Digital Communications
Homework #1
Due 8/31/2007

SOLUTION

1. (a) Determine the convolution of a sinc function with an impulse (δ(t)).
   (b) What is the spectrum?
   (c) Use Matlab to generate this result and plots. (Use the `conv()` function.)

(a) The convolution of any function with an impulse is

$$x(t) * \delta(t) = x(t)$$

Thus,

$$\mathrm{sinc}(2Wt) * \delta(t) = \mathrm{sinc}(2Wt)$$

(b) Clearly, then the spectrum of the result is the spectrum of the sinc function or 1/2W Π(f/2W). The spectrum can be determined from the fact that convolution in time is multiplication in frequency. Since,

$$\mathrm{sinc}(2Wt) \Leftrightarrow \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$$

$$\delta(t) \Leftrightarrow 1$$

Then,

$$\mathrm{sinc}(2\pi Wt) * \delta(t) \Leftrightarrow \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$$

(c) Matlab code

```
delta = zeros(1, 10001);
delta(5001) = 1;
t = -5:0.001:5;
Sinc = sin(2*pi*t)./(2*pi*t);
y = conv(Sinc,delta);
Y = fft(y);
```

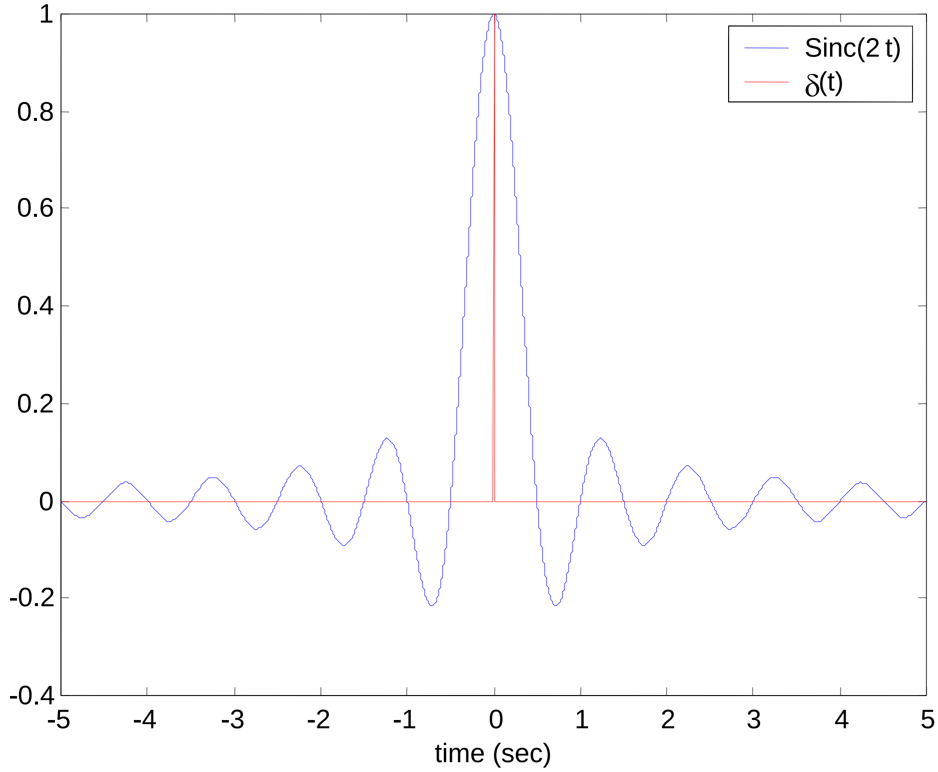

Figure 1: Time plots for Problem 1 (c).

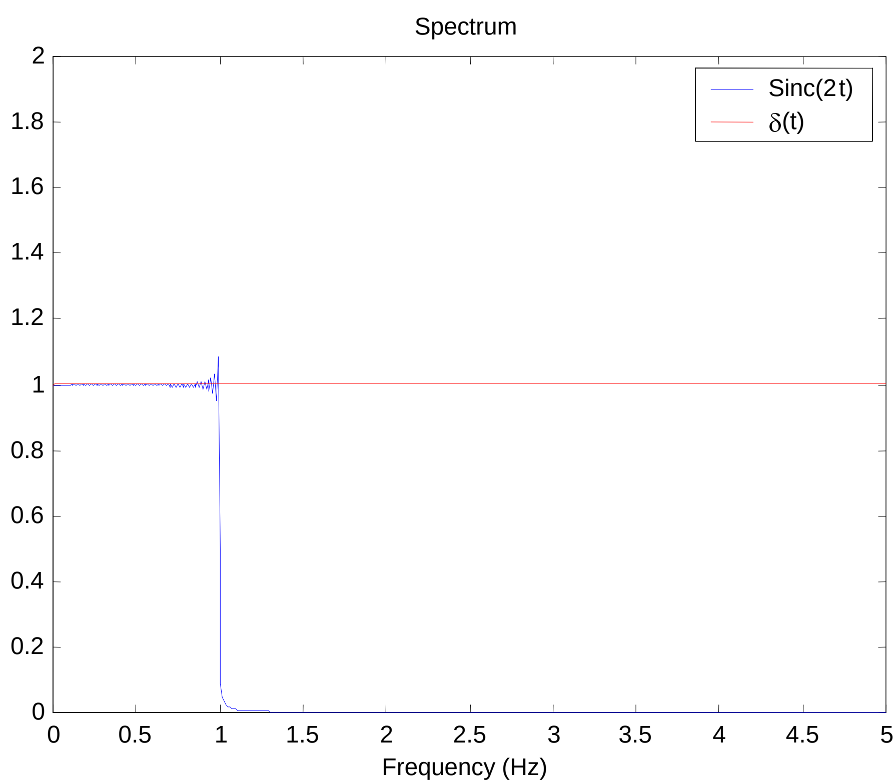

Figure 2: Spectrum of *sinc(2Wt)* and *δ(t)* for Problem 1(c)

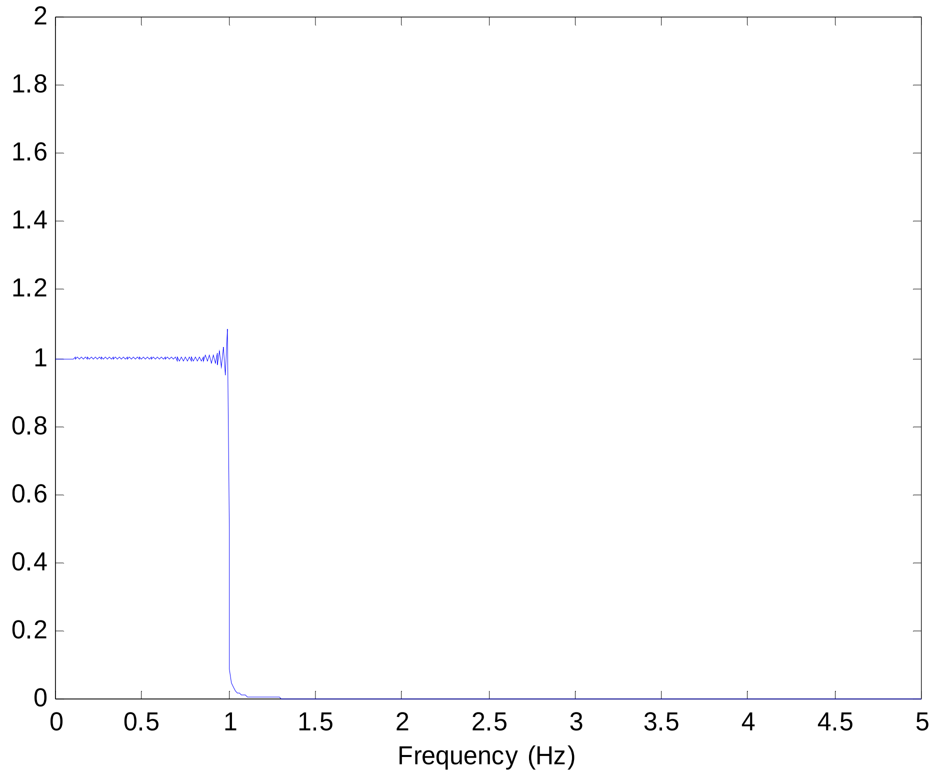

Figure 3: Resulting Spectrum from the Convolution in Problem 1 (c)

2. Convolve the two square pulses (in the time domain) of width *T*, Π(t/T). Plot the time waveform and the spectrum. You can either analysis or Matlab. Be sure to label all axes.

Graphically, the convolution of two rectangular pulses Π(t/T) results in TΛ(t/T) as shown in Figure 4. The spectrum is then $T\Im\left\{\Lambda\left(\frac{t}{T}\right)\right\} = T^2 \text{sinc}^2\left(fT\right)$.

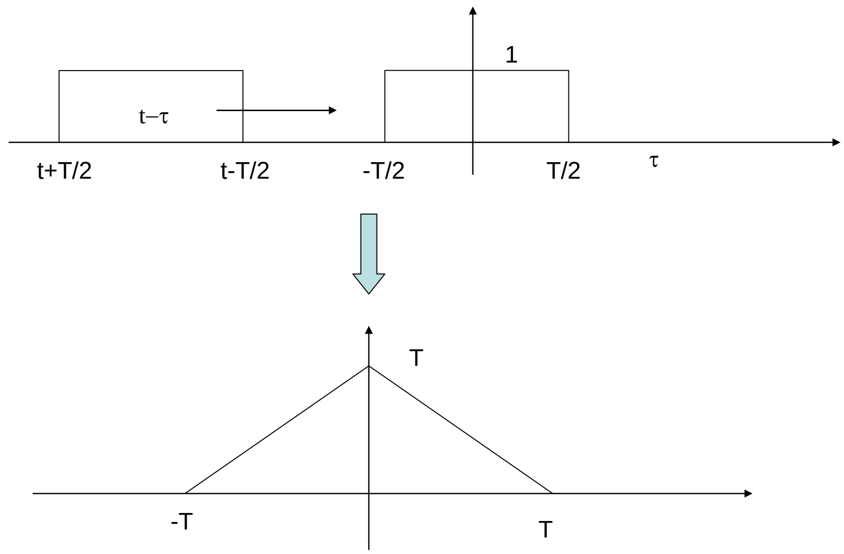

Figure 4: The convolution of two rectangular pulses

Alternatively, we could find the spectrum by multiplying the spectra of the two convolved pulses:

$$\begin{aligned} Y(f) &= X(f)X(f) \\ &= T\mathrm{sinc}(fT)T\mathrm{sinc}(fT) \\ &= T^2\mathrm{sinc}^2(fT) \end{aligned}$$

The spectrum is plotted in Figure 5.

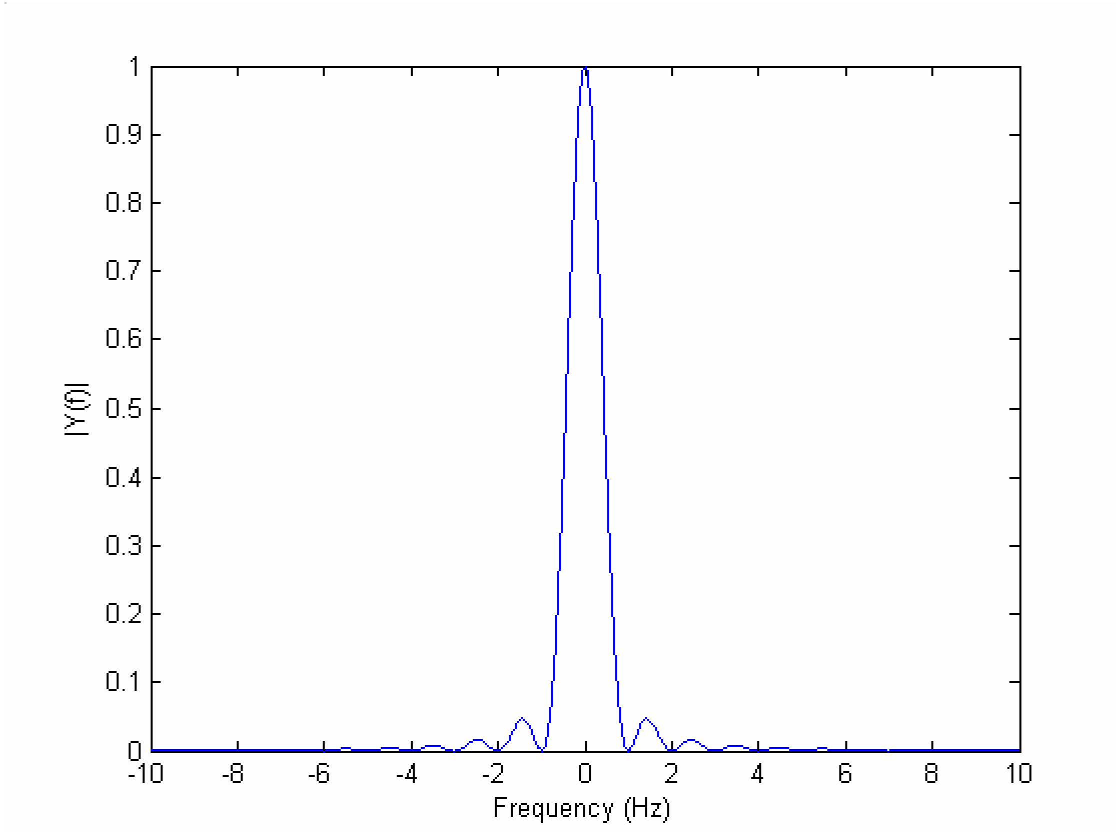

Figure 5: Spectrum of y(t)=x(t)*x(t) in Problem 2 (T=1)

3. Let x(t) = sinc(2Wt) and y(t) = t. If z(t) = x(t)y(t), what is Z(f)?

Since z(t) = t x(t) we can use the Fourier Transform property $Z(f) = (-j2\pi)^{-1}\frac{d}{df}X(f)$:

$$z(t) = x(t)y(t) = t\,\text{sinc}(2Wt)$$

$$Z(f) = (-j2\pi)^{-1}\frac{d}{df}\left\{\frac{1}{2W}\prod\left(\frac{f}{2W}\right)\right\}$$

$$= \frac{1}{-j4\pi W}\frac{d}{df}\{u(f+W) - u(f-W)\}$$

$$= \frac{1}{-j4\pi W}\{\delta(f+W) - \delta(f-W)\}$$

$$= \frac{1}{2\pi W}\left\{\frac{\delta(f+W) - \delta(f-W)}{-j2}\right\}$$

$$= \frac{1}{2\pi W}\left\{\frac{\delta(f-W) - \delta(f+W)}{j2}\right\}$$

$$z(t) = \frac{1}{2\pi W}\sin(2\pi Wt)$$

This can be seen in the time domain in Figure 6 and the frequency domain in Figure 7.

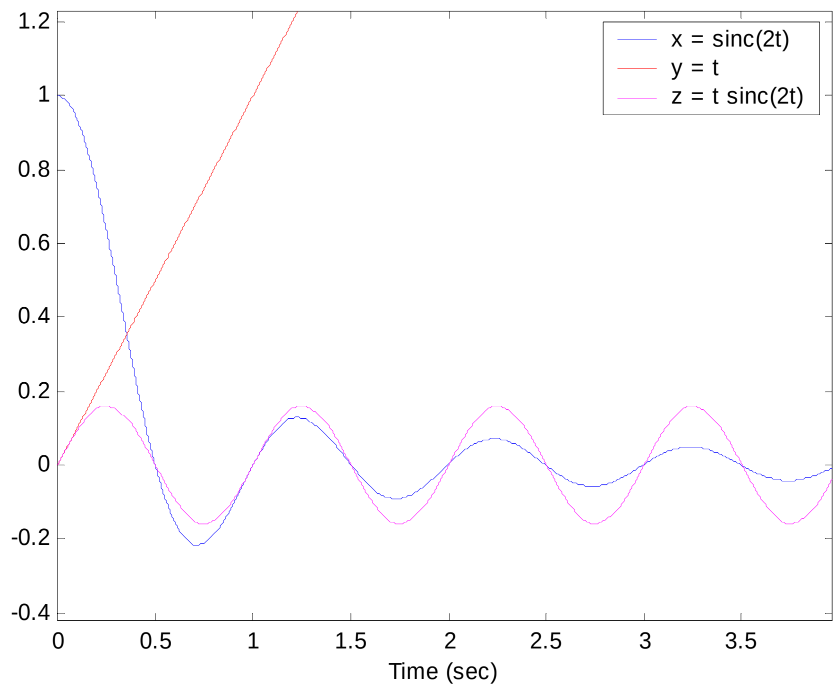

Figure 6: Time plots for Problem 5.

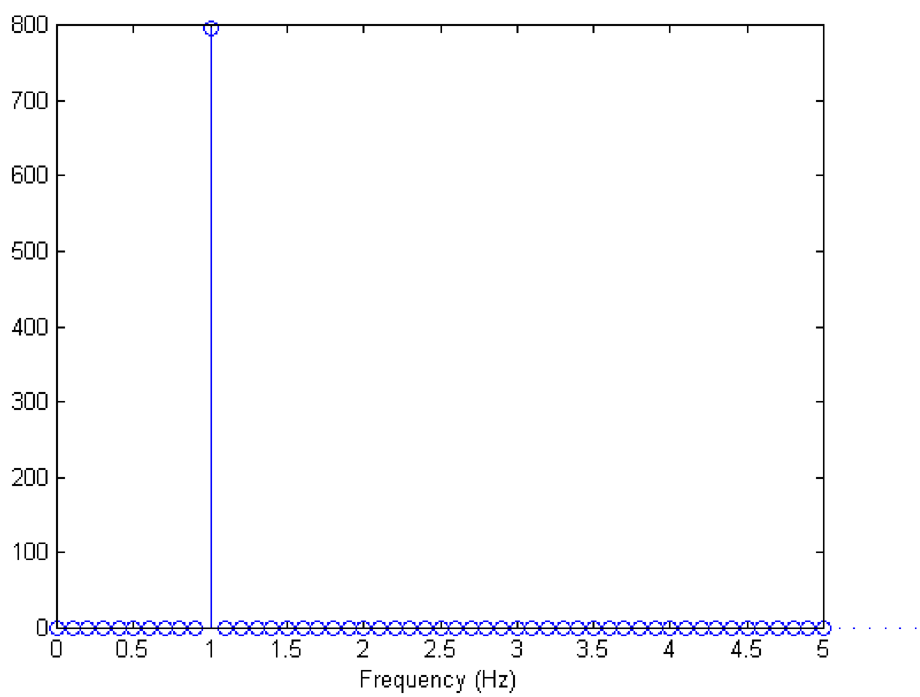

Figure 7: Spectrum for Problem 5

4. Determine whether each one of the following signals are energy or power signals and determine their average energy or power:

(a) $w(t) = \Pi(t/T_o)$

We examine the energy:

$$E = \int_{-\infty}^{\infty} w^2(t)dt$$
$$= \int_{-T_o/2}^{T_o/2} 1dt$$
$$= T_o$$

Since the energy is finite, this is an energy signal.

(b) $w(t) = \Pi(t/T_o)\cos(2\pi f_o t)$

Again we examine the energy:

$$E = \int_{-\infty}^{\infty} w^2(t)dt$$
$$= \int_{-T_o/2}^{T_o/2} \cos^2(2\pi f_o t)dt$$
$$= \frac{T_o}{2}$$

since the energy is finite, it is an energy signal.

(c) $w(t) = \cos^2(2\pi f_o t)$

This time we examine the power

$$P = \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} w^2(t)dt$$
$$= \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} \cos^4(2\pi f_o t)dt$$
$$= \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} \frac{1}{8}\left(3 + 4\cos(4\pi f_o t) + \cos(8\pi f_o t)\right)dt$$
$$= \frac{3}{8}$$

Since the power is finite and non-zero, the signal is a power signal.

5. Use the convolution property to find the spectrum for

$$w(t) = \sin\left(2\pi f_1 t\right)\cos\left(2\pi f_2 t\right)$$
$$= w_1(t)w_2(t)$$
$$W(f) = W_1(f) * W_2(f)$$

Now, we know that

$$W_1(f) = \left(\frac{j}{2}\delta(f+f_1) - \frac{j}{2}\delta(f-f_1) \right)$$

$$W_2(f) = \left(\frac{1}{2}\delta(f+f_2) + \frac{1}{2}\delta(f-f_2) \right)$$

Further, from the sifting property of the delta function we know that

$$\delta(f+f_1) * \delta(f+f_2) = \delta(f+f_1+f_2)$$

Thus,

$$W(f) = \frac{j}{4}\left[\delta(f+f_1+f_2) - \delta(f-f_1+f_2) + \delta(f+f_1-f_2) - \delta(f-f_1-f_2)\right]$$

6. The signal $x(t) = \cos(10t)$ is input to a linear system that has an impulse response

$$h(t) = \begin{cases} e^{-t/10} & t > 0 \\ 0 & t < 0 \end{cases}.$$

(a) What is the output time signal?

We know that the output of a linear system can be found as

$$y(t) = x(t) * h(t)$$

or in the frequency domain

$$Y(f) = X(f)H(f)$$

The Fourier Transform of the input signal is:

$$\begin{aligned} X(f) &= \Im\{x(t)\} \\ &= \Im\{\cos(10t)\} \\ &= \frac{1}{2}\left(\delta\left(f - \frac{10}{2\pi} \right) + \delta\left(f + \frac{10}{2\pi} \right) \right) \end{aligned}$$

Similarly, the system transfer function H(f)is

$$\begin{aligned} H(f) &= \Im\{h(t)\} \\ &= \Im\{e^{-t/10}u(t)\} \\ &= \frac{10}{1 + j2\pi f(10)} \end{aligned}$$

The output in the frequency domain is then:

$$
\begin{aligned}
Y(f) &= X(f)H(f) \\
&= \frac{1}{2}\left(\delta\left(f-\frac{10}{2\pi}\right)+\delta\left(f+\frac{10}{2\pi}\right)\right)\frac{10}{1+j2\pi f(10)} \\
&= \frac{10}{2}\left(\frac{1}{1+j2\pi\frac{10}{2\pi}10}\delta\left(f-\frac{10}{2\pi}\right)+\frac{1}{1-j2\pi\frac{10}{2\pi}10}\delta\left(f+\frac{10}{2\pi}\right)\right) \\
&= \frac{10}{2}\left(\frac{1}{1+j100}\delta\left(f-\frac{10}{2\pi}\right)+\frac{1}{1-j100}\delta\left(f+\frac{10}{2\pi}\right)\right) \\
&\approx \frac{1}{20}\left(e^{-j\pi/2}\delta\left(f-\frac{10}{2\pi}\right)+e^{j\pi/2}\delta\left(f+\frac{10}{2\pi}\right)\right)
\end{aligned}
$$

In the time domain this becomes:

$$
\begin{aligned}
y(t) &= \Im^{-1}\left\{\frac{1}{20}\left[e^{-j\pi/2}\delta\left(f-\frac{10}{2\pi}\right)+e^{j\pi/2}\delta\left(f+\frac{10}{2\pi}\right)\right]\right\} \\
&= \Im^{-1}\left\{\frac{1}{20}\left[-j\delta\left(f-\frac{10}{2\pi}\right)+j\delta\left(f+\frac{10}{2\pi}\right)\right]\right\} \\
&= \Im^{-1}\left\{\frac{1}{10}\left[\frac{\delta\left(f-\frac{10}{2\pi}\right)-\delta\left(f+\frac{10}{2\pi}\right)}{j2}\right]\right\} \\
&= \frac{1}{10}\sin(10t)
\end{aligned}
$$

(b) Due to interference, the signal $z(t)=\cos(15t+\pi/8)$ is added to the input signal prior to entering the system. Provide the impulse response *h(t)* of a brick-wall (rectangular pulse in frequency) filter that eliminates the interferer but not effect signal of interest.

The input signal is now x(t) + z(t) and the spectrum is X(f)+Z(f):

$$
\begin{aligned}
X(f)+Z(f) &= \Im\{x(t)+z(t)\} \\
&= \Im\{\cos(10t)+\cos(15t+\pi/8)\} \\
&= \frac{1}{2}\left\{\delta\left(f-\frac{10}{2\pi}\right)+\delta\left(f+\frac{10}{2\pi}\right)+e^{j\pi/8}\delta\left(f-\frac{15}{2\pi}\right)+e^{-j\pi/8}\delta\left(f+\frac{15}{2\pi}\right)\right\}
\end{aligned}
$$

The filter needs to be a rectangular function in frequency

$$H(f) = \Pi\left(\frac{f}{2W}\right)$$

where $\frac{10}{2\pi} < W < \frac{15}{2\pi}$. *We arbitrarily choose* $W = \frac{12}{2\pi}$. *Thus,*

$$\begin{aligned} H(f) &= \Pi\left(\frac{f}{2\left[12/(2\pi)\right]}\right) \\ &= \Pi\left(\frac{f}{12/\pi}\right) \end{aligned}$$

The impulse response is then:

$$\begin{aligned} h(t) &= \Im^{-1}\left\{H(f)\right\} \\ &= \Im^{-1}\left\{\Pi\left(\frac{f}{12/\pi}\right)\right\} \\ &= 2W\text{sinc}\left(2Wt\right) \\ &= \frac{12}{\pi}\text{sinc}\left(2\frac{6}{\pi}t\right) \\ &= \frac{12}{\pi}\text{sinc}\left(12t\right) \end{aligned}$$