Homework 3

SOLUTIONS

Solution to problem 5.10

In linear delta modulation, if we make the step-size Δ too small, then the system suffers from slope overload distortion. On the other hand, if we make the step-size Δ too large relative to the local slope characteristic of the message signal, then the system suffers from granular distortion. Thus, for a fixed sampling rate $1/T_s$ and with Δ as the only variable, the best that a linear delta modulator can do is to choose a step-size Δ that will provide a compromise (or trade-off) between these two forms of quantization noise. (*2 points*)

Solution to problem 5.18

(a) The Nyquist rate = 2 $\times$ Signal bandwidth = 2 $\times$15 KHz = 30 KHz. (*2 points*)

(b) Since we use binary encoding scheme and $512 = 2^9$, the number of bits per sample is equal to 9. Thus, given a sampling rate of 30 KHz, the minimum bit rate is 30$\times$9 = 270 Kbps. (*2 points*)

Solution to problem 5.21

The modulating waveform is $m(t) = A_m \cos(2\pi f_m t)$. The slope of $m(t)$ is given by $\frac{dm(t)}{dt} = -2\pi f_m A_m \sin(2\pi f_m t)$ and the maximum slope is $\max\left\{\left|-2\pi f_m A_m \sin(2\pi f_m t)\right|\right\} = 2\pi f_m A_m$. To ensure there is no distortion, the value of D should be greater than the maximum change of the signal during one sample interval. That is $\Delta \geq T_s \max\left\{\left|\frac{dm(t)}{dt}\right|\right\}$. Thus, the following should be satisfied

$$\frac{\Delta}{T_s} \geq \max\left(\frac{dm(t)}{dt}\right) = 2\pi f_m A_m$$

This gives

$$A_m \le \frac{\Delta}{2\pi f_m T_s} \qquad (1)$$

If A_m exceeds this value, distortion will occur. (*2 points*)

For $m(t) = A_m \cos(2\pi f_m t)$, its power is given by (**assume a load of 1 ohm**)

$$P = \frac{1}{T_m}\int_0^{T_m} m^2(t)dt = \frac{1}{T_m}\int_0^{T_m} A_m^2 \cos^2(2\pi f_m t)dt = \frac{A_m^2}{2}$$

Using (1), we can obtain

$$P_{\max} = \frac{\Delta^2}{8\pi^2 f_m^2 T_s^2} \quad (2\ points)$$

Solution to problem 5.22

(a) For a general analog signal $\Delta \ge 2\pi W T_s A_m = 2\pi \times 3.1 \times 10/64 = 3.04\text{v}$. Note that you can also use the formula specifically for voice signals, $\Delta \ge 2\pi 800 T_s V_p$. Using that formula: $\Delta \ge 2\pi 800 T_s V_p = 2\pi 800 \cdot (1/64k) \cdot 10 = 0.785\,\text{v}$. (*2 points*)

(b) The average power of granular noise is $\frac{\Delta^2}{3} \times \frac{B}{f_s} = 0.15\text{w}$. Or if you used the second formula in part (a), average power of granular noise = 9.96mW. Note that this has taken the effect of filtering process into account. (*2 points*)

(c) Since delta modulation uses one bit for each sample, the data rate is 64 kbps. If we use an ideal pulse with a duration of $\frac{1}{64000}$ s, the system bandwidth is 32 kHz which corresponds to the use of a sinc pulse. (*2 points*). However, if you choose to use a square pulse, your minimum (first-null) bandwidth is 64 kHz. We will accept this answer only if you clearly stated a square pulse has been used.

Solution to problem 5.24

The transmitter prediction filter operates on exact samples of the signal while the receiving filter operates on quantized samples. Since there is always quantization error, the two prediction filters operate on slightly different inputs. (*2 points*)

Solution to the homemade problem

(a) Assume the signal is uniformly distributed between $[-A, A]$ with the probability of $1/2A$. Using a uniform quantization, we assume there are totally N quantization levels. The average SNR is defined as

$$\left(\frac{S}{N}\right)_{avg} = \frac{E\{x^2\}}{E[n_Q^2]} = \frac{\int_{-A}^{A} x^2 f(x)dx}{\sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} (x-\tilde{x}_k)^2 f(x)dx} = \frac{A^2/3}{D}$$

where $k = 1, 2, \ldots, N$

$$x_k = -A + \frac{2A}{N}k$$

$$\tilde{x}_k = \frac{x_k + x_{k-1}}{2}$$

Because we use a uniform quantization for a uniformly distributed signal, the quantization error is same for all intervals $[x_k, x_{k-1}]$. So, we have $D = ND_1$ where D_1 represents the quantization error for $k = 1$. Specifically,

$$D_1 = \int_{-A}^{-A+\frac{2A}{N}} (x-(-A+\frac{A}{N}))^2 \times \frac{1}{2A} dx$$

$$= \frac{1}{2A}\left[\frac{x^3}{3} - (-A+\frac{A}{N})x^2 + (-A+\frac{A}{N})^2 x\right]_{-A}^{-A+\frac{2A}{N}}$$

$$= \frac{A^2}{2} \times \frac{2}{3N^3}$$

$$= \frac{A^2}{3N^3}$$

So, the average SNR is given by

$$\left(\frac{S}{N}\right)_{avg} = \frac{A^2/3}{D} = \frac{A^2/3}{ND_1} = N^2$$

To maintain a SNR of 35 dB, we have $10\log_{10}(N^2) \geq 35$. This gives $N \geq 10^{3.5/2} = 56.23$. We round it the nearest power of 2, $N = 64 = 2^6$. So, we should have 6 bits for quantization. Now, since the signal bandwidth is 200 KHz, the Nyquist rate is 400 KHz. Thus, the data rate is given by $R = 400 \times 6 \;= 2400\text{kbps} \;= 2.4\text{Mbps}$. (*2 points*)

(b) Due to the bit rate of 2.4Mbps, the duration T_0 of each square pulse is $\frac{1}{2.4\text{MHz}} = \frac{1}{f_0}$.

The pulse shape is given in Figure 6.1 below:

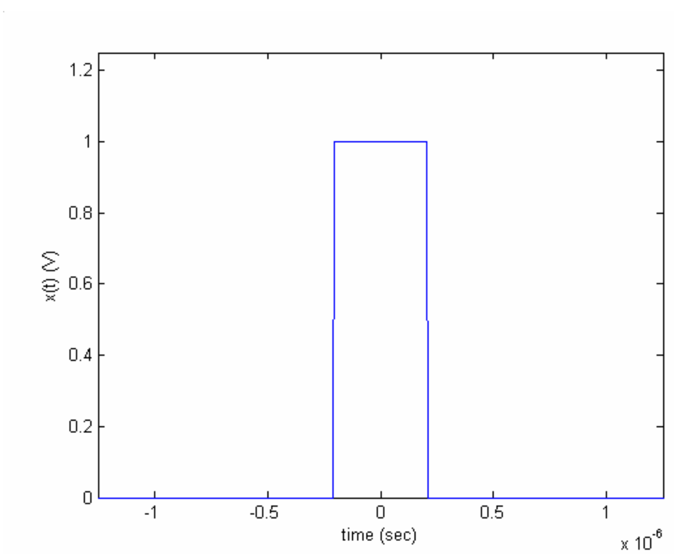

Figure 6.1 – Pulse shape used in Problem 6(b)

Using binary polar NRZ signaling, the power spectral density is

$$S_x(f) = \frac{1}{T_o}\left|F(f)\right|^2$$
$$= T_o \operatorname{sinc}^2(fT_0)$$

To determine the 30dB bandwidth we plot $10\log_{10}\left(\frac{\left|F(f)\right|^2}{\max\left\{\left|F(f)\right|^2\right\}}\right)$ which is shown in Figure 6.2 below.

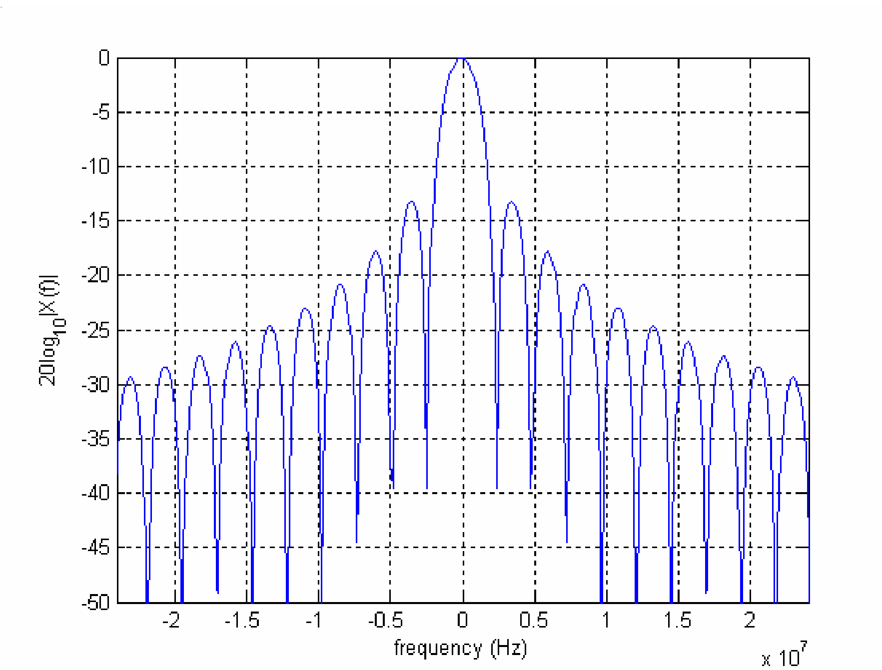

Figure 6.2 – Power Spectral Density with Pulse shape used in Problem 6(b)

From Figure 6.2 we see that the 30dB bandwidth is approximately, $B_{30\text{dB}} = 24MHz$. Any answer reasonably close to that is acceptable. Please check Lecture # 6 for the definition of X-dB bandwidth. (*2 points*)

(c) The plot of the appropriate triangular pulse is given in Figure 6.3. The corresponding power spectral density is shown in Figure 6.4.

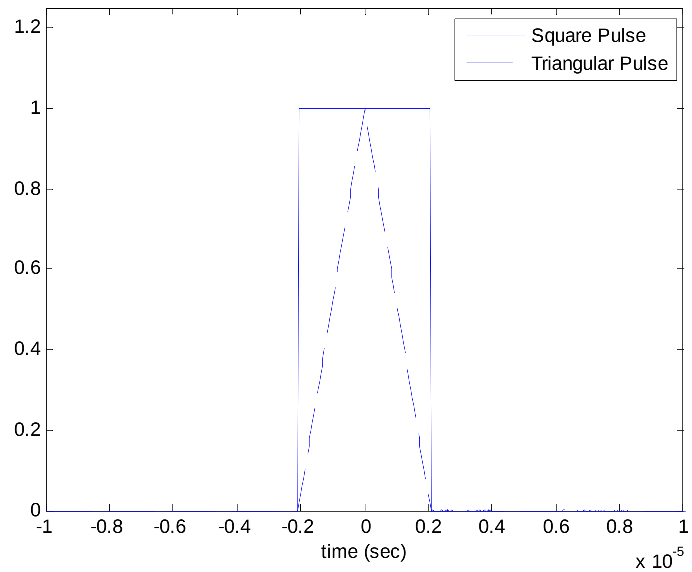

Figure 6.3 – Pulse shapes used in Problem 6(b) and 6(c)

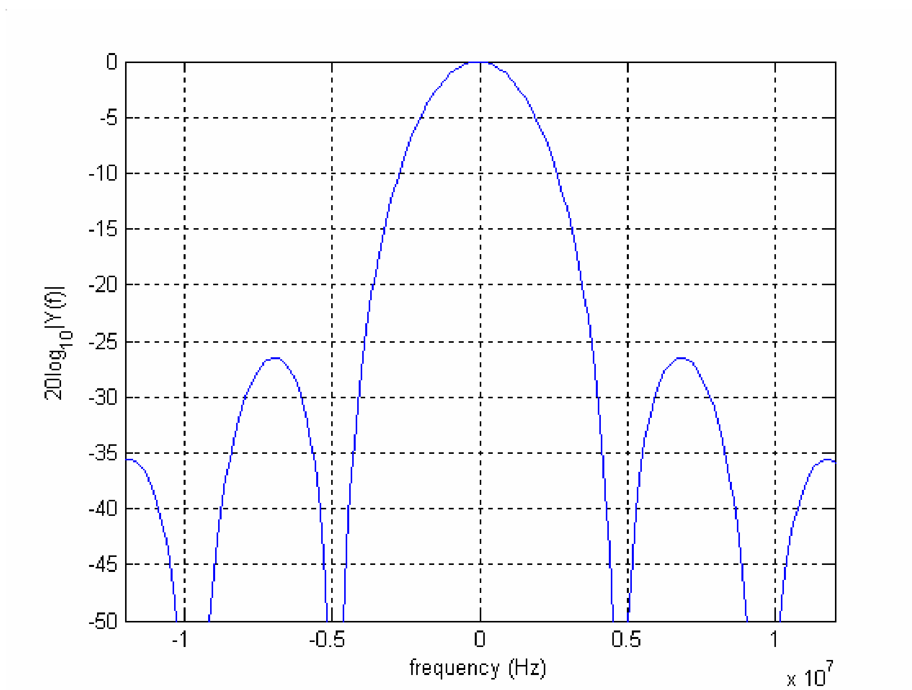

Figure 6.4 – Power Spectral Density with Pulse shape used in Problem 6(c)

From Figure 6.4 we see that the 30dB bandwidth is approximately, $B_{30\text{dB}} = 8MHz$. Any answer reasonably close to that is acceptable. Please check Lecture # 6 for the definition of X-dB bandwidth. (*2 points*)

(d) From the lecture notes we know that the SNR for Delta Modulation can be written as (assuming that we filter the quantization noise)

$$SNR = \frac{3f_s^3}{4\pi^2 B^2 B_f}\left(\frac{P_{avg}}{V_p^2}\right)$$

Now, making the reasonable assumption that $B_f = B$, we have

$$SNR = \frac{3f_s^3}{4\pi^2 B^3}\left(\frac{P_{avg}}{V_p^2}\right)$$

Now, we need to find f_s in order to satisfy SNR $=10^{3.5}=3162$. However, before we can do this we must determine the peak to average power ratio. Since the signal is uniformly distributed on some range –A to A, the peak power is A2. The average power is found as

$$P_{avg} = E\left\{x^2\right\} = \int_{-\infty}^{\infty} x^2 f_X\left(x\right)dx$$

Where $f_X(x)$ is the probability density function of the input x. Now, the signal is uniformly distributed so $f_X(x)=1/(2A)$:

$$\begin{aligned} E\left\{x^2\right\} &= \int_{-\infty}^{\infty} x^2 \frac{1}{2A} dx \\ &= \frac{1}{2A}\frac{x^3}{3}\bigg|_{-A}^{A} \\ &= \frac{A^2}{3} \end{aligned}$$

Thus, we have $SNR = \frac{P_{avg}}{V_p^2} = \frac{1}{3}$. Thus, plugging this value in we have

$$3162 = \frac{3 f_s^3}{4\pi^2 (200000)^3} \left(\frac{1}{3} \right)$$

$$f_s = \sqrt[3]{4\pi^2 (200000)^3 * 3162}$$

$$= 10 Msps$$

Now, since we must transmit one pulse per bit with delta modulation, the bit (and thus symbol) rate is 10Mbps and T = 10^{-7}. The bandwidth from Figure 6.5 is 3.25/T or 32.5MHz.

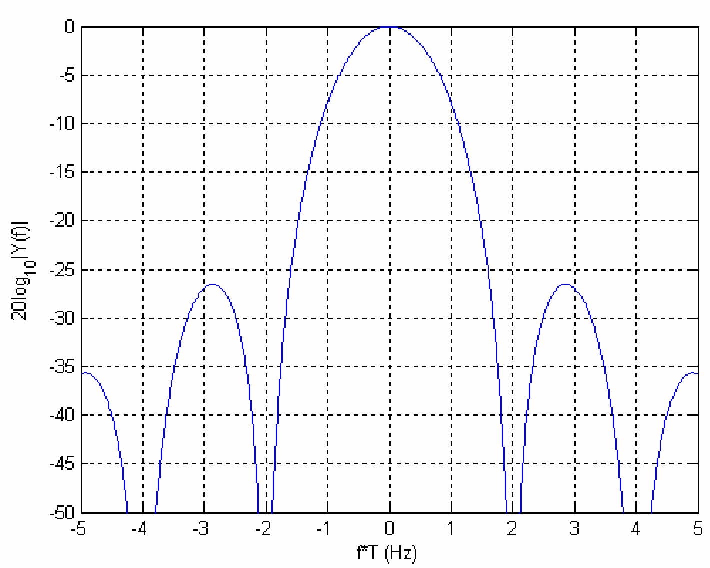

Figure 6.5 – Power Spectral Density of Triangular Pulse Parameterized by symbol duration *T*