ECE4634
Digital Communications
Fall 2007

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Lecture #22: Probability and Random Variables

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Why are Random Variables and Processes important?

- Random Variables and Random Processes let us talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random (otherwise no information is being sent!)
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability.
- What to read
  - 8.1-8.5 (8.3 is less important for our purposes)

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Course Objectives

- **Design digital communication systems, given constraints on data rate, bandwidth, power, fidelity, and complexity;**
- **Analyze the performance of a digital communication link when additive noise is present in terms of the signal-to-noise ratio and bit error rate;**
- Compute the power and bandwidth requirements of modern communication systems, including those employing ASK, PSK, FSK, and QAM modulation formats;
- Design a scalar quantizer for a given source with a required fidelity and determine the resulting data rate;
- **Determine the auto-correlation function of a line code and determine its power spectral density;**
- **Determine the power spectral density of bandpass digital modulation formats.**

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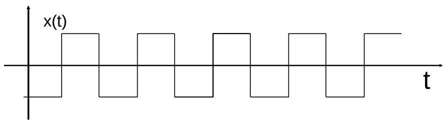
Example 22.1

- A square wave is a deterministic waveform

$$x(t) = \sum_{n=-\infty}^{\infty} p(t - nT) \qquad p(t) = \begin{cases} -1 & t \le \frac{T}{2} \\ 1 & t > \frac{T}{2} \end{cases}$$



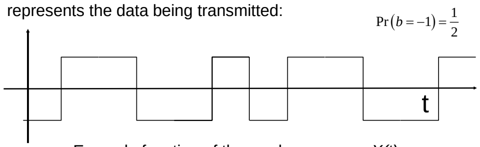
Example 22.1 (cont.)

- However, we cannot describe communications signals using a simple square wave. Instead, an example communications signal is

$$X(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

- Where b_n is a binary random variable and represents the data being transmitted:

$$\Pr(b = 1) = \frac{1}{2}$$
$$\Pr(b = -1) = \frac{1}{2}$$



Example function of the random process *X(t)*

Random Events

- When we conduct a random <u>experiment</u>, we can use set notation to describe possible outcomes.
- Example: Roll a six-sided die.
  Possible Outcomes: $S = \{1,2,3,4,5,6\}$
- An <u>event</u> is any subset of possible outcomes: $A = \{1,2\}$
- The <u>complementary event</u>: $\overline{A} = S - A = \{3,4,5,6\}$
- The set of all outcomes is the <u>certain event</u>: S
- The <u>null event</u>: ϕ
- Transmitting a data bit is also an experiment

Probability

- The probability *P(A)* is a number which measures the likelihood of the event *A*.

Axioms of Probability:

- No event has probability less than zero: $P(A) \geq 0$
- Further: $P(A) \leq 1$ and $P(A) = 1 \Leftrightarrow A = S$
- Union = $\bigcup$ Intersection = $\bigcap$
- Let *A* and *B* be two events such that: $A \cap B = \phi$

  Then: $P(A \cup B) = P(A) + P(B)$

U (union) represents the probability that *either* event occurs

Relationships Between Random Events

- Joint Probability: $P(A,B) = P(A \cap B)$
  - Probability that both A and B occur
- Conditional Probability: $P(A|B) = \dfrac{P(A,B)}{P(B)}$
  - Probability that A will occur given that B has occurred
- Statistical Independence: $P(A,B) = P(A) \cdot P(B)$
  - Events A and B are statistically independent if:

    $P(A|B) = P(A)$ and $P(B|A) = P(B)$
  - If A and B are independent then:

$$P(A,B) = P(B \mid A)P(A)$$
$$= P(B)P(A)$$

Random Variables

- A random variable *X(s)* is a real-valued function of the underlying event space:
  - $s \in S$
- A random variable may be:
  - Discrete-valued: range is finite (e.g. {0,1}) or countably infinite (e.g., {1,2,3,...})
  - Continuous-valued - range is uncountably infinite
    - (e.g. $\Re$)
- A random variable may be described by:
  - A name: *X*
  - It's range: $X \in \Re$
  - A description of its distribution (pdf or cdf)

Cumulative Distribution Function (CDF)

$$F_X(x) = F(x) = P(X \le x)$$

Function Argument (x); Random Variable (X)

- Properties:

$$F(x) \text{ is monotonically nondecreasing}$$
$$F(-\infty) = 0$$
$$F(\infty) = 1$$
$$P(a < X \le b) = F(b) - F(a)$$

- While the CDF completely defines the distribution of a random variable, we will usually work with the probability distribution function (pdf) or probability mass function (pmf)

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Probability Density Function (PDF)

- Defn: $f_X(x) = \frac{dF_X(x)}{dx}$ or $f(x) = \frac{dF(x)}{dx}$
- Interpretations:
  - PDF measures how fast CDF is increasing or how likely a random variable is to lie at a particular value
- Properties:

$$f(x) \ge 0$$
$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$P(a < X \le b) = \int_a^b f(x)dx$$

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Expected Values

- Expected values are a shorthand way of describing a random variable
- The most important examples are:
  - Mean: $E(X) = m_X = \int_{-\infty}^{\infty} x f(x)dx$
  - Variance: $E\left([X - m_X]^2\right) = \int_{-\infty}^{\infty} (x - m_X)^2 f(x)dx$
- The expectation operator works with any function:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)dx$$

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Probability Mass Functions (pmf)

- A discrete random variable can be described with a pdf if we allow impulse functions

$$P(x) = P(X = x)\delta(X - x)$$

- Also can use probability mass functions (pmf):

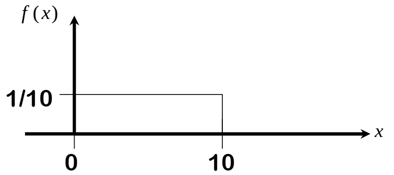
$$P(x) = P(X = x)$$

- Properties are analogous to pdf:

$$\boxed{\sum_X P(x) = 1} \qquad \boxed{P(x) \geq 0} \qquad \boxed{P(a \leq X \leq b) = \sum_{x=a}^{b} P(x)}$$

Example 22.2: Uniform pdf

- $f(x) = \begin{cases} 1/10, & 0 \leq x \leq 10 \\ 0, & \text{else} \end{cases}$



Example 22.2 (continued)

- Mean: $m_X = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^{10} x \cdot \frac{1}{10}dx = \left[\frac{x^2}{20}\right]_0^{10} = 5$
- Variance:

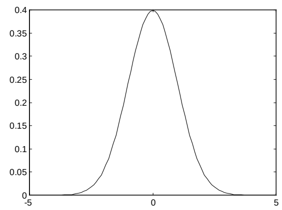
$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-5)^2 \cdot f(x)dx = \int_0^{10} (x-5)^2 \cdot \frac{1}{10}dx = \frac{25}{3}$$

- Probability Calculation:

$$P(6 \leq x \leq 9) = \int_6^9 f(x)dx = \int_6^9 \frac{1}{10}dx = 0.3$$

Example 22.3: Gaussian pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}$$



- A Gaussian random variable is completely determined by its mean and variance
- This is one of the most important random variables we will use

Example 22.4 - Rayleigh pdf

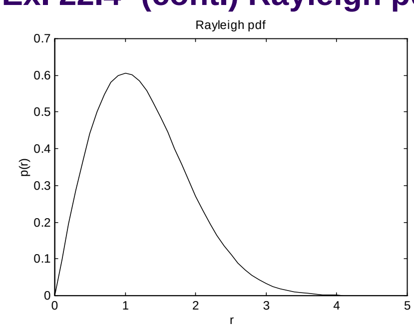
- Let: $R = \sqrt{X_1^2 + X_2^2}$

  where X_1 and X_2 are Gaussian with mean 0 and variance σ^2
- Then *R* is a Rayleigh random variable with pdf:

$$p_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} u(r)$$

- Rayleigh pdf's are frequently used to model fading when no line of site signal is present (*Rayleigh Fading)*

Ex. 22.4- (cont.) Rayleigh pdf



- No negative values
- Extends to positive infinity

$$mean = \sqrt{\frac{\pi}{2}}\sigma$$

$$\text{var} = \frac{\sigma^2(4-\pi)}{2}$$

σ^2 = variance of underlying Gaussian RV

Example 22.5: Binary Distribution

- $P(x) = \begin{cases} 1/2, & x = 0 \\ 1/2, & x = 1 \end{cases}$
- This is frequently used to model binary data
- Mean: $m_X = \sum_x x \cdot P(x) = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$
- Variance:
  $\sigma_x^2 = \sum_x (x - m_x)^2 \cdot P(x) = (-1/2)^2 \cdot 1/2 + (1/2)^2 \cdot 1/2 = 1/4$
- If X_1 and X_2 are independent binary random variables, then
  $P_{X_1 X_2}(0,0) = P_{X_1}(0) \cdot P_{X_2}(0) = 1/2 \cdot 1/2 = 1/4$

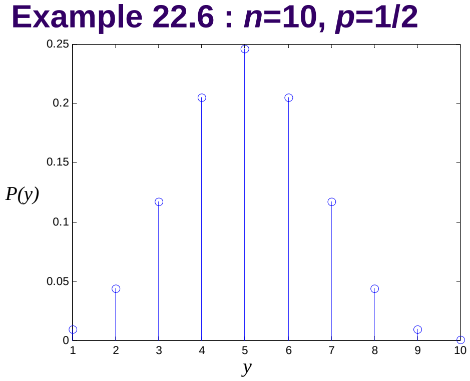
Example 22.6: Binomial Distribution

- Let $Y = \sum_{i=1}^{n} X_i$ where $\{X_i, i = 1, \ldots, n\}$ are independent

binary RVs with: $P_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \end{cases}$

- Then $P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $\binom{n}{y} = \dfrac{n!}{y! \cdot (n-y)!}$
- Mean: $m_X = n \cdot p$
- Variance: $\sigma_X^2 = n \cdot p \cdot (1-p)$

Example 22.6 : *n*=10, *p*=1/2



Example 22.6 (continued)

- Suppose that we transmit a 31 bit sequence with error correction capable of correcting up to 3 errors.
- If the probability of a bit error is *p*=0.001, what is the probability that the codeword is received in error?

$$P(\text{codeword error}) = 1 - P(\text{correct codeword})$$
$$= 1 - \sum_{i=0}^{3} \binom{31}{i} (0.999)^{31-i} (0.001)^{i} \approx 3\times 10^{-8}$$

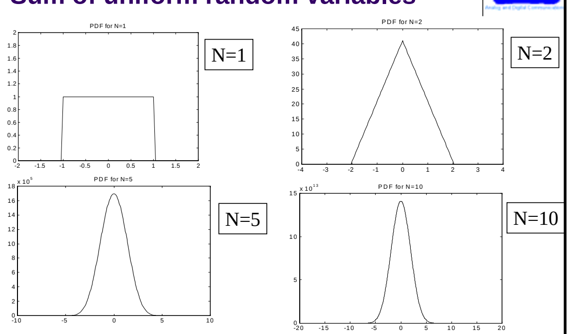
- If no error correction is used, the error probability is:

$$1 - \Pr[\text{No Errors}] = 1 - (1 - P_e)^{31}$$
$$1 - (1 - 0.001)^{31} = 0.0305$$

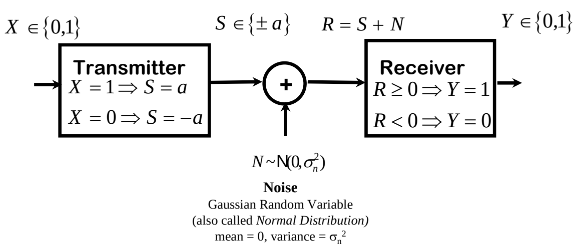
Central Limit Theorem

- Let $X_1, X_2, \ldots, X_N$ be a set of independent random variables with identical pdfs
- Let: $Y = \sum_{i=1}^{N} X_i$
- Then as $N \to \infty$, the distribution of Y will tend towards a Gaussian distribution
- In practice, N = 10 is usually enough to see this effect
- Thermal noise results from the random movement of many electrons - it is well modeled by a Gaussian distribution

Example of Central Limit Theorem: Sum of uniform random variables



Example 22.7: A Communication System with Gaussian Noise



Description of Communications System Model

- $X \in \{0,1\}$
  - binary random variable representing one bit of input data
  - either value is equally likely $p_X(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{2} & x = 1 \end{cases}$
- $S \in \{-a, +a\}$
  - binary random variable representing transmitted signal level $p_S(x) = \begin{cases} \frac{1}{2} & x = a \\ \frac{1}{2} & x = -a \end{cases}$

$$X = 1 \Rightarrow S = a, X = 0 \Rightarrow S = -a$$

- $N \sim \mathcal{N}(0, \sigma_n^2)$
  - Gaussian random variable representing noise
  - Mean = 0, Variance = σ_n^2

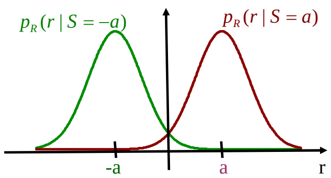
Description of Communications System Model (continued)

- $R = S + N \quad R \in \{\Re\}$
  - random variable representing the received signal
  - sum of signal plus noise components
- $Y \in \{0,1\}$
  - random variable representing bit decision by receiver

$$R \geq 0 \Rightarrow Y = 1$$
$$R < 0 \Rightarrow Y = 0$$

Description of Communications System Model (continued)

- $R = S + N$ $\quad N \sim N(0,\sigma_n^2) \rightarrow \begin{cases} R \mid (S=a) \sim N(a,\sigma_n^2) \\ R \mid (S=-a) \sim N(-a,\sigma_n^2) \end{cases}$
  - random variable representing the received signal
  - sum of signal plus noise components
  - random variable representing bit decision by receiver



Bit Error Calculation

$Y \in \{0,1\}$	$R \geq 0 \Rightarrow Y = 1$ $R < 0 \Rightarrow Y = 0$

$$
\begin{aligned}
P_e &= \Pr[Y=1 \mid X=0]\Pr[X=0] + \Pr[Y=0 \mid X=1]\Pr[X=1] \\
&= 0.5\Pr[Y=1 \mid X=0] + 0.5\Pr[Y=0 \mid X=1] \\
&= \Pr[Y=1 \mid X=0] \quad \text{(Because of symmetry)} \\
&= \Pr[R \geq 0 \mid X=0] \\
&= \Pr[S+N \geq 0 \mid S=-a] \\
&= \Pr[-a+N \geq 0] \\
&= \Pr[N \geq a]
\end{aligned}
$$

$$P_e = \Pr[N \geq a] = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x^2}{2\sigma_n^2}} dx = Q\left(\frac{a}{\sigma_n}\right)$$

The *Q*-function

- The *Q*-function is a standard form for expressing error probabilities without a closed form:

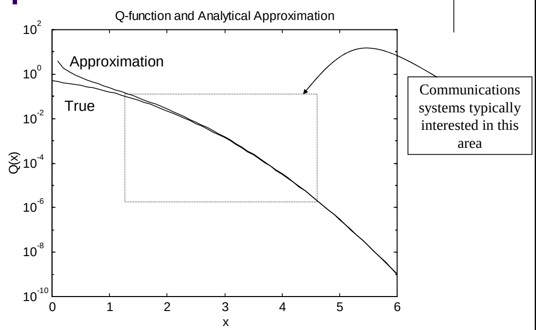
$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

- Numerical Calculation of *Q*-function:

$$Q(x) = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}\left[1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} + \cdots + \frac{(-1)^n \cdot 1\cdot 3 \cdots (2n-1)}{x^{2n}}\right]$$

$$\approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, \text{ for } x \geq 3$$

The Q-function and its Approximation



Summary

- Deterministic signals may be handled through linear system theory
- Random signals require understanding of probability and random processes
- Probability theory follows from a few basic assumptions
- Random variables may be described by:
  - A name
  - A set of values the variable can take
  - A probability distribution
- We will use random variables heavily in discussing the performance of digital communication systems