Digital Communications
Homework #4
SOLUTION

1. Show that if v(t) = Re{g(t)$e^{j\omega ct}$} then v(t) = R(t)cos(ω_ct + θ(t)) and v(t) = x(t)cos(ω_ct)-y(t) sin(ω_ct). Note that g(t) = x(t) +j y(t) = R(t)$e^{j\theta(t)}$

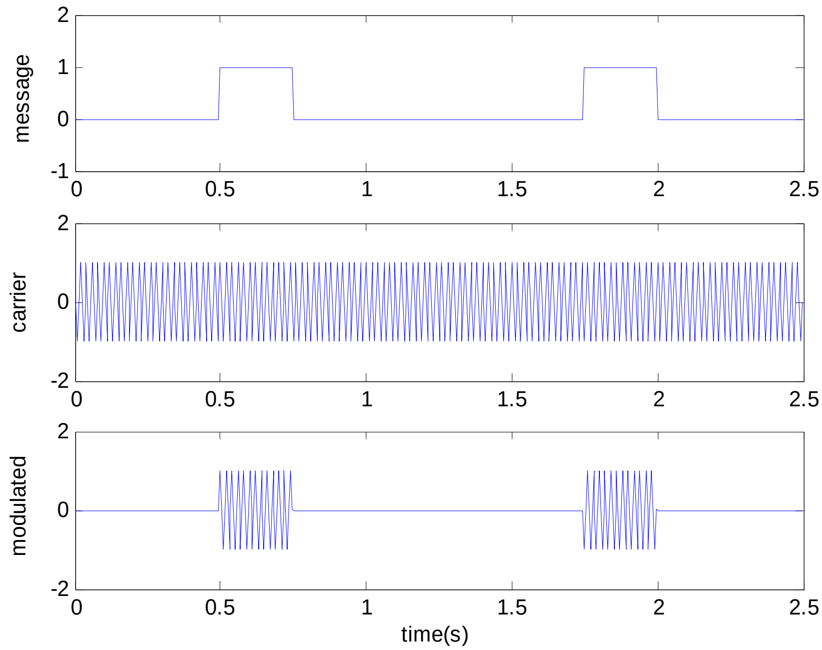
$$\begin{aligned} v(t) &= \mathrm{Re}\{g(t)\exp(j\omega_c t)\} = \mathrm{Re}\{R(t)\exp(j[\omega_c t + \theta(t)])\} \\ &= \mathrm{Re}\{R(t)\cos(\omega_c t + \theta(t)) + jR(t)\sin(\omega_c t + \theta(t))\} \\ &= R(t)\cos(\omega_c t + \theta(t)) \qquad (2\ \ points) \end{aligned}$$

$$\begin{aligned} v(t) &= \mathrm{Re}\{g(t)\exp(j\omega_c t)\} = \mathrm{Re}\{[x(t) + jy(t)][\cos\omega_c t + j\sin\omega_c t]\} \\ &= \mathrm{Re}\{[x(t)\cos\omega_c t - y(t)\sin\omega_c t] + j[x(t)\sin\omega_c t + y(t)\cos\omega_c t]\} \\ &= x(t)\cos\omega_c t - y(t)\sin\omega_c t \qquad (2\ \ points) \end{aligned}$$

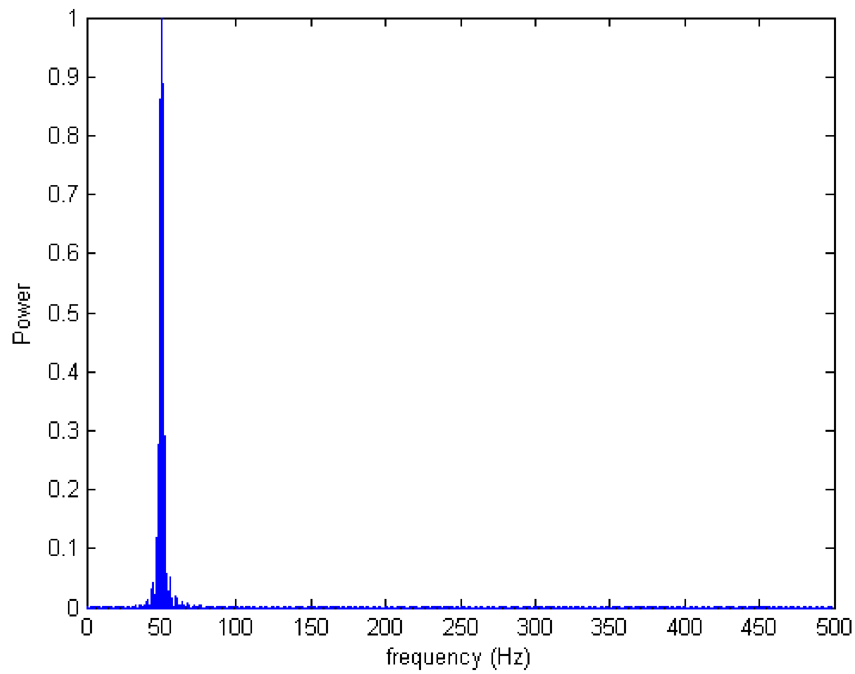
2. Plot time domain and frequency domain examples of a BASK signal using Matlab. Specifically plot 10 bits of 4bps data stream when the carrier frequency is 50Hz. Repeat for BPSK.

Data bits are 0 0 1 0 0 0 0 1 0 0.

Time domain representation of BASK signal (*2 points*)



Frequency domain (*2 points*)



3. Problem 6.2 in the text.

We have, for positive frequency $f > 0$, and $\alpha = 1 - f_1/B_0$

$$\frac{P(f)}{\sqrt{E}/2B_0} = \begin{cases} 1, & 0 \le f/B_0 < 1-\alpha \\ \frac{1}{2}\left(1+\cos\left(\frac{\pi}{2}\times\frac{f/B_0 - 1 + \alpha}{\alpha}\right)\right), & 1-\alpha \le f/B_0 < 1+\alpha \\ 0, & 1+\alpha \le f/B_0 \end{cases}$$

Changing variable $f/B_0 = x$ and integrating the above function, the area can thus be calculated as

$$\begin{aligned} A &= \int_0^{1-\alpha} 1\,dx + \int_{1-\alpha}^{1+\alpha} \frac{1}{2}\left(1+\cos\left(\frac{\pi}{2}\times\frac{x-1+\alpha}{\alpha}\right)\right)dx \\ &= 1-\alpha + \frac{1}{2}\times 2\alpha + \frac{2\alpha}{\pi}\sin\left(\frac{\pi(x-1+\alpha)}{2\alpha}\right)\Bigg|_{1-\alpha}^{1+\alpha} \\ &= 1 \end{aligned}$$

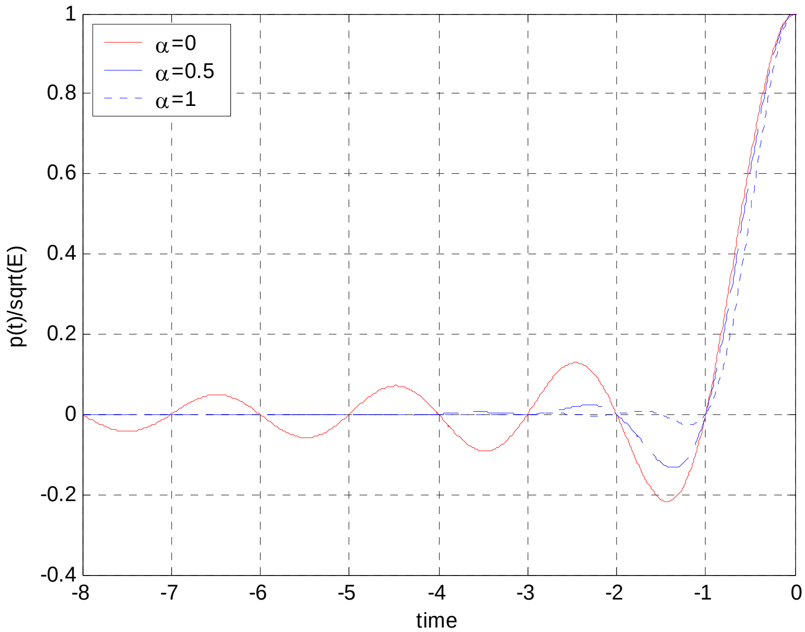
(*2 points*)

4. Problem 6.4 in the text. For part (b), try plotting the signal for the given values of α in order to determine the delay that would be appropriate provided that we don't want to cut off values greater than 5% of the peak value.

(a) Using the Fourier transform property shown below, you can easily prove it.

$$P(f)\exp(-j2\pi f t_0) \Leftrightarrow p(t-t_0)$$ (*2 points*)

(b) To make a system causal, we ideally need to the impulse response equal to zero for $t < 0$. However, $p(t)$ has infinite extension at negative time axis. So, we would select a delay t_0 which results in $p(t-t_0)$ has a very small amplitude. If we select $B_0 = 0.5\text{Hz}$, the pulse repetition time is $T_p = \frac{1}{2B_0} = 1\text{s}$. We can plot $p(t)$ corresponding to different roll off factors as below (we are plot the negative time axis)



Apparently, to bring the signal amplitude down to an acceptable small level, when $a=1$, $t_0=2\text{s}$ is good enough; when $a=1/2$, $t_0=3\text{s}$ is good enough; when $a=0$, $t_0=6\text{s}$ is good enough (5% of the maximum). Any well justified answer will be accepted.

(*2 points*)

5. Problem 6.8 in the text. You may use the known Fourier relationship between equation (6.19) and (6.17).

Using Eq. (6.19) and substituting $\alpha=1$, we have

$$p(t)=\text{sinc}(2B_0\text{t})\frac{\cos(2\pi B_0\text{t})}{1\text{-}16B_0^2t^2}$$

Because,

$$\text{sinc}(2B_0\text{t})\cos(2\pi B_0\text{t})=\frac{\sin(2\pi B_0\text{t})\cos(2\pi B_0\text{t})}{2\pi B_0 t}=\frac{\sin(4\pi B_0\text{t})}{4\pi B_0 t}=\text{sinc}(4B_0\text{t})$$

Plug it back in $p(t)$, we get the desired result. (*2 points*)

6. Problem 6.9 in the text.

$$T_b=\frac{1}{R_b}=\frac{1}{56\text{ k}}\text{s}$$

Since binary PAM is used, the pulse rate is equal to bit rate, so we have

$$T_p=T_b$$

The transmission bandwidth is

$$B_T=B_0(1+\alpha)=\frac{1}{2T_p}(1+\alpha)$$

(a) 35 kHz
(b) 42 kHz
(c) 49 kHz
(d) 56 kHz (*2 points*)

7. Problem 6.15 in the text. Note that quarternary PAM uses four levels (i.e., symbols). Also, the bandwidth of the digital transmit signal equals the bandwidth of the channel.

(a) The transmission bandwidth of a raise-cosine pulse is given by

$$B_T = B_0(1+\alpha) = 13\,\text{kHz}$$

So, the pulse repetition time is

$$T_p = \frac{1}{2B_0} = \frac{1+\alpha}{2B_T} = \frac{1}{B_T} = \frac{1}{13\,\text{kHz}}$$

Using quaternary PAM, each pulse is carrying two bits of information, thus the bit rate is give by

$$R_b = 2\times\frac{1}{T_p} = 26\,\text{kbps}$$ (*2 points*)

(b) Assume the sampling rate is f_s (sample/second), each sample is represented by totally 7+1=8 (bits/sample). Thus the output of PCM has a bit rate of $8f_s$ bits/second. So,

$$f_s = R_b/8 = 26/8 = 3.25\,\text{kHz}$$

The highest frequency of the analog signal is half of sampling frequency

$$f_H = f_s/2 = 1.625\,\text{kHz}$$ (*2 points*)

8. Problem 6.16 in the text. (use Matlab with the function `eyediagram`.) Also, the bit rate should be *$2f_o$* rather than *$2B_o$*.

To generate the low pass filter impulse response, we know that

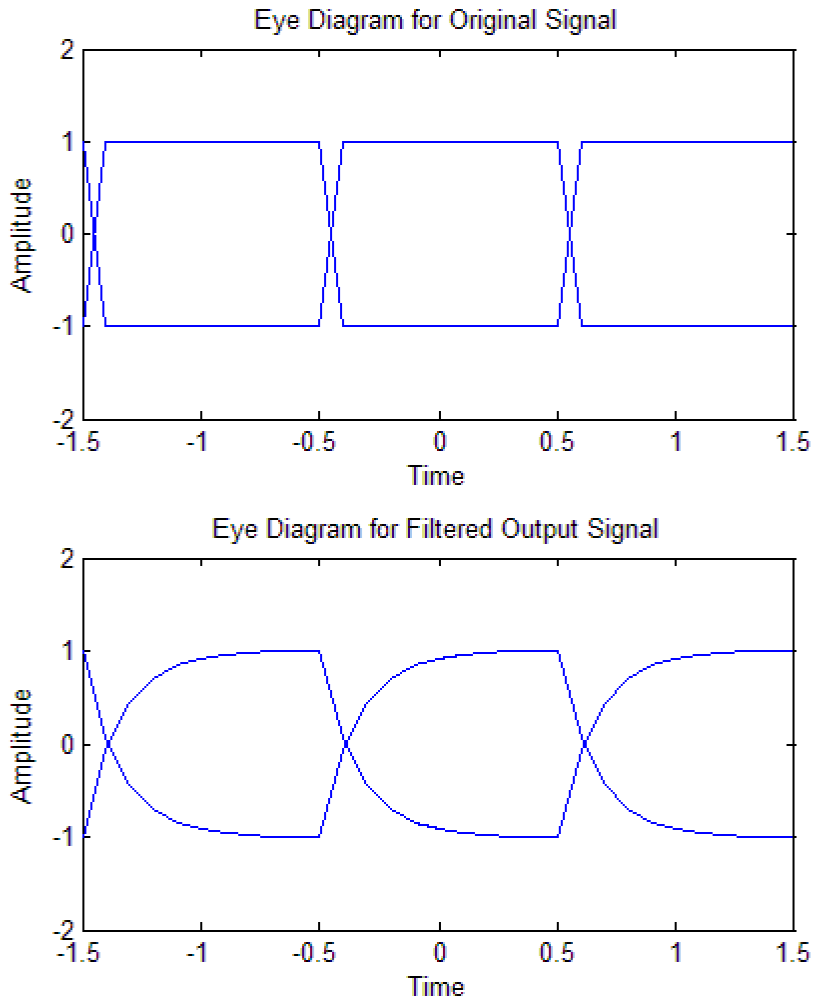
$$H(f) = \frac{1}{1+jf/f_0} \Leftrightarrow h(t) = 2\pi f_0 u(t)\exp(-2\pi f_0 t),$$ (*2 points*)

where $u(t)$ is the unit-step function. Then, what you need to do is to first generate source sequences and convolve it with a truncated version of the impulse response $h(t)$.

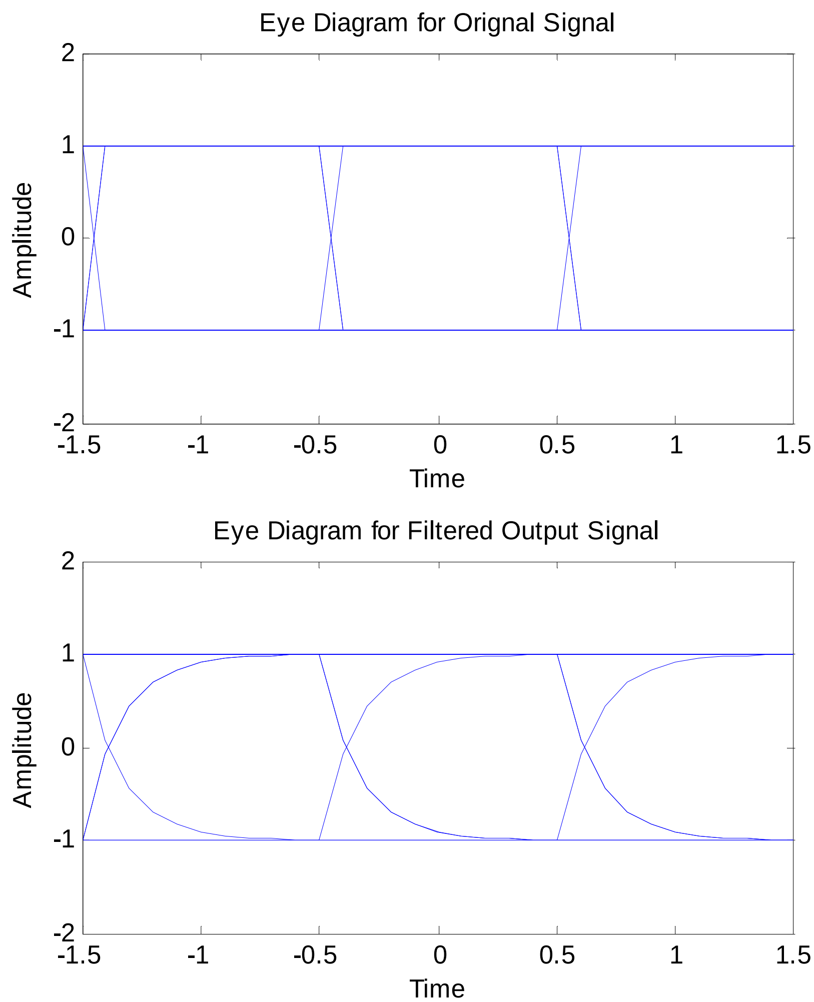
Matlab codes for generating 1st sequence's eyediagram:

```
>> f0 = 1; t0 = 1/f0;
>> a = (-1).^(1 : 1000);
>> s1 = zeros(1, 10000);
>> s1(1 : 10 : 10000) = a;          % source sequence
>> p = ones(1,10);                  %  square pulse
>> x1 = conv(s1, p);                % generate source signal waveform
>> t = 0: 1/10 : 5*t0;              % truncate up to 5*t0
>> h=2*pi*f0*exp(-2*pi*f0*t);  % generate filter impulse response
```

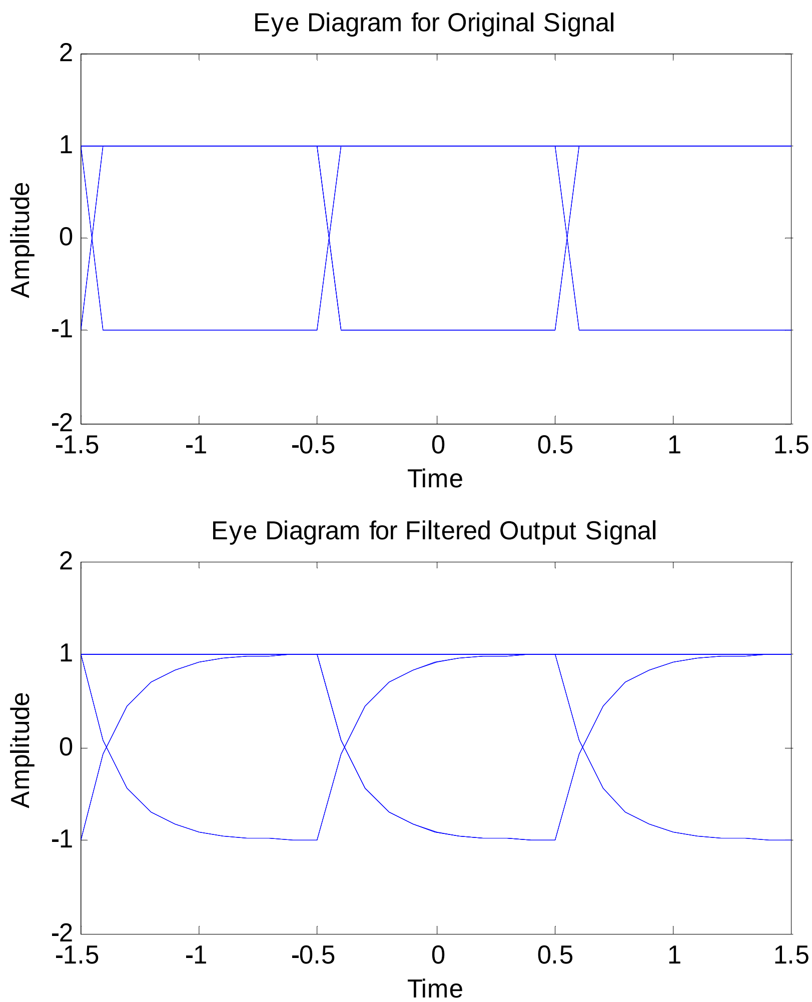
```
>> y1 = conv(x1, h);    % convolving source signal with filter impulse response
>> y1 = y1/max(y1);    % normalize y1 to unit amplitude, only for visualization purpose
>> y1(1:10) = y1(21:30);  % this is to simply remove the edge effect caused by filtering
>> z1 = [x1; y1]';         % arrange x1 and y1 into a two column matrix
>> eyediagram(z1, 30, 3*t0, 4);   % plot 3 eyes, you can also plot only one
```



(*2 points*)


Eye Diagram for Orignal Signal
Amplitude
Time
2
1
0
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
Eye Diagram for Filtered Output Signal
Amplitude
Time

(*2 points*)



(*2 points*)

9. Derive the power spectral density for Binary Phase Shift Keying.

We know that we can relate the power spectral density of a bandpass signal to the complex baseband signal by

$$P_v(f) = \frac{1}{4}\left[P_g(f - f_c) + P_g(-f - f_c)\right]$$

Further, we know that the complex baseband signal for a BPSK bandpass signal is a polar NRZ signal. We can find the PSD of a polar NRZ to zero signal from

$$\frac{|F(f)|^2}{T_s} \sum_{k=-\infty}^{\infty} R(k) e^{-j2\pi fkT_s}$$

Now for polar NRZ

$$R(k) = \begin{cases} A^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and

$$F(f) = T_b \frac{\sin(\pi f T_b)}{\pi f T_b}$$

Thus,

$$P(f) = A^2 T_b \left(\frac{\sin(\pi f T_b)}{\pi f T_b} \right)^2 \qquad \textit{(2 points)}$$

Substituting the result into the equation for bandpass signals results in

$$\begin{aligned} P_v(f) &= \frac{1}{4}\left[P_g(f - f_c) + P_g(-f - f_c) \right] \\ &= \frac{A^2 T_b}{4}\left[\left(\frac{\sin(\pi(f - f_c)T_b)}{\pi(f - f_c)T_b} \right)^2 + \left(\frac{\sin(\pi(-f - f_c)T_b)}{\pi(-f - f_c)T_b} \right)^2 \right] \\ &= \frac{A^2 T_b}{4}\left[\left(\frac{\sin(\pi(f - f_c)T_b)}{\pi(f - f_c)T_b} \right)^2 + \left(\frac{-\sin(\pi(f + f_c)T_b)}{\pi(-f - f_c)T_b} \right)^2 \right] \qquad \textit{(2 points)} \\ &= \frac{A^2 T_b}{4}\left[\left(\frac{\sin(\pi(f - f_c)T_b)}{\pi(f - f_c)T_b} \right)^2 + \left(\frac{\sin(\pi(f + f_c)T_b)}{\pi(f + f_c)T_b} \right)^2 \right] \end{aligned}$$

10. Derive the power spectral density for Binary Amplitude Shift Keying.

We know that we can relate the power spectral density of a bandpass signal to the complex baseband signal by

$$P_v(f) = \frac{1}{4}\left[P_g(f - f_c) + P_g(-f - f_c) \right]$$

Further, we know that the complex baseband signal for a BASK bandpass signal is a unipolar NRZ signal. We can find the PSD of a polar NRZ to zero signal from

$$\frac{|F(f)|^2}{T_s}\sum_{k=-\infty}^{\infty} R(k)e^{-j2\pi fkT_s}$$

Now for unipolar NRZ

$$R(k)=\begin{cases}\dfrac{A^2}{2} & k=0\\[4pt] \dfrac{A^2}{4} & k\neq 0\end{cases}$$

and

$$F(f)=T_b\frac{\sin(\pi fT_b)}{\pi fT_b}$$

Thus,

$$P(f)=\frac{A^2T_b}{4}\underbrace{\left(\frac{\sin(\pi fT_b)}{\pi fT_b}\right)^2}_{\text{due to pulse shape}}\left[1+\underbrace{\frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_b}\right)}_{\substack{\text{discrete terms due to}\\ \text{correlation in data}}}\right]$$

(2 points)

$$=\frac{A^2T_b}{4}\underbrace{\left(\frac{\sin(\pi fT_b)}{\pi fT_b}\right)^2}_{\text{due to pulse shape}}\left[1+\underbrace{\frac{1}{T_b}\delta(f)}_{\substack{\text{discrete term due to}\\ \text{correlation in data}}}\right]$$

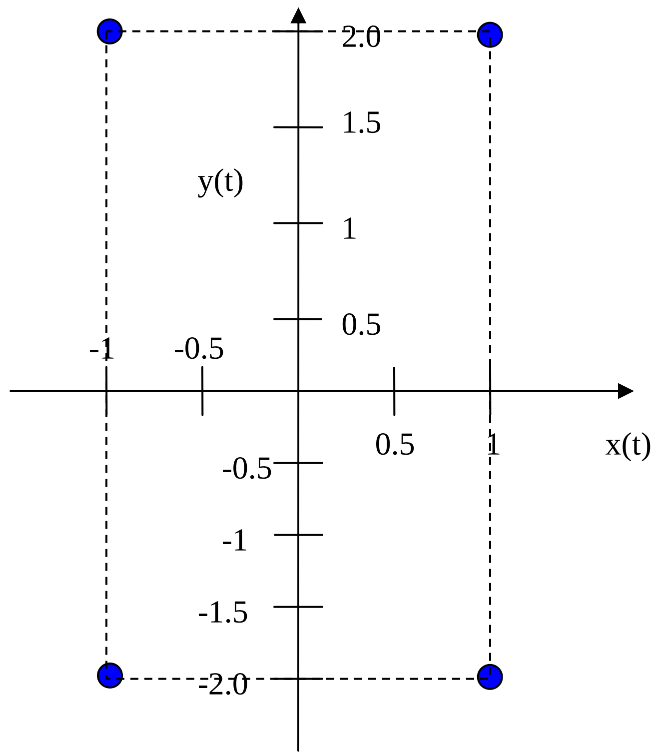
Substituting the result into the equation for bandpass signals results in

$$\begin{aligned}
P_v(f)&=\frac{1}{4}\left[P_g(f-f_c)+P_g(-f-f_c)\right]\\
&=\frac{A^2T_b}{16}\left[\left(\frac{\sin(\pi(f-f_c)T_b)}{\pi(f-f_c)T_b}\right)^2+\left(\frac{\sin(\pi(-f-f_c)T_b)}{\pi(-f-f_c)T_b}\right)^2+\frac{1}{T_b}\delta(f-f_c)+\frac{1}{T_b}\delta(-f-f_c)\right]\\
&=\frac{A^2T_b}{16}\left[\left(\frac{\sin(\pi(f-f_c)T_b)}{\pi(f-f_c)T_b}\right)^2+\left(\frac{-\sin(\pi(f+f_c)T_b)}{\pi(-f-f_c)T_b}\right)^2+\frac{1}{T_b}\delta(f-f_c)+\frac{1}{T_b}\delta(f+f_c)\right]\\
&=\frac{A^2T_b}{16}\left[\left(\frac{\sin(\pi(f-f_c)T_b)}{\pi(f-f_c)T_b}\right)^2+\left(\frac{\sin(\pi(f+f_c)T_b)}{\pi(f+f_c)T_b}\right)^2+\frac{1}{T_b}\delta(f-f_c)+\frac{1}{T_b}\delta(f+f_c)\right]
\end{aligned}$$

(2 points)

11. Suppose that a transmitter modulates a sinusoid by using an in-phase component consisting of a square wave that takes on values of either +1 or -1 and a quadrature component consisting of a square wave that takes on values of either +2 or -2. How is the resulting sinusoid being modulated (amplitude, phase or frequency)? How many values can the (amplitude/phase/frequency) assume and what are they?

Since the In-Phase component is ± 1 *and the quadrature component is* ± 2*, the constellation diagram is:*



From the constellation diagram, we can see that the phase is modulated but not the amplitude or the frequency. *(2 points)*

The amplitude is constant for all four symbols with phase values 63^o, 117^o, -63^o, -117^o. *(2 points)*

Note that this is not an efficient constellation since the top and bottom pairs of symbols are closer together than they need to be. Ideally, they should be equally spaced in phase (45^o, 135^o, -45^o, -135^o).

12. An analog voice signal is to be sampled, quantized and transmitted using BPSK modulation. The original analog signal has a bandwidth of 4kHz. A quantization SNR of 50dB is desired. If the peak to average power ratio is 10dB, what is the second-null-to-null bandwidth of the transmitted signal?

The second null-to-null bandwidth of a BPSK signal is

$$\begin{aligned} B_{2nd} &= 4R_s \\ &= 4R_b \end{aligned}$$

The bit rate can be determined from

$$R_b = nf_s$$

where n is the number of bits per quantization level and f_s is the sampling rate. The sampling rate is determined from the analog signal bandwidth using the Nyquist rate:

$$\begin{aligned} f_s &= 2B \\ &= 2*4kHz \qquad \text{(2 points)} \\ &= 8kHz \end{aligned}$$

The number of bits required is determined by the SNR requirement from the quantizer. Now, the formula for SNR for a voice signal quantized using a uniform quantizer is

$$\frac{S}{N}(dB) = 6.02n + 4.77 - 20\log\left(\frac{V_{peak}}{V_{rms}}\right)$$

We know that the voice signal's peak-to-average power ratio is 10dB. That is $20\log\left(\frac{V_{peak}}{V_{rms}}\right) = 10dB$ *and the requirement is* $\frac{S}{N}(dB) \geq 50dB$. *Thus we have*

$$\begin{aligned} 6.02n + 4.77 - 20\log\left(\frac{V_{peak}}{V_{rms}}\right) &\geq 50dB \\ 6.02n + 4.77 - 10 &\geq 50dB \\ 6.02n &\geq 55.23 \qquad \text{(2 points)} \\ n &\geq 9.17 \\ n &= 10 \end{aligned}$$

Going back to our bandwidth formula:

$$\begin{aligned} B_{2nd} &= 4R_b \\ &= 4*n*f_s \qquad \text{(2 points)} \\ &= 4*10*8kHz \\ &= 320kHz \end{aligned}$$