ECE4634
Digital Communications
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Lecture #2: Review of Signals and the Fourier Transform

Overview

- Information in communication systems is transferred through the use of EM waves
- At each point in the system, we observe signals. These signals can be described mathematically using both the time and the frequency domains.
- While the time domain is more familiar to most students, often the frequency domain is more intuitive for understanding certain signal characteristics
- At the receiver we observe both the desired waveform as well as undesired waveforms such as *noise* and *interference*.
- Reading
  - Sections 2.1-2.3

Objectives for this Lecture

- Review important properties of signals
- Review a key mathematical tool for analyzing communication systems the *Fourier Transform*
  - Motivation for Fourier Theory
  - Common Fourier Transforms
  - Fourier Transform Properties

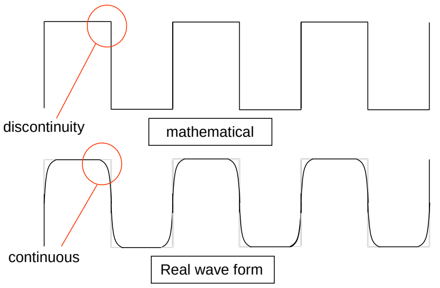
Course Objectives

- **Design digital communication systems, given constraints on data rate, bandwidth, power, fidelity, and complexity;**
- Analyze the performance of a digital communication link when additive noise is present in terms of the signal-to-noise ratio and bit error rate;
- **Compute the power and bandwidth requirements of modern communication systems, including those employing ASK, PSK, FSK, and QAM modulation formats;**
- Design a scalar quantizer for a given source with a required fidelity and determine the resulting data rate;
- **Determine the auto-correlation function of a line code and determine its power spectral density;**
- **Determine the power spectral density of bandpass digital modulation formats**.

Physically Realizable Waveforms

- Have finite time duration (finite energy!)
- Occupy finite frequency spectrum
- Are continuous
- Have finite peak value
- Are real-valued
- All real-world signals will have these properties, although sometimes we use mathematical models which violate these conditions.

Mathematical Representations



•Extends to infinity in time.
•Has infinite frequency extent

•Ends after T seconds
•Has finite frequency content

Mathematical Representations

- Thus, when we analyze communication systems we use mathematical models.
- These models allow for convenient analysis but are not completely accurate concerning the real world.
- Fortunately, they provide close enough approximation that the conclusions reached using the models are still valid.

Energy and Power

- **Energy:** $E = \int_{-\infty}^{\infty} w^2(t)dt$
- A signal *w(t)* is an Energy Signal if $0 < E < \infty$
- **Power:** $P = \left\langle w^2(t) \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} w^2(t)dt$
- For periodic signals, power can be computed by integrating over one period
- A signal *w(t)* is a Power Signal if $0 < P < \infty$

Decibels

- Base 10 logarithmic measure of *power* ratios
- Useful when:
  - Power and energy levels vary over orders of magnitude
  - It is the ratio of two powers that is important
- When comparing power or energy:

$$dB = 10\log_{10}\left(P_1/P_2\right)$$

- Sometimes it is useful to compare a power with 1 W or with 1 mW:

$$dBW = 10\log_{10}\left(P_1/1W\right)$$

$$dBm = 10\log_{10}\left(P_1/1mW\right)$$

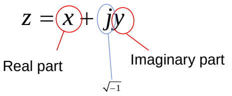
For voltages or currents:

$$dB = 20\log_{10}\left(V_1/V_2\right)$$

$$= 20\log_{10}\left(I_1/I_2\right)$$

Complex Numbers

- A complex number is a number composed of two real numbers, one which represents the “real” part and one which represents the “imaginary” part (originally created for defining roots of a polynomial)

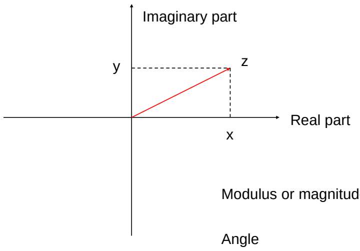


$$z = x + jy$$

- We define addition and multiplication as

$$z = x + jy$$
$$c = a + jb$$
$$z + c = (x + a) + j(y + b)$$
$$z * c = (xa - yb) + j(ya + xb)$$

Complex Plane



Modulus or magnitude $|z| = \sqrt{x^2 + y^2}$

Angle $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\begin{aligned} z &= x + jy \\ &= |z|(\cos\theta + j\sin\theta) \end{aligned}}$$

Why do we care?

- **Answer**: Because of phase!
- In communications we often deal with sinusoidal signals
  - Further we may always represent a signal as the sum (or integral) of sinusoidal signals
  - Sinusoids are conveniently represented using complex numbers because of phase

$$\begin{aligned} x(t) &= A\cos(\omega_c t + \theta) \\ &= A\cos\theta\cos(\omega_c t) - A\sin\theta\sin(\omega_c t) \end{aligned}$$

A convenient short hand is: $\tilde{x}(t) = Ae^{j\theta}$ where the frequency ω_c is assumed

Using phasors

- The original signal is then represented as

$$x(t) = \mathrm{Re}\left\{\tilde{x}(t)e^{j\omega_c t}\right\}$$

- Note that while $\tilde{x}(t)$ is complex, the true signal *x(t)* is always real. The complex nature is simply a convenient mathematical construct to readily handle phase components
- Further note that

$$\begin{aligned}\tilde{x}(t) &= Ae^{j\theta}\\ &= A\cos\theta + jA\sin\theta\end{aligned}$$

- This form will be particularly convenient when we analyze bandpass communication systems
- Note that while physical signals are always "real", the imaginary part of $\tilde{x}(t)$ will have physical meaning as we will see later in the course

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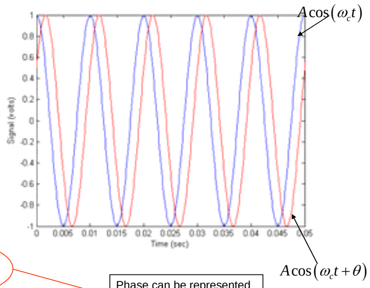
Phase

- Phase represents time delay of a sinusoid

$$\begin{aligned}x(t) &= A\cos(\omega_c t)\\ x(t-t_o) &= A\cos(\omega_c(t-t_o))\\ &= A\cos(\omega_c t-\omega_c t_o)\\ &= A\cos(\omega_c t-\theta)\end{aligned}$$

$$\begin{aligned}x(t) &= A\cos(\omega_c t)\\ x(t-t_o) &= A\cos(\omega_c t-\theta)\\ &= A\cos(\theta)\cos(\omega_c t)+\\ &\quad A\sin(\theta)\sin(\omega_c t)\end{aligned}$$



Phase can be represented by a weighted sum of cos and sin

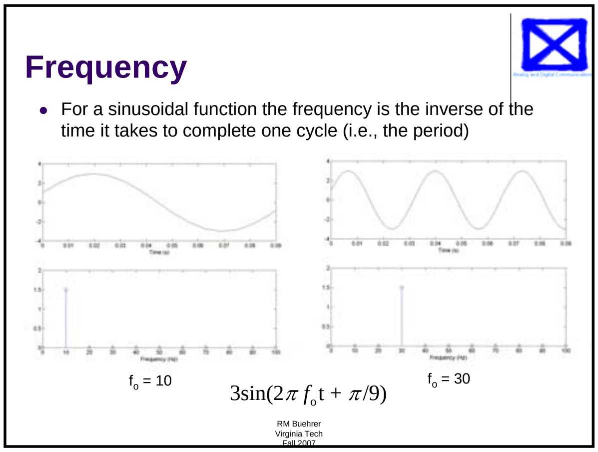
Euler's Identities

- $e^{jx} = \cos(x) + j\sin(x)$
- $\sin(x) = \dfrac{e^{jx} - e^{-jx}}{2j}$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

- **Note:** $\left|e^{jx}\right| = 1$

This is termed a *phasor* representation of sinusoidal signals. We will use this representation extensively.



Frequency

- For a sinusoidal function the frequency is the inverse of the time it takes to complete one cycle (i.e., the period)

$f_o = 10$ $f_o = 30$

$$3\sin(2\pi f_o t + \pi/9)$$

Frequency

- Any signal can be expressed as the weighted sum of sinusoids of varying frequencies and phases.
- Many physical objects respond to EM waves based on the frequency of the wave. Thus, we are interested in the frequency of signals.
- Fourier Theory allows us to view the ‘frequency content’ of a signal by decomposing the signal into an infinite sum (or an integral) of sinusoids.
- The Fourier Transform tells us how much of each frequency is needed.
  - Magnitude tells us the amount of each frequency
  - The phase tells us how much cos vs sin

The Fourier Transform

- One of the most common mathematical tools for analyzing a signal or waveform is the Fourier Transform.
- The Fourier Transform provides us with information concerning the *frequency content*.
- This is useful for:
  - Determining bandwidth
  - Demodulating frequency modulated signals
  - Understanding how objects or systems will respond to a signal (Transfer Function)
  - Equalization

The Fourier Transform

- The Fourier Transform of a signal *w(t)* is given by:

$$W(f) = F\{w(t)\} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi ft} dt$$

We correlate the signal with a complex sinusoid of frequency *f* to determine how much of that frequency is present.

- The Inverse Fourier Transform is given by:

$$w(t) = F^{-1}\{W(f)\} = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df$$

- We denote a Fourier Transform pair by: $w(t) \Leftrightarrow W(f)$
- The Fourier Transform always exists if *w(t)* is an Energy Signal

We sum complex sinusoids of different frequencies *f*, weighting them by the amount of each frequency contained by the signal.

Why do we use *complex* sinusoids?

- Answer: We are interested in *phase*
- If we defined the Fourier Transform as

$$W(f) = F\{w(t)\} = \int_{-\infty}^{\infty} w(t)\cos(2\pi f)\,dt$$

we would lose any signal information related to *sin(ωt)*

- If we only used *sin(ωt)* in the transform, we would lose any signal information related to *cos(ωt)*
- Thus we must have both: $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

Interpretation of Fourier Transform

- The Fourier Transform may be thought of as a tool for looking at a signal from a different perspective
  - Consider how different a chair might look when viewed from different angles

- Frequency measures the rate of change
  - High frequency corresponds to rapid change with time
  - Low frequency corresponds to slow change with time

Fourier Transform Example 2.1: Rectangular Pulse

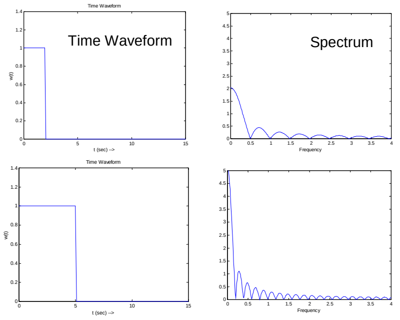
$$rect\left(\frac{t}{T}\right)=\begin{cases}1, & |t|\le T/2\\ 0, & |t|>T/2\end{cases}$$

$$W(f)=\int_{-T/2}^{T/2}1\cdot e^{-j2\pi ft}dt \quad =\frac{e^{-j\pi fT}-e^{j\pi fT}}{-j2\pi f}$$

$$\sin(x)=\left(e^{jx}-e^{-jx}\right)/2j$$

$$\boxed{W(f)=\frac{\sin(\pi fT)}{\pi f}=T\frac{\sin(\pi fT)}{\pi fT}=T\mathrm{sinc}(fT)}$$

Time - Frequency



As we expand the duration of the pulse, the signal changes less rapidly.
Thus the signal has more low frequency content.
Time expands → Frequency compresses

Approach to Finding Fourier Transform Pairs

- We could continue to find transform pairs according to the definition, but this is inefficient
- In general, we compile a table of known transform pairs
- We also compile a table of simple rules for modifying transform pairs.
- Using the known pairs and transform properties we can find most transforms needed.

Fourier Transform Pairs

Rectangular Pulse	$\text{rect}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]$
Triangular Pulse	$\text{tri}\left(\frac{t}{T}\right)$	$T\left[\text{sinc}(fT)\right]^2$
Unit Step	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
Signum	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at t_o	$\delta(t-t_o)$	$e^{-j2\pi f t_o}$
Sinc	$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
Phasor	$e^{j\omega_o t+\varphi}$	$e^{j\varphi}\delta(f-f_o)$
Sinusoid	$\cos(2\pi ft+\varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f-f_o)+\frac{1}{2}e^{-j\varphi}\delta(f+f_o)$
Gaussian	$e^{-\pi(t/t_o)^2}$	$t_o e^{-\pi(f t_o)^2}$

Note: Think of a constant as a sinusoid with an infinite period (f = 0). Does the transform make sense?

Important Transform Properties

- Linearity
- Time Delay
- Scale Change
- Duality
- Modulation
- Convolution
- Differentiation
- Integration

Fourier Transform Properties

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t)+a_2 w_2(t)$	$a_1 W_1(f)+a_2 W_2(f)$
Time Scale	$w(at)$	$\frac{1}{\lvert a\rvert}W\left(\frac{f}{a}\right)$
Delay	$w(t-\tau)$	$W(f)e^{-j2\pi f\tau}$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation	$w(t)\cos\left(2\pi ft+\theta\right)$	$\frac{1}{2}\left[e^{j\theta}W(f-f_c)+e^{-j\theta}W(f+f_c)\right]$
Complex Signal Frequency Translation	$w(t)e^{j2\pi f_c t}$	$W(f-f_c)$
Bandpass Signal	$\text{Re}\left[g(t)e^{j2\pi f_c t}\right]$	$\frac{1}{2}\left[G(f-f_c)+G^*(-f-f_c)\right]$
Differentiation	$\frac{d^n}{dt^n}w(t)$	$(j2\pi f)^n W(f)$
Integration	$\int_{-\infty}^{t} w(s)ds$	$(j2\pi f)^{-1}W(f)+{}^1\!/_2\,W(0)\delta(f)$
Convolution	$w_1(t)*w_2(t)$	$W_1(f)W_2(f)$
Multiplication	$w_1(t)w_2(t)$	$W_1(f)*W_2(f)$
Multiplication by *t*	$t^n w(t)$	$(-j2\pi)^{-n}\frac{d^n W(f)}{df^n}$

In-class drill

- Find a mathematical expression for

Summary

- Today we have reviewed important concepts from signal and system theory that are particularly useful to communication system analysis
- Most importantly we will extensive use the Fourier Transform to analyze signals and systems in the *frequency domain*
- You should review
  - Fourier Transform pairs
  - Fourier Transform properties
- Next class we will examine useful functions termed *singularity functions,* review the concepts of Energy and Power Spectral Density, and the application of Fourier Theory to linear systems

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Appendix

Additional Examples

Fourier Transform Example 2.2: Exponential

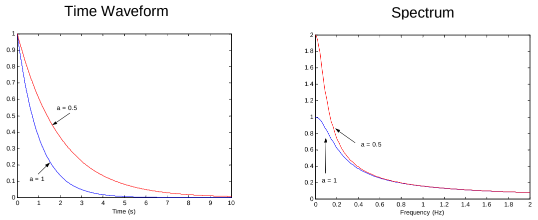
$$w(t) = u(t)e^{-at} = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$W(f) = \int_{-\infty}^{\infty} u(t)e^{-at}e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(a+j2f\pi)t}dt$$

$$= \left[\frac{e^{-(a+j2f\pi)t}}{a+j2f\pi}\right]_{\infty}^{0} = \frac{1}{a+j2f\pi}$$

- $|W(f)| = \dfrac{1}{\sqrt{a^2+(2f\pi)^2}}$

Example #2 (cont.)


Time Waveform

Spectrum

As time waveform decreases more slowly, the *more low frequency content in the wave.*

Example 2.3: Cosine Waveform

- Easier to verify some transforms pairs from the inverse transform

  $$W(f) = \frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)$$

- $w(t) = \int_{-\infty}^{\infty} \frac{1}{2}\delta(f - f_c)e^{j2ft\pi}df + \int_{-\infty}^{\infty} \frac{1}{2}\delta(f + f_c)e^{j2ft\pi}df$

$$= \frac{e^{j2f_c t\pi} + e^{-j2f_c t\pi}}{2}$$

$$= \cos(2\pi f_c t)$$