# Digital Communications
# Midterm Exam II

November 3, 2006

SOLUTION

I pledge that I have neither given nor received any assistance on this exam.

________________________________________

(signed)

______________________________ ____________________

Name (print) Student Number

1. (20 points) Multiple Choice – Choose the answer that best completes the sentence.

(a) [5 points] Differential PSK modulation

(a) requires a non-coherent receiver
(b) requires a coherent receiver
(c) can use either a coherent or a non-coherent receiver
(d) All of the above
(e) None of the above

(b) [5 points] Non-coherent demodulation of M-FSK

(a) is possible since there is no information in the phase
(b) performs worse than coherent demodulation of M-FSK
(c) requires 2*M* correlator branches in the receiver
(d) All of the above
(e) None of the above

(c) [5 points] An eye diagram

(a) shows the amount of bandwidth being used by a pulse shape
(b) shows the amount of ISI introduced by a pulse shape
(c) shows the impact of phase error in a coherent receiver
(d) All of the above
(e) None of the above

(d) [5 points] Which pulse shape causes the least degradation due to timing error?

(a) square pulse
(b) raised cosine pulse
(c) sinc pulse
(d) All of the above would show equal degradation
(e) None of the above

2. (20 points) Bandpass Representation

Consider the following modulation scheme:

$$s(t) = Ap(t)\cos(2\pi f_c t) + Bp(t)\sin(2\pi f_c t)$$

where

$$A \in \{-3,-1,1,3\}$$
$$B \in \{-4,-2,2,4\}$$

and *p(t)* is an arbitrary pulse shape.

(a) [5 points] Write the in-phase and quadrature representation of this signal.

*The in-phase quadrature representation is written as*

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

*Thus,*

$$x(t) = Ap(t)$$
$$y(t) = -Bp(t)$$

*where*

$$A \in \{-3,-1,1,3\}$$
$$B \in \{-4,-2,2,4\}$$

(b) [ 5 points] Write the complex baseband version of this signal

*The complex baseband representation is*

$$\begin{aligned} s(t) &= \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \\ &= \mathrm{Re}\left\{\left(x(t)+jy(t)\right)e^{j2\pi f_c t}\right\} \end{aligned}$$

*The complex baseband* $\tilde{s}(t)$ *can be written from the in-phase and quadrature representation as*

$$\begin{aligned} \tilde{s}(t) &= x(t) + jy(t) \\ &= Ap(t) - jBp(t) \end{aligned}$$

*where*

$$A \in \{-3,-1,1,3\}$$
$$B \in \{-4,-2,2,4\}$$

(c) [ 5 points] Write the amplitude/phase representation of this signal.

*The amplitude/phase representation can be written as*

$$s(t) = R(t)\cos\left(2\pi f_c t + \theta(t)\right)$$

*where*

$$\begin{aligned} R(t) &= \sqrt{x^2(t) + y^2(t)} \\ &= \sqrt{A^2 p^2(t) + B^2 p^2(t)} \\ &= \sqrt{A^2 + B^2}\,\left|p(t)\right| \\ \theta(t) &= \tan^{-1}\left\{\frac{y(t)}{x(t)}\right\} \\ &= \tan^{-1}\left\{\frac{-Bp(t)}{Ap(t)}\right\} \\ &= \tan^{-1}\left\{\frac{-B}{A}\right\} \end{aligned}$$

*Thus, there four different amplitudes:*

$$\begin{aligned} \sqrt{A^2 + B^2} &= \sqrt{1^2 + 2^2} = \sqrt{5} \\ &= \sqrt{1^2 + 4^2} = 3 \\ &= \sqrt{3^2 + 2^2} = \sqrt{13} \\ &= \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

*There are 16 different phases. In the first quadrant:*

$$\begin{aligned} \theta(t) &= \tan^{-1}\left\{\frac{-B}{A}\right\} \\ &= \tan^{-1}\left\{\frac{2}{3}\right\} = 34^o \\ &= \tan^{-1}\left\{\frac{4}{3}\right\} = 53^o \\ &= \tan^{-1}\left\{\frac{4}{3}\right\} = 63^o \\ &= \tan^{-1}\left\{\frac{4}{3}\right\} = 76^o \end{aligned}$$

*The other 12 phases are then easily found as: $104^o$,$117^o$,$127^o$, $146^o$, $-34^o$,$-53^o$, $-63^o$, $-76^o$,$-104^o$, $-117^o$, $-127^o$, $-146^o$.*

(d) [5 points] What is the bandwidth efficiency (in bps/Hz) of this modulation scheme if *p(t)* has a bandwidth of $1.25\ R_s$ Hz where $R_s$ is the pulse rate ?
?

*Since the bandwidth of the baseband pulse is 1.25 $R_s$ Hz, the bandwidth of the bandpass signal is 2.5$R_s$ Hz. The scheme has 16 symbols and thus four bits per symbol. Thus, the bit rate is 4Rs. The bandwidth efficiency is then:*

$$\begin{aligned}\eta_{BW} &= \frac{R_b}{B} \\ &= \frac{4R_s}{2.5R_s} \\ &= 1.6 bps / Hz\end{aligned}$$

3. (25 points) Binary Bandpass Modulation / Pulse Shaping

Consider the following modulation scheme:

$$x(t) = \sum_{k=-\infty}^{\infty} A_k p(t-kT)\cos(2\pi f_c t)$$

where $A_k \in \{0,1\}$

$$p(t) = \text{sinc}(t/T)$$

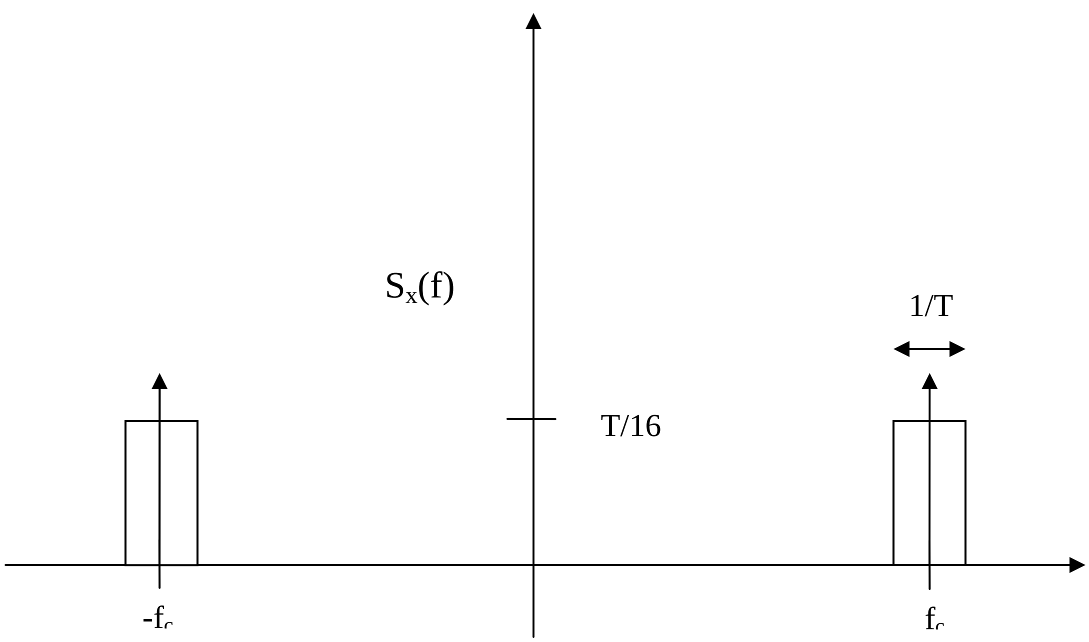
(a) (10 points) Plot the power spectral density of this modulation scheme

*This is a BASK scheme with a sinc pulse shape. Thus, the PSD of the complex baseband is found to be:*

$$\tilde{S}_x(f) = \frac{1}{4T}|P(f)|^2 + \frac{1}{2T^2}\sum_{n=-\infty}^{\infty}\left|P\left(\frac{n}{T_s}\right)\right|^2 \delta\left(f - \frac{n}{T_s}\right)$$
$$= \frac{1}{4T}T^2 rect(fT) + \frac{1}{2}\delta(f)$$

*This means that the PSD of the bandpass signal is*

$$S_x(f) = \frac{T}{16}rect((f-f_c)T) + \frac{1}{8}\delta((f-f_c)) + \frac{T}{16}rect((f+f_c)T) + \frac{1}{8}\delta((f+f_c))$$




(b (5 points) Do you see any practical difficulties with this scheme? If so, suggest a fix (please be specific!)

*The problem is that the pulse is infinitely long and non-causal. The solution is to insert delay into the system and truncate the pulse to an integer number of bits. This will prevent ISI and allow for a practical system at a slight bandwidth penalty.*

(c) (10 points) One of you colleagues suggests reducing the bandwidth of this system. Specifically, he suggests passing the baseband signal through a low-pass filter with a bandwidth of 1/4T Hz where $1/T$ is the symbol rate. Would this reduce the bandwidth of the transmit signal? Even if it did reduce the bandwidth of the signal, does it introduce any difficulties at the receiver?

*Yes, this would reduce the bandwidth by a factor of 2. However, it distorts the pulse and introduces ISI which would degrade performance at the receiver.*

4. (35 points) M-ary Modulation / Signal Space Representation

Consider the following modulation scheme

$$
\begin{aligned}
x_1(t) &= \sqrt{2}A\cos(2\pi f_1 t) \\
x_2(t) &= 0 \\
x_3(t) &= A\cos(2\pi f_2 t) \\
x_4(t) &= -A\cos(2\pi f_2 t)
\end{aligned}
$$

where $f_2 = f_1 + R_s$ and $R_s$ is the symbol rate.

(a) [15 points] If this modulation scheme requires two basis functions, suggest two appropriate functions and show that they are indeed an orthonormal basis.

*Two logical basis function would be*

$$
\begin{aligned}
\varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_1 t) \\
\varphi_2(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)
\end{aligned}
$$

*They are clearly orthogonal since*

$$
\begin{aligned}
\int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)\sqrt{\frac{2}{T}}\cos(2\pi f_2 t)dt &= \frac{1}{T}\int_0^T \left\{\cos\left(2\pi(f_1 - f_2)t\right) + \cos\left(2\pi(f_1 + f_2)t\right)\right\}dt \\
&\approx \frac{1}{T}\int_0^T \cos\left(2\pi(f_1 - f_2)t\right)dt \\
&= \frac{1}{T2\pi(f_1 - f_2)}\sin\left(2\pi(f_1 - f_2)t\right)\Big|_0^T \\
&= \frac{1}{T2\pi(f_1 - f_2)}\sin\left(2\pi(f_1 - f_2)T\right) \\
&= \frac{1}{T2\pi R_s}\sin(2\pi R_s T) \\
&= 0
\end{aligned}
$$

*They also have unit energy:*

$$\int_0^T \left\{ \sqrt{\frac{2}{T}} \cos(2\pi f_1 t) \right\}^2 dt = \frac{1}{T} \int_0^T \{1 + \cos(4\pi f_1 t)\} dt$$
$$\approx \frac{1}{T} \int_0^T dt$$
$$= 1$$

*Now we simply need to show that the basis is complete. This is easily done by noting that*

$$\mathbf{x}_1 = \begin{bmatrix} \sqrt{T}A \\ 0 \end{bmatrix}$$
$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ \sqrt{T/2}A \end{bmatrix}$$
$$\mathbf{x}_4 = \begin{bmatrix} 0 \\ -\sqrt{T/2}A \end{bmatrix}$$

(b) [10 points] Draw the signal space diagram in terms of energy per symbol. Be careful to label all axes and points.

*In terms of the basis functions we have*

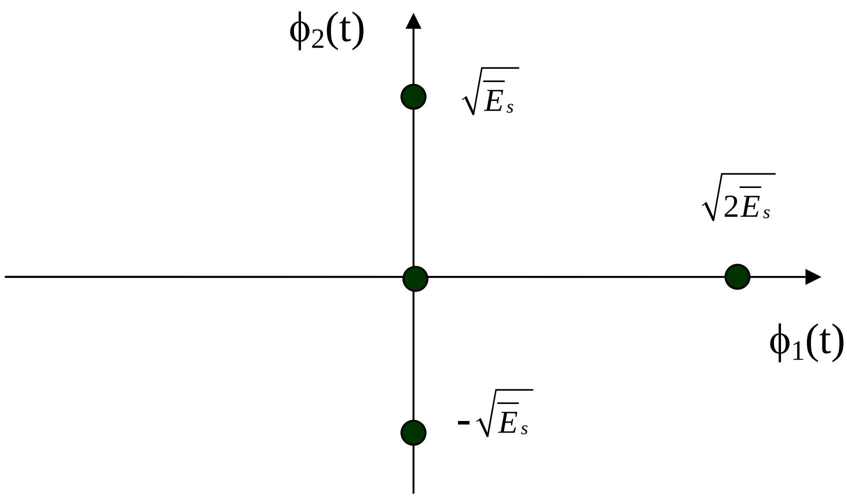
$$x_1(t) = A\sqrt{T}\varphi_1(t)$$
$$x_2(t) = 0$$
$$x_3(t) = A\sqrt{\frac{T}{2}}\varphi_2(t)$$
$$x_4(t) = -A\sqrt{\frac{T}{2}}\varphi_2(t)$$

*The average energy per symbol is:*

$$\overline{E}_s = \frac{1}{4}\left\{ \left(A\sqrt{T}\right)^2 + 0 + \left(A\sqrt{\frac{T}{2}}\right)^2 + \left(-A\sqrt{\frac{T}{2}}\right)^2 \right\}$$
$$= \frac{1}{4}\left\{ A^2T + 0 + A^2\frac{T}{2} + A^2\frac{T}{2} \right\}$$
$$= \frac{A^2T}{2}$$

*Rewriting in terms of the average energy per symbol:*

$$x_1(t) = \sqrt{2E_s}\varphi_1(t)$$
$$x_2(t) = 0$$
$$x_3(t) = \sqrt{E_s}\varphi_2(t)$$
$$x_4(t) = -\sqrt{E_s}\varphi_2(t)$$




(c) [5 points] What is the minimum distance between signal points in terms of energy per bit?

*From the diagram above we can see that the minimum distance between points is* $\sqrt{E_s}$ *. Since there are four symbols, there are two bits per symbol and* $E_s = 2E_b$*. Thus, the minimum distance in terms of energy per bit is* $\sqrt{2E_b}$

(d) [5 points] Is this scheme more or less bandwidth efficient than QPSK? Support your answer.

*This scheme is less BW efficient than QPSK. QPSK has a bandwidth efficiency of 2.0b/s/Hz if bandwidth is defined as the first-null-to-null BW. This on the other hand will have a null-to-null BW of* $3R_s$*. Since* $R_b = 2R_s$*, we have* $2R_s/3R_s = 0.67$ *b/s/Hz .*

| | | |
|---|---|---|
| Rectangular Pulse | $\text{rect}\left(\frac{t}{T}\right)$ | $T\left[\text{sinc}(fT)\right]$ |
| Triangular Pulse | $\text{tri}\left(\frac{t}{T}\right)$ | $T\left[\text{sinc}(fT)\right]^2$ |
| Unit Step | $u(t)$ | $\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$ |
| exponential | $\text{sgn}(t)$ | $\frac{1}{j\pi f}$ |
| Constant | $1$ | $\delta(f)$ |
| Impulse at $t_o$ | $\delta(t-t_o)$ | $e^{-j2\pi f t_o}$ |
| Sinc | $\text{sinc}(2Wt)$ | $\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$ |
| Phasor | $e^{j\omega_o t+\varphi}$ | $e^{j\varphi}\delta(f-f_o)$ |
| Sinusoid | $\cos(2\pi ft+\varphi)$ | $\frac{1}{2}e^{j\varphi}\delta(f-f_o)+\frac{1}{2}e^{-j\varphi}\delta(f+f_o)$ |
| Gaussian | $e^{-\pi(t/t_o)^2}$ | $t_o e^{-\pi(f\,t_o)^2}$ |

| **Property** | |
|---|---|
| Conjugation | $x^*(t) \rightleftharpoons X^*(-f)$ |
| Linearity | $\alpha x(t) + \beta y(t) \rightleftharpoons \alpha X(f) + \beta Y(f)$ |
| Time-shifting | $x(t - t_o) \rightleftharpoons e^{-j2\pi f t_o} X(f)$ |
| Frequency-shifting | $e^{j2\pi f_o t} x(t) \rightleftharpoons X(f - f_o)$ |
| Time reversal | $x(-t) \rightleftharpoons X(-f)$ |
| Time-differentiation | $\frac{d}{dt}\{x(t)\} \rightleftharpoons (j2\pi f) X(f)$ |
| Time-integration | $\int_{-\infty}^{t} x(\tau) d\tau \overset{*}{\rightleftharpoons} \frac{1}{j2\pi f} X(f)$ |
| Time/freq-scaling | $x(at) \rightleftharpoons \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$ |
| Multiplication | $x(t) y(t) \rightleftharpoons X(f) * Y(f)$ |
| Convolution | $x(t) * y(t) \rightleftharpoons X(f) Y(f)$ |