear to be straightforward

$$\begin{aligned}R_x(\tau) &= \int_{-\infty}^{\infty} x_1(t) x_1^*(t - \tau) dt \\ &= \int_{-\infty}^{\infty} \text{sinc}(t) \text{sinc}(t - \tau) dt\end{aligned}$$

- How else might we do this problem?

Example 8.1 (cont.)

- Since we know that the autocorrelation function is the Fourier Transform of the ESD, let us examine that approach. Perhaps it is easier.
- The ESD is simply

$$\begin{aligned}\psi_x(f) &= |X(f)|^2 \\ &= |\text{rect}(f)|^2 \\ &= \text{rect}(f)\end{aligned}$$

- Taking the inverse Fourier Transform we have

$$R_x(\tau) = \text{sinc}(\tau)$$

- Thus, the autocorrelation function of a sinc pulse is simply a sinc pulse

Power Spectral Density

- Recall the definition of power as the time-average energy of a signal

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

- Define

$$x_T(t) = \text{rect}\left(\frac{t}{2T}\right) x(t)$$

- Thus,

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x_T(t)|^2 dt$$

- From the Rayleigh energy theorem

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

PSD – cont.

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

- Interchanging the limiting and integration operations

$$\begin{aligned} P &= \int_{-\infty}^{\infty} \left[\lim_{T \rightarrow \infty} \frac{1}{2T} |X_T(f)|^2 \right] df \\ &= \int_{-\infty}^{\infty} S_x(f) df \end{aligned}$$

where $S_x(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} |X_T(f)|^2$ is termed the Power Spectral Density

Autocorrelation Function

- The autocorrelation function of a power signal is

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x^*(t - \tau) dt$$

- Further, just as with energy signals we can show that the autocorrelation function of power signals and the power spectral density are a Fourier Transform pair

$$R_x(\tau) \xLeftrightarrow{\quad} S_x(f)$$

Periodic Signals

- While the previous definitions are helpful for general power signals, for the important subclass of power signals, periodic signals we can find a more direct expression for the power spectral density
- Recall that we can write a periodic signal in terms of the Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

- Further, the autocorrelation function is

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x^*(t - \tau) dt$$

Periodic Signals (cont.)

- Substituting the Fourier Series representation into the autocorrelation function definition

$$\begin{aligned} R_x(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_o t} \right] \left[\sum_{m=-\infty}^{\infty} c_m^* e^{-j2\pi m f_o (t-\tau)} \right] dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n e^{j2\pi n f_o t} c_m^* e^{-j2\pi m f_o (t-\tau)} \right] dt \\ &= \left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{j2\pi n f_o t} e^{-j2\pi m f_o (t-\tau)} dt \right) \right] \\ &= \left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi m f_o \tau} \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{j2\pi(n-m)f_o t} dt \right) \right] \end{aligned}$$

$R_x(\tau)$ of Periodic Signals – cont.

□ Continuing...

$$\begin{aligned} R_x(\tau) &= \left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi n f_o \tau} \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{j2\pi(n-m)f_o t} dt \right) \right] \\ &= \left[\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{j2\pi n f_o \tau} \delta(n-m) \right] \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau} \end{aligned}$$

□ The Power Spectral Density can then be found using the Fourier Transform

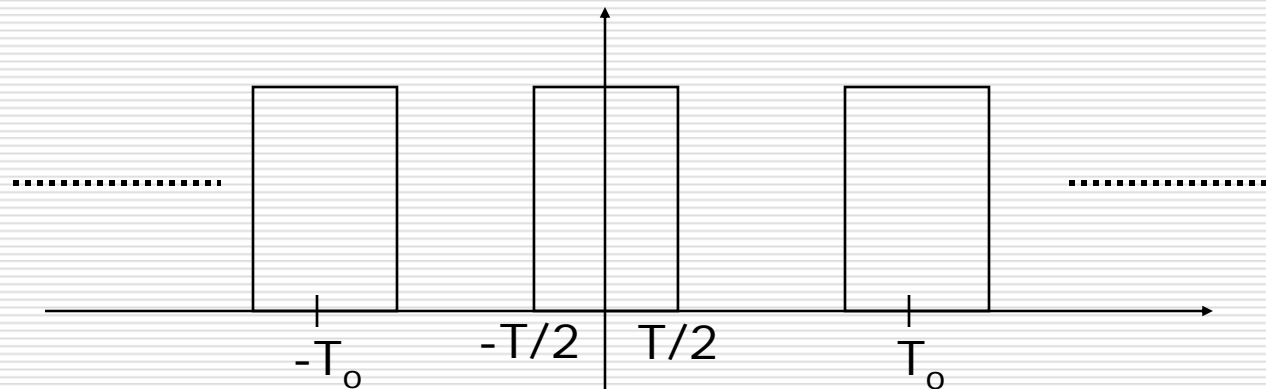
PSD of Periodic Signals

- We can find the PSD using the Fourier Transform

$$\begin{aligned}R_x(\tau) &= \sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau} \\S_x(f) &= F\{R_x(\tau)\} \\&= F\left\{\sum_{n=-\infty}^{\infty} |c_n|^2 e^{j2\pi n f_o \tau}\right\} \\&= \sum_{n=-\infty}^{\infty} |c_n|^2 F\{e^{j2\pi n f_o \tau}\} \\&= \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_o)\end{aligned}$$

Example 8.2

□ Find the PSD of the pulse train



Example 8.2 (cont.)

- We know that the PSD of a periodic signal can be written as

$$S_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_o)$$

- Further, we can determine the Fourier Series coefficients as

$$\begin{aligned} c_n &= \frac{1}{T_o} \int_{-T/2}^{T/2} e^{-j2\pi n f_o t} dt \\ &= \frac{T}{T_o} \text{sinc}(n f_o T) \end{aligned}$$

Example 8.2 (final)

□ Thus, we have

$$\begin{aligned} S_x(f) &= \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_o) \\ &= \sum_{n=-\infty}^{\infty} \left| \frac{T}{T_o} \operatorname{sinc}(nf_o T) \right|^2 \delta(f - nf_o) \end{aligned}$$

Filtering Energy Signals

- We know from system theory that the Fourier Transform of the output of a system can be written as

$$Y(f) = H(f)X(f)$$

- The ESD can be written as

$$\begin{aligned}\psi_Y(f) &= |Y(f)|^2 \\ &= |H(f)X(f)|^2 \\ &= |H(f)|^2 |X(f)|^2 \\ &= |H(f)|^2 \psi_X(f)\end{aligned}$$

- Which tells us that the ESD at the output of a system is the ESD of the input multiplied by the magnitude of the transfer function squared

Filtering Power Signals

- The Fourier Transform of the output of a system when the input is periodic can be written as

$$\begin{aligned} Y(f) &= H(f) X(f) \\ &= H(f) \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_o) \\ &= \sum_{n=-\infty}^{\infty} H(nf_o) c_n \delta(f - nf_o) \end{aligned}$$

- The PSD can be written as

$$\begin{aligned} S_Y(f) &= \sum_{n=-\infty}^{\infty} |H(nf_o) c_n|^2 \delta(f - nf_o) \\ &= \sum_{n=-\infty}^{\infty} |H(nf_o)|^2 |c_n|^2 \delta(f - nf_o) \end{aligned}$$

Summary

- In this lecture we have briefly reviewed the concepts of Energy Spectral Density and Power Spectral Density
- The ESD or PSD of the output of a LTI system can be readily found using the ESD or PSD of the input and the transfer function of the system
- The autocorrelation function and the power spectral density (energy spectral density) form a Fourier Transform pair for power (energy) signals