Digital Communications
Homework #6

SOLUTION

1. The noise equivalent bandwidth of a filter is defined as the width of a fictitious rectangular filter transfer function such that the power in the rectangular band is equal to the power associated with the actual filter over positive frequencies. In other words if we define

$$P_{pos} = \int_0^\infty \left|H(f)\right|^2 df$$

Then we wish to find B_{eq} such that

$$P_{pos} = B_{eq}\left|H(f_0)\right|^2$$

where f_0 is the frequency where the transfer function is the largest. Combining the two equations we find that the noise equivalent bandwidth for any filter can be written as

$$B_{eq} = \frac{1}{\left|H(f_0)\right|^2}\int_0^\infty \left|H(f)\right|^2 df$$

If square pulses with unit energy ($E = \int_{-\infty}^{\infty} \left|p(t)\right|^2 dt = 1$) are used to transmit PCM data, find the noise equivalent bandwidth of the matched filter. Assume that the symbol duration is T_s.

Square pulses

$$p(t) = \frac{1}{\sqrt{T_s}} rect\left(\frac{t}{T_s}\right) \Leftrightarrow P(f) = \sqrt{T_s}\,\mathrm{sinc}(\mathrm{fT})$$

$$B_{eq} = \frac{1}{\left|H(f_o)\right|^2}\int_0^\infty \left|H(f)\right|^2 dF = \frac{T_s}{T_s^2}\int_0^\infty \frac{T_s^{\,2}}{T_s}\mathrm{sinc}^2(fT)dF = \int_0^\infty \mathrm{sinc}^2(fT_s)dF$$

let $\begin{matrix} x = fT_s \\ dx = T_s df \end{matrix} \Rightarrow \frac{1}{T_s}\int_0^\infty \mathrm{sinc}^2(x)dx = \underline{\underline{\frac{1}{2T_s}}}$ *(2 points)*

2. Repeat problem 1 for sinc pulses. Note that T_s is the time between pulses not the pulse duration. What does this say about a matched filter?

Sinc Pulses

$$p(t) = \frac{1}{\sqrt{T_s}} \operatorname{sinc}\left(\frac{\mathrm{t}}{\mathrm{T_s}}\right) \Leftrightarrow P(f) = \frac{1}{\sqrt{T_s}\,(1/T_s)} rect(fT) = \sqrt{T_s}\, rect(fT)$$

$$B_{eq} = \frac{1}{\left|H(f_o)\right|^2} \int_0^\infty \left|H(f)\right|^2 df = \frac{1}{T_s} \int_0^\infty T_s rect(fT) df = \underline{\frac{1}{2T_s}} \qquad \textit{(2 points)}$$

A matched filter will pass the same noise power regardless of pulse shape since it has a normalized pulse energy.

3. Use the results in problems 2 and 3 to determine the signal-to-noise ratio at the output of the matched filter.

Matched filter output:

noise power = $N_o B_{eq} = \frac{N_o}{2T_s}$

Assuming all signal power is captured (because matched filter is used)

$$\frac{S}{N} = \frac{S}{\left(N_o / 2T_s\right)} = \underline{\frac{2E_s}{N_o}} \qquad \textit{(2 points)}$$

4. A random variable *x* has a PDF

$$f(x) = \begin{cases} \frac{3}{32}\left(-x^2 + 8x - 12\right) & 2 < x < 6 \\ 0 & else \end{cases}$$

(a) Demonstrate that it is a valid PDF.
(b)Find the mean
(c) Find the second moment.
(d) Find the variance.

(a) To show that we have a valid pdf, we must show that it integrates to 1:

$$\int_2^6 f(x)dx = \int_2^6 \frac{3}{32}\left(-x^2+8x-12\right)dx$$

$$= \frac{3}{32}\left[-\frac{x^3}{3}+4x^2-12x\right]_2^6$$

$$= \frac{3}{32}\left[-\frac{6^3}{3}+4*6^2-12*6+\frac{2^3}{3}-4*2^2+12*2\right]$$

$$= \frac{3}{32}\left[-\frac{216}{3}+\frac{432}{3}-\frac{216}{3}+\frac{8}{3}-\frac{48}{3}+\frac{72}{3}\right]$$

$$= \frac{3}{32}\frac{32}{3}$$

$$= 1$$

(2 points)

(b) We can easily find the mean as:

$$\bar{x} = \int_2^6 xf(x)dx$$

$$= \int_2^6 x\frac{3}{32}\left(-x^2+8x-12\right)dx$$

$$= \frac{3}{32}\left[-\frac{x^4}{4}+\frac{8}{3}x^3-6x^2\right]_2^6$$

$$= \frac{3}{32}\left[-\frac{6^4}{4}+\frac{8}{3}*6^3-6*6^2+\frac{2^4}{4}-\frac{8}{3}*2^3+6*2^2\right]$$

$$= \frac{3}{32}\left[-\frac{3888}{12}+\frac{6912}{12}-\frac{2592}{12}+\frac{48}{12}-\frac{256}{12}+\frac{288}{12}\right]$$

$$= \frac{3}{32}\frac{512}{12}$$

$$= 4$$

(2 points)

(c) *The second moment is then:*

$$\int_a^b x^2 f(x)dx = \int_2^6 x^2 \left(\frac{3}{32}\right)\left(-x^2+8x-12\right)dx$$

$$= \int_2^6 \left(\frac{3}{32}\right)\left(-x^4+8x^3-12x^2\right)dx$$

$$= \left(\frac{3}{32}\right)\left[-\frac{x^5}{5}+2x^4-4x^3\right]_2^6$$

$$= \left(\frac{3}{32}\right)\left[-\frac{6^5}{5}+2*6^4-4*6^3+\frac{2^5}{5}-2*2^4+4*2^3\right]$$

$$= \left(\frac{3}{32}\right)\left[-\frac{23328}{15}+\frac{38880}{15}-\frac{12960}{15}+\frac{96}{15}-\frac{480}{15}+\frac{480}{15}\right]$$

$$= 16.8$$

(2 points)

(d) *The variance is the second moment about the mean:*

$$\int_a^b \left(x-\overline{x}\right)^2 f(x)dx = \int_2^6 \left(x-4\right)^2 \left(\frac{3}{32}\right)\left(-x^2+8x-12\right)dx$$

$$= \int_2^6 \left(\frac{3}{32}\right)\left(x^2-8x+16\right)\left(-x^2+8x-12\right)dx$$

$$= \left(\frac{3}{32}\right)\int_2^6 \left(-x^4+16x^3-92x^2+224x-192\right)dx$$

$$= \left(\frac{3}{32}\right)\left[-\frac{x^5}{5}+4x^4-\frac{92}{3}x^3+112x^2-192x\right]_2^6$$

$$= \left(\frac{3}{32}\right)\left[-\frac{6^5}{5}+4*6^4-\frac{92}{3}*6^3+112*6^2-192*6+\frac{2^5}{5}-4*2^4+\frac{92}{3}*2^3-112*2^2+192*2\right]$$

$$= \left(\frac{3}{32}\right)[-1555.2+5184-6624+4032-1152+6.4-64+245.3-448+384]$$

$$= 0.797$$

(2 points)

5. A constant voltage signal of *V* volts is corrupted by noise signal. In other words, a sample of the received signal voltage is modeled as $R = V + N$ where *N* is a random variable with a Gaussian pdf. Given that *N* has a mean of 0 volts and a standard deviation of *V*/10 volts. What is the probability that the random variable *R* is less than zero? (Note that if you would like a version of the *Q*-function, it is available on the Matlab page of the course website.)

$$\Rightarrow R = V + N, \quad N = N\left(V, (V/10)^2\right)$$

$$\Rightarrow \Pr[R < 0] = \int_{-\infty}^{0} p_r(x)dx$$

where $p_r(x)dx = \frac{1}{\sqrt{2\pi}(V/10)} e^{\frac{-(x-V)^2}{(2V^2/100)}}$ (Gaussian pdf with mean *V* and variance $(V/10)^2$)

$$\Rightarrow \Pr[R < 0] = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}(V/10)} e^{\frac{-(x-V)^2}{(2V^2/100)}} dx$$

let $y = \frac{x - V}{(V/10)}$, $dy = \frac{dx}{(v/10)}$ $\Rightarrow \Pr[R < 0] = \int_{-\infty}^{\frac{-V}{(V/10)}}$

by symmetry = $\int_{\frac{V}{(V/10)}=10}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = Q(10) \approx \underline{1e^{-22}}$ *(2 points)*