Digital Communications
Homework #6
Due 10/27/2006
SOLUTION

1: (a) Draw the signal constellation diagram for a 16-QAM signal with maximum amplitude *A*. (b) What is the average energy per symbol? (c) What is the distance between neighboring signal points?

Solution: (a) We know that the points are located at x={-3c,-c,c,3c}, y={-3c,-c,c,3c}for some constant c. The maximum amplitude is thus

$$A = \sqrt{(3c)^2 + (3c)^2}$$
$$= 3\sqrt{2}c$$
$$c = \frac{A}{3\sqrt{2}}$$

The resulting signal constellation is given in Figure 1.

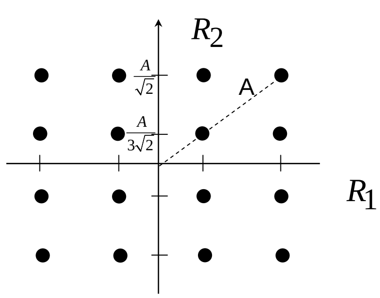

Figure 1: Signal Constellation Diagram for 16-QAM with maximum amplitude A.

(b) The average energy per symbol can be calculated by examining one quadrant of the constellation diagram:

$$
\begin{aligned}
E_s &= \frac{1}{4}\left[\left(\frac{A^2}{9*2}+\frac{A^2}{9*2}\right)+2\left(\frac{A^2}{9*2}+\frac{A^2}{2}\right)+\left(\frac{A^2}{2}+\frac{A^2}{2}\right)\right] \\
&= \frac{1}{4}\left[\left(\frac{A^2}{9}\right)+2\left(\frac{10A^2}{18}\right)+\left(A^2\right)\right] \\
&= \frac{20A^2}{9*4} \\
&= \frac{5A^2}{9}
\end{aligned}
$$

(c) From the signal constellation diagram we can see that the minimum distance between signal points is $\sqrt{\left(\frac{A}{\sqrt{2}}-\frac{A}{3\sqrt{2}}\right)^2} = \frac{A}{\sqrt{2}}-\frac{A}{3\sqrt{2}} = \frac{\sqrt{2}A}{3} \approx 0.47A$.

2. (a) Draw the signal constellation diagram for a 16-PSK signal with maximum amplitude *A*. (b) What is the average energy per symbol? (c) What is the distance between neighboring signal points? (d) Change the amplitude so that the average energy is the same as in 16-QAM presented in problem 1. How do the minimum distance between signal points compare?

(a)

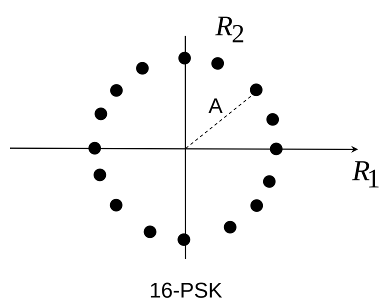

Figure 2: Signal Constellation Diagram for 16-PSK with maximum amplitude A.

(b)The average energy per symbol is equal to the average energy in a single symbol which is A^2.

(c) The distance between neighboring signal points can be calculated using the law of cosines $c^2=a^2+b^2-2ab\cos(C)$ *where* $a=b=A$ *our maximum amplitude and* $C = \pi/8$:

$$
\begin{aligned}
d^2 &= A^2 + A^2 - 2A * A\cos(\pi/8) \\
&= 2A^2\left(1-\cos(\pi/8)\right)\backslash \\
d &= \sqrt{2\left(1-\cos(\pi/8)\right)}A \\
&= 0.39A
\end{aligned}
$$

(d) To make the average energy the same in the two constellations, we simply have to change the amplitude of the PSK scheme to $\sqrt{\frac{5}{9}}A$. *The average distance between points in 16PSK now becomes*

$$
\begin{aligned}
d &= 0.39 * \sqrt{\frac{5}{9}}A \\
&= 0.29A
\end{aligned}
$$

which can be compared to 16QAM which has a minimum distance of 0.47A.

3. In class we said that for differential encoding and decoding of *M*-ary PSK, we can obtain the proper decision statistics using

$$
\begin{aligned}
R_1 &= R_1^i R_1^{i-1} + R_2^i R_2^{i-1} \\
R_2 &= R_2^i R_1^{i-1} - R_1^i R_2^{i-1}
\end{aligned}
$$

Show that this eliminates the unknown phase. (*Hint: use the definitions for* R_1^i *and* R_2^i*.)*

Recall the definitions for the two terms:

$$
\begin{aligned}
R_1^i &= \frac{A_c T_b}{2}\cos\left[\frac{2\pi}{M}m_i(t) + \theta_o\right] \\
R_2^i &= \frac{A_c T_b}{2}\sin\left[\frac{2\pi}{M}m_i(t) + \theta_o\right]
\end{aligned}
$$

Now, returning to the expression:

$$
\begin{aligned}
R_1 &= R_1^i R_1^{i-1} + R_2^i R_2^{i-1} \\
&= \frac{A_c T_b}{2}\cos\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]\frac{A_c T_b}{2}\cos\left[\frac{2\pi}{M} m_{i-1}(t) + \theta_o\right] \\
&\quad + \frac{A_c T_b}{2}\sin\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]\frac{A_c T_b}{2}\sin\left[\frac{2\pi}{M} m_{i-1}(t) + \theta_o\right] \\
&= \frac{A_c^2 T_b^2}{8}\left(\cos\left[\frac{2\pi}{M} m_i(t) + \frac{2\pi}{M} m_{i-1}(t) + 2\theta_o\right] + \cos\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right] + \cos\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right] - \cos\left[\frac{2\pi}{M} m_i(t) + \frac{2\pi}{M} m_{i-1}(t) + 2\theta_o\right]\right) \\
&= \frac{A_c^2 T_b^2}{4}\cos\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right] \\
R_2 &= R_2^i R_1^{i-1} - R_1^i R_2^{i-1} \\
&= \frac{A_c T_b}{2}\sin\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]\frac{A_c T_b}{2}\cos\left[\frac{2\pi}{M} m_{i-1}(t) + \theta_o\right] \\
&\quad - \frac{A_c T_b}{2}\cos\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]\frac{A_c T_b}{2}\sin\left[\frac{2\pi}{M} m_{i-1}(t) + \theta_o\right] \\
&= \frac{A_c^2 T_b^2}{8}\left(\sin\left[\frac{2\pi}{M} m_i(t) + \frac{2\pi}{M} m_{i-1}(t) + 2\theta_o\right] + \sin\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right] - \sin\left[\frac{2\pi}{M}\left(m_{i-1}(t) - m_i(t)\right)\right] - \sin\left[\frac{2\pi}{M} m_i(t) + \frac{2\pi}{M} m_{i-1}(t) + 2\theta_o\right]\right) \\
&= \frac{A_c^2 T_b^2}{8}\left(\sin\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right] + \sin\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right]\right) \\
&= \frac{A_c^2 T_b^2}{4}\sin\left[\frac{2\pi}{M}\left(m_i(t) - m_{i-1}(t)\right)\right]
\end{aligned}
$$

4. Explain the difference between coherent reception, non-coherent reception and differentially coherent reception.

Coherent reception requires an absolute phase reference at the receiver. Non-coherent reception does not need a phase reference at the receiver. Differentially coherent reception requires a phase reference, but it can be obtained from the signal itself. In other words, it does not need a locally generated reference, since it uses the previous symbol as the phase reference for the current symbol. This can be done since the information is encoded in the phase difference between consecutive symbols.

5. (a) Explain the trade-off between energy efficiency and bandwidth efficiency for MPSK and MASK. (b) Does this trade-off differ for MFSK?

(a) As we increase M for both PSK and ASK, we increase the spectral efficiency (more bits per symbol or equivalently more bits/sec/Hz) and we decrease the energy efficiency (more energy is required to obtain the same error performance).
(b) For MFSK the trade-off is the opposite. As we increase M we actually improve energy efficiency since the symbols do not get any farther apart while we are able to put more bits on each symbol. However, since FSK is an M-ary orthogonal modulation scheme, we require more dimensions in signal space to maintain orthogonality as M grows. This means that we must increase our bandwidth with M which is exponentially related to the number of bits per symbol.

6. Assume that a QPSK signal is used to send data at a rate of 30 Mbps over a satellite transponder. The transponder has a bandwidth of 24MHz.

(a) If the satellite signal is equalized to have an equivalent raised cosine filter characteristic what is the roll-off factor required?
(b) Could a rolloff factor *r* be found so that a 50-Mbps data rate could be supported?

Solution:

(a) The transmission bandwidth is $B_T = (1+r)\frac{R}{l}$ *where r is the rolloff factor, R is the bit rate and l is the number of bits per symbol. Thus, solving for r with R=30Mbps,* B_T*=24MHz, and l=2:*

$$
\begin{aligned}
B_T &= (1+r)\frac{R}{l} \\
r &= \frac{B_T l}{R} - 1 \\
&= \frac{24*10^6*2}{30*10^6} - 1 \\
&= 0.6
\end{aligned}
$$

(b) The maximum data rate allowable will occur when r=0

$$
\begin{aligned}
B_T &= (1+r)\frac{R}{l} \\
R &= \frac{l}{1+r} B_T \\
&= 2*24*10\wedge 6 \\
&= 48 Mbps
\end{aligned}
$$

Thus, a rolloff factor could not be found to support a 50Mbps link.

7. (a) Using Matlab generate 10,000 Gaussian random variables with mean 1 and variance ½. Use the "**randn**" function. Create and plot a histogram. Hint: Use the following commands

```
range = -4:0.01:6;
h = hist(g, range);
plot(range, h/sum(h));
```

where *g* is the vector of Gaussian random variables. (b) Superimpose a plot of the Gaussian density distribution on the histogram. (The command `hold on;` will allow you to put multiple plots on one graph.)

The above commands will provide a plot of the histogram of a GRV with mean 1 and variance ½ as shown in Figure 1. The theoretical plot can be obtained by using

```
hold on;
thy = 1/sqrt(2*pi*1/2)*exp(1/(2*1/2)*(range-1).^2);
plot(range, thy*0.01, 'r')
legend('Histogram', 'Theoretical PDF')
xlabel('x')
ylabel('f_X(x)')
```

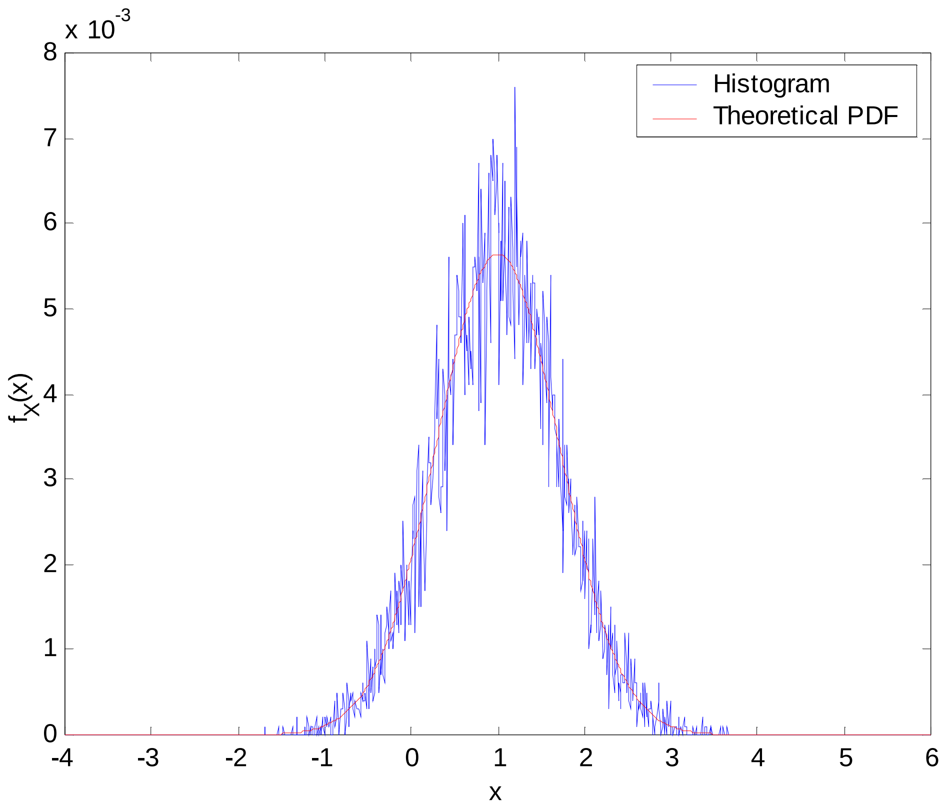

Figure 1: Measured Histogram and Theoretical Gaussian Probability Density Function

8. (a) Using Matlab create 10,000 instances of the random variable $Y = \sqrt{X_1^2 + X_2^2}$ where X_1 and X_2 are Gaussian random variables with mean 0 and variance = ½. Plot the histogram. (b) What type of random variable is Y? (c) Superimpose a plot of the probability density function.

The square root of the sum of two squared GRV's is a Rayleigh random variable. The plot of the histogram (using the same procedure as in (1) and the theoretical pdf are given in Figure 2 A Rayleigh distribution is given as:

$$p_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}.$$

Where σ^2 = ½ in this case.

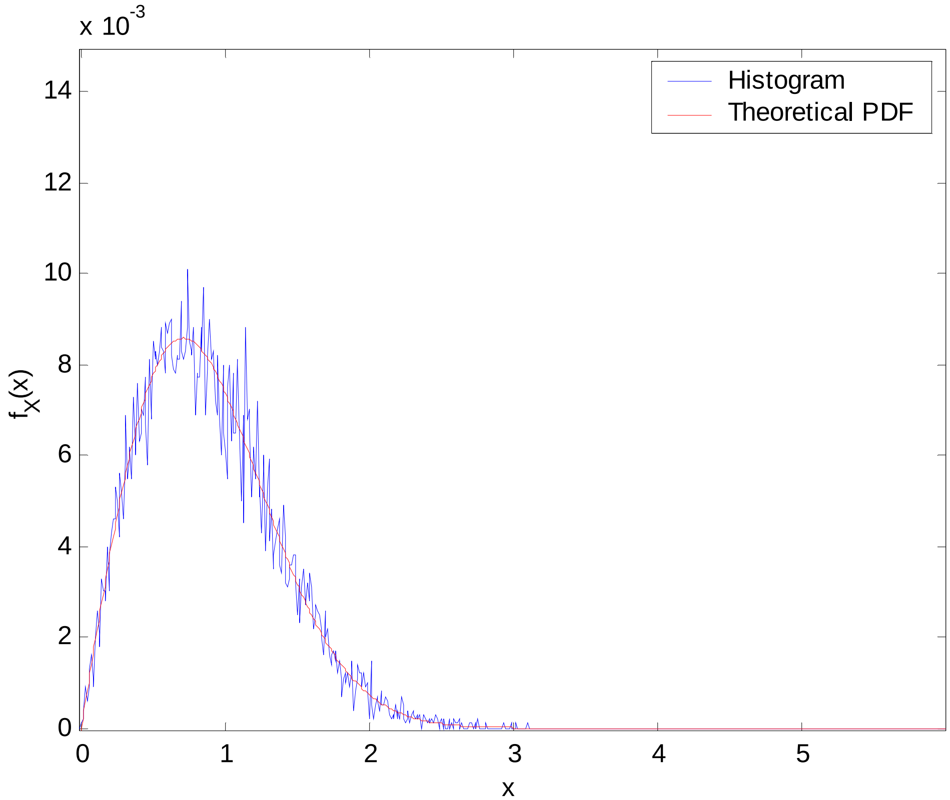

Figure 2: Measured Histogram and Theoretical Rayleigh Probability Density Function

9. Using Matlab create 10,000 sample functions of the random process $X(t) = 2\cos(2\pi t + \theta)$ over the time period $0< t <10$ where θ is a uniform random variable distributed on (0,2π). Find the histogram of *X(t=1).*

The histogram can be generated by using the Matlab commands:

```
x = zeros(10000,1000);
t = 0:0.01:10-0.01;
for k=1:10000
   x(k,:) = 2*cos(2*pi*t+rand*2*pi);
end
z = x(:,101);
range = -2.5:0.01:2.5;
h = hist(z, range);
plot(range, h/sum(h));
thy = 1/pi./sqrt(2^2-range.^2);
hold on
plot(range, thy*0.01, 'r');
```

The pdf of the random variable x(t=1) is a sinusoidal distribution or

$$f(x) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - x^2}} & |x| \le A \\ 0 & else \end{cases}$$

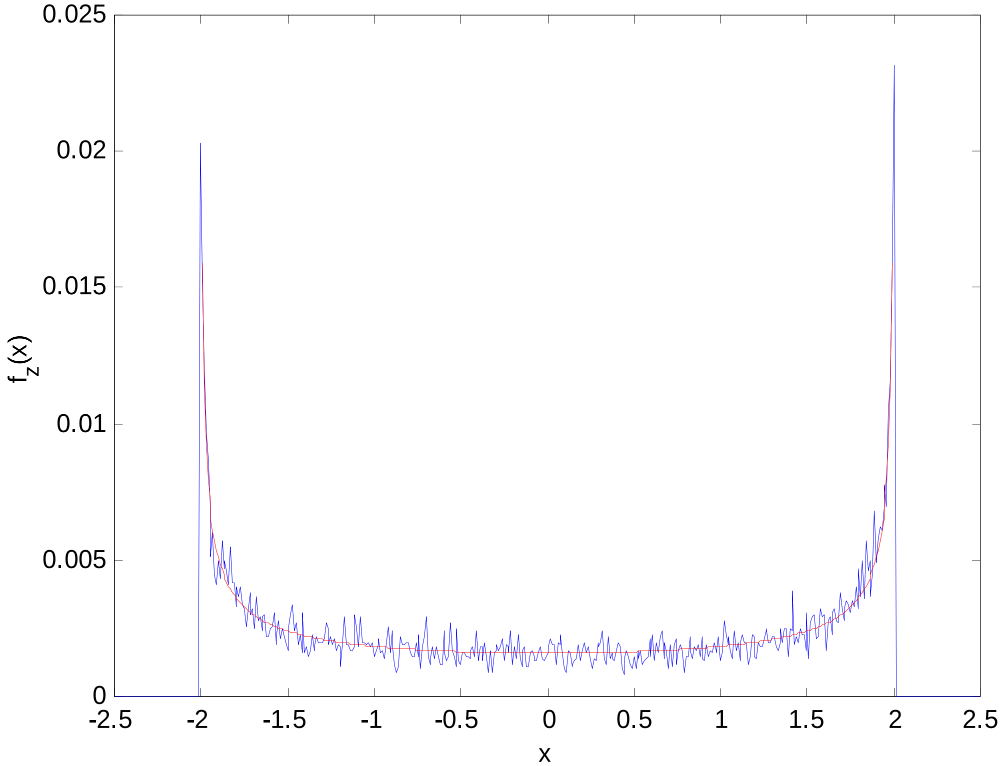

Figure 3: Measured Histogram and Theoretical Probability Density Function for Problem 3

10. A random variable *x* has a PDF

$$f(x) = \begin{cases} \frac{3}{32}\left(-x^2 + 8x - 12\right) & 2 < x < 6 \\ 0 & else \end{cases}$$

(a) Demonstrate that it is a valid PDF.
(b) Find the mean
(c) Find the second moment.
(d) Find the variance.

(a) To show that we have a valid pdf, we must show that it integrates to 1:

$$\int_2^6 f(x)dx = \int_2^6 \frac{3}{32}\left(-x^2 + 8x - 12\right)dx$$

$$= \frac{3}{32}\left[-\frac{x^3}{3} + 4x^2 - 12x\right]_2^6$$

$$= \frac{3}{32}\left[-\frac{6^3}{3} + 4*6^2 - 12*6 + \frac{2^3}{3} - 4*2^2 + 12*2\right]$$

$$= \frac{3}{32}\left[-\frac{216}{3} + \frac{432}{3} - \frac{216}{3} + \frac{8}{3} - \frac{48}{3} + \frac{72}{3}\right]$$

$$= \frac{3}{32}\frac{32}{3}$$

$$= 1$$

(b) We can easily find the mean as:

$$\overline{x} = \int_2^6 xf(x)dx$$

$$= \int_2^6 x\frac{3}{32}\left(-x^2 + 8x - 12\right)dx$$

$$= \frac{3}{32}\left[-\frac{x^4}{4} + \frac{8}{3}x^3 - 6x^2\right]_2^6$$

$$= \frac{3}{32}\left[-\frac{6^4}{4} + \frac{8}{3}*6^3 - 6*6^2 + \frac{2^4}{4} - \frac{8}{3}*2^3 + 6*2^2\right]$$

$$= \frac{3}{32}\left[-\frac{3888}{12} + \frac{6912}{12} - \frac{2592}{12} + \frac{48}{12} - \frac{256}{12} + \frac{288}{12}\right]$$

$$= \frac{3}{32}\frac{512}{12}$$

$$= 4$$

(c) *The second moment is then:*

$$\int_a^b x^2 f(x)dx = \int_2^6 x^2 \left(\frac{3}{32}\right)\left(-x^2+8x-12\right)dx$$

$$= \int_2^6 \left(\frac{3}{32}\right)\left(-x^4+8x^3-12x^2\right)dx$$

$$= \left(\frac{3}{32}\right)\left[-\frac{x^5}{5}+2x^4-4x^3\right]_2^6$$

$$= \left(\frac{3}{32}\right)\left[-\frac{6^5}{5}+2*6^4-4*6^3+\frac{2^5}{5}-2*2^4+4*2^3\right]$$

$$= \left(\frac{3}{32}\right)\left[-\frac{23328}{15}+\frac{38880}{15}-\frac{12960}{15}+\frac{96}{15}-\frac{480}{15}+\frac{480}{15}\right]$$

$$= 16.8$$

(d) *The variance is the second moment about the mean:*

$$\int_a^b \left(x-\overline{x}\right)^2 f(x)dx = \int_2^6 \left(x-4\right)^2 \left(\frac{3}{32}\right)\left(-x^2+8x-12\right)dx$$

$$= \int_2^6 \left(\frac{3}{32}\right)\left(x^2-8x+16\right)\left(-x^2+8x-12\right)dx$$

$$= \left(\frac{3}{32}\right)\int_2^6 \left(-x^4+16x^3-92x^2+224x-192\right)dx$$

$$= \left(\frac{3}{32}\right)\left[-\frac{x^5}{5}+4x^4-\frac{92}{3}x^3+112x^2-192x\right]_2^6$$

$$= \left(\frac{3}{32}\right)\left[-\frac{6^5}{5}+4*6^4-\frac{92}{3}*6^3+112*6^2-192*6+\frac{2^5}{5}-4*2^4+\frac{92}{3}*2^3-112*2^2+192*2\right]$$

$$= \left(\frac{3}{32}\right)[-1555.2+5184-6624+4032-1152+6.4-64+245.3-448+384]$$

$$= 0.797$$