

ECE4634

Digital Communications

Fall 2007

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Lecture #22: Probability and
Random Variables



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Why are Random Variables and Processes important?



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- Random Variables and Random Processes let us talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random (otherwise no information is being sent!)
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability.
- What to read
 - 8.1-8.5 (8.3 is less important for our purposes)



Course Objectives

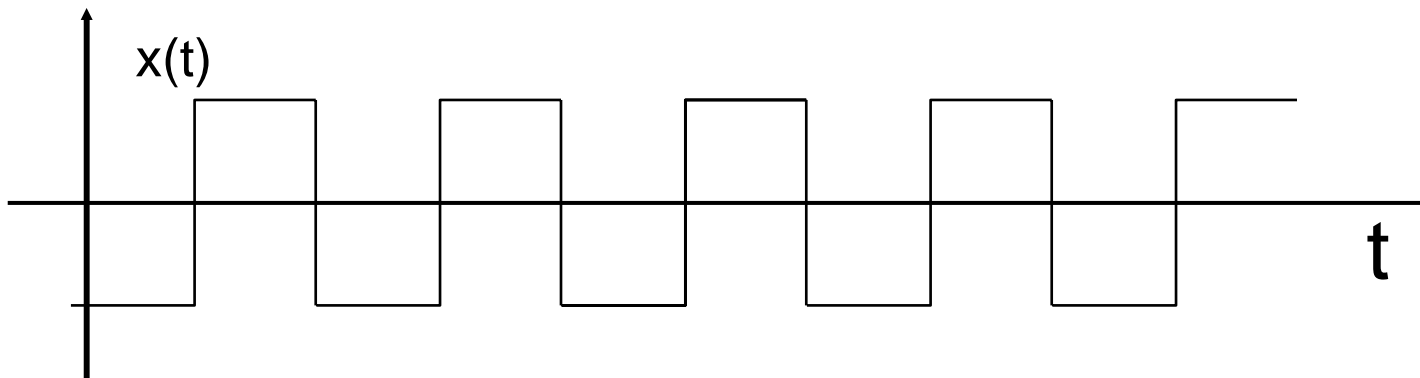
- **Design digital communication systems, given constraints on data rate, bandwidth, power, fidelity, and complexity;**
- **Analyze the performance of a digital communication link when additive noise is present in terms of the signal-to-noise ratio and bit error rate;**
- Compute the power and bandwidth requirements of modern communication systems, including those employing ASK, PSK, FSK, and QAM modulation formats;
- Design a scalar quantizer for a given source with a required fidelity and determine the resulting data rate;
- **Determine the auto-correlation function of a line code and determine its power spectral density;**
- **Determine the power spectral density of bandpass digital modulation formats.**



Example 22.1

- A square wave is a deterministic waveform

$$x(t) = \sum_{n=-\infty}^{\infty} p(t - nT) \quad p(t) = \begin{cases} -1 & t \leq \frac{T}{2} \\ 1 & t > \frac{T}{2} \end{cases}$$



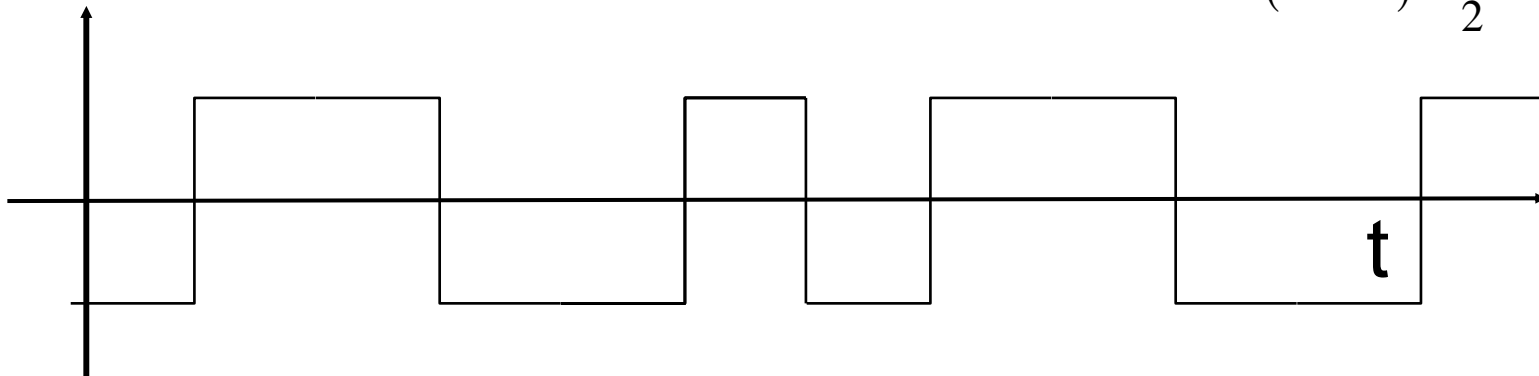
Example 22.1 (cont.)

- However, we cannot describe communications signals using a simple square wave. Instead, an example communications signal is

$$X(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

- Where b_n is a binary random variable and represents the data being transmitted:

$$\Pr(b = 1) = \frac{1}{2}$$
$$\Pr(b = -1) = \frac{1}{2}$$



Example function of the random process $X(t)$

Random Events



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- When we conduct a random experiment, we can use set notation to describe possible outcomes.
- Example: Roll a six-sided die.
Possible Outcomes: $S = \{1,2,3,4,5,6\}$
- An event is any subset of possible outcomes: $A = \{1,2\}$
- The complementary event: $\bar{A} = S - A = \{3,4,5,6\}$
- The set of all outcomes is the certain event: S
- The null event: ϕ
- Transmitting a data bit is also an experiment

Probability



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- The probability $P(A)$ is a number which measures the likelihood of the event A .

Axioms of Probability:

- No event has probability less than zero: $P(A) \geq 0$
- Further: $P(A) \leq 1$ and $P(A) = 1 \Leftrightarrow A = S$
- Union = \cup Intersection = \cap
- Let A and B be two events such that: $A \cap B = \phi$
Then: $P(A \cup B) = P(A) + P(B)$

\cup (union) represents the probability that *either* event occurs

Relationships Between Random Events



- Joint Probability: $P(A, B) = P(A \cap B)$

- Probability that both A and B occur

- Conditional Probability: $P(A|B) = \frac{P(A, B)}{P(B)}$

- Probability that A will occur given that B has occurred

- Statistical Independence: $P(A, B) = P(A) \cdot P(B)$

- Events A and B are statistically independent if:

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

- If A and B are independent then:

$$\begin{aligned} P(A, B) &= P(B | A)P(A) \\ &= P(B)P(A) \end{aligned}$$



Random Variables

- A random variable $X(s)$ is a real-valued function of the underlying event space:
 - $s \in S$
- A random variable may be:
 - Discrete-valued: range is finite (e.g. $\{0,1\}$) or countably infinite (e.g., $\{1,2,3,\dots\}$)
 - Continuous-valued - range is uncountably infinite
 - (e.g. \mathcal{R})
- A random variable may be described by:
 - A name: X
 - It's range: $X \in \mathcal{R}$
 - A description of its distribution (pdf or cdf)

Cumulative Distribution Function (CDF)



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$$F_X(x) = F(x) = P(\underbrace{X}_{\text{Random Variable}} \leq \underbrace{x}_{\text{Function Argument}})$$

- Properties:

$F(x)$ is monotonically nondecreasing

$$F(-\infty) = 0$$

$$F(\infty) = 1$$

$$P(a < X \leq b) = F(b) - F(a)$$

- While the CDF completely defines the distribution of a random variable, we will usually work with the probability distribution function (pdf) or probability mass function (pmf)

Probability Density Function (PDF)



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- Defn: $f_X(x) = \frac{dF_X(x)}{dx}$ or $f(x) = \frac{dF(x)}{dx}$
- Interpretations:
 - PDF measures how fast CDF is increasing or how likely a random variable is to lie at a particular value
- Properties:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$P(a < X \leq b) = \int_a^b f(x)dx$$

Expected Values



- Expected values are a shorthand way of describing a random variable
- The most important examples are:

- Mean:

$$E(X) = m_x = \int_{-\infty}^{\infty} xf(x)dx$$

- Variance:

$$E\left([X - m_x]^2\right) = \int_{-\infty}^{\infty} (x - m_x)^2 f(x)dx$$

- The expectation operator works with any function:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Probability Mass Functions (pmf)



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- A discrete random variable can be described with a pdf if we allow impulse functions

$$P(x) = P(X = x)\delta(X - x)$$

- Also can use probability mass functions (pmf):

$$P(x) = P(X = x)$$

- Properties are analogous to pdf:

$$\sum_X P(x) = 1$$

$$P(x) \geq 0$$

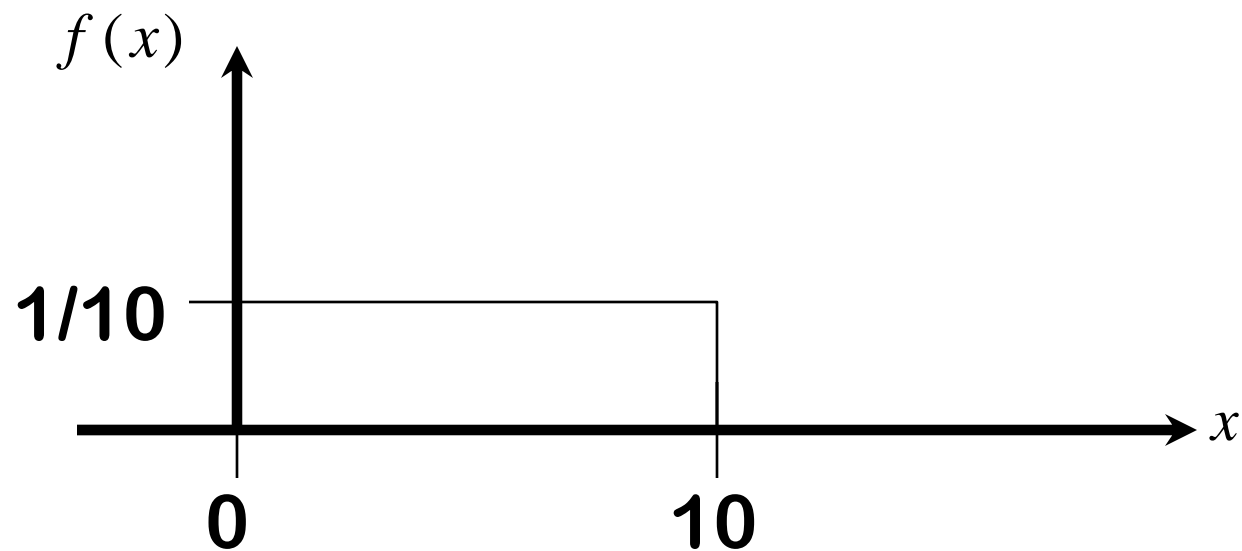
$$P(a \leq X \leq b) = \sum_{x=a}^b P(x)$$

Example 22.2: Uniform pdf



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- $$f(x) = \begin{cases} 1/10, & 0 \leq x \leq 10 \\ 0, & \text{else} \end{cases}$$



Example 22.2 (continued)



- Mean:
$$m_x = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^{10} x \cdot \frac{1}{10} dx = \left[\frac{x^2}{20} \right]_0^{10} = 5$$

- Variance:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-5)^2 \cdot f(x) dx = \int_0^{10} (x-5)^2 \cdot \frac{1}{10} dx = \frac{25}{3}$$

- Probability Calculation:

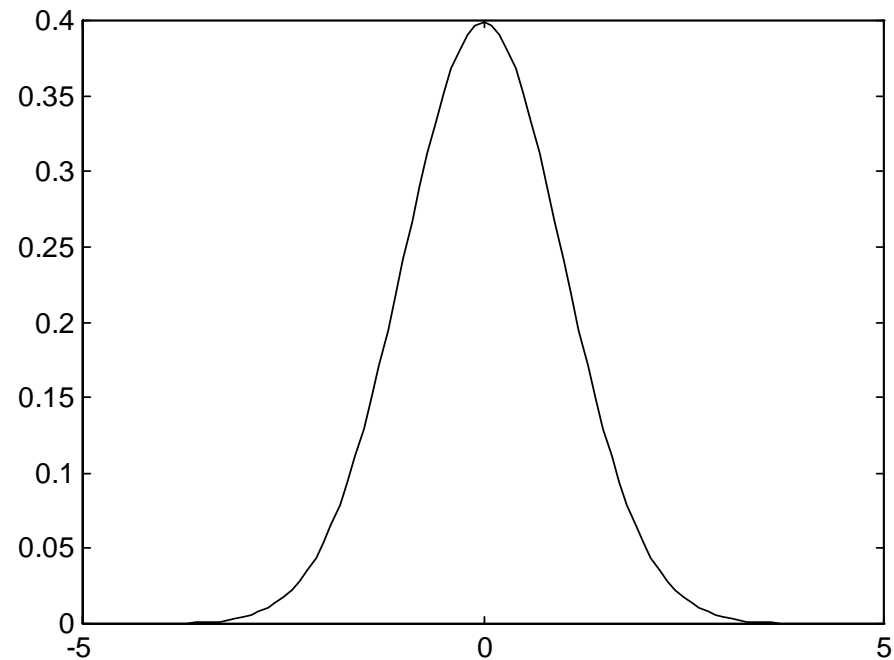
$$P(6 \leq x \leq 9) = \int_6^9 f(x) dx = \int_6^9 \frac{1}{10} dx = 0.3$$

Example 22.3: Gaussian pdf



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$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}$$



- A Gaussian random variable is completely determined by its mean and variance
- This is one of the most important random variables we will use



Example 22.4 - Rayleigh pdf

- Let: $R = \sqrt{X_1^2 + X_2^2}$

where X_1 and X_2 are Gaussian with mean 0 and variance σ^2

- Then R is a Rayleigh random variable with pdf:

$$p_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} u(r)$$

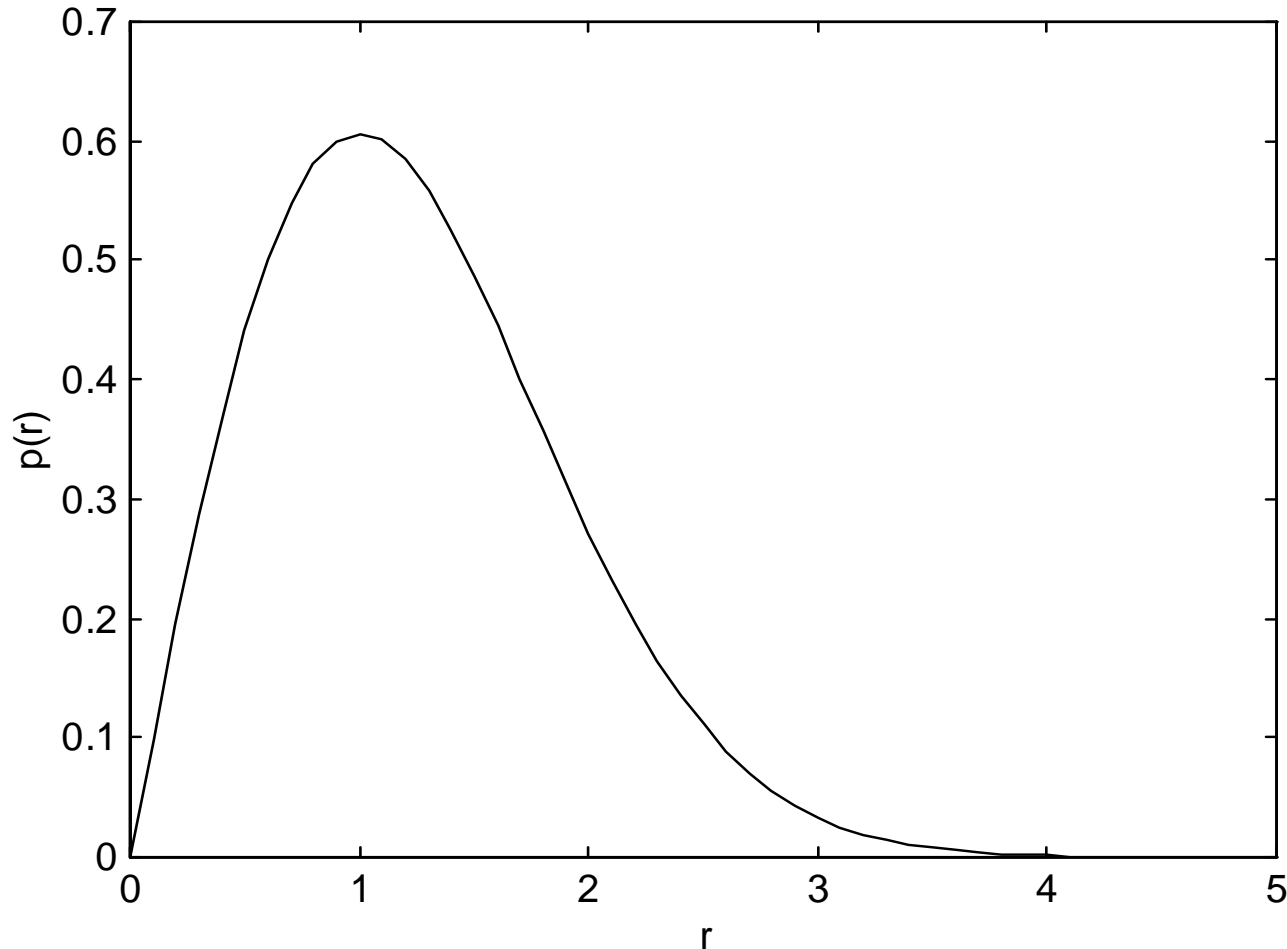
- Rayleigh pdf's are frequently used to model fading when no line of site signal is present (*Rayleigh Fading*)

Ex. 22.4- (cont.) Rayleigh pdf



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Rayleigh pdf



- No negative values
- Extends to positive infinity

$$\text{mean} = \sqrt{\frac{\pi}{2}}\sigma$$

$$\text{var} = \frac{\sigma^2(4 - \pi)}{2}$$

σ^2 = variance of
underlying Gaussian
RV

Example 22.5: Binary Distribution



- $$P(x) = \begin{cases} 1/2, & x = 0 \\ 1/2, & x = 1 \end{cases}$$

- This is frequently used to model binary data

- Mean:
$$m_x = \sum_x x \cdot P(x) = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

- Variance:

$$\sigma_x^2 = \sum_x (x - m_x)^2 \cdot P(x) = (-1/2)^2 \cdot 1/2 + (1/2)^2 \cdot 1/2 = 1/4$$

- If X_1 and X_2 are independent binary random variables, then

$$P_{X_1 X_2}(0,0) = P_{X_1}(0) \cdot P_{X_2}(0) = 1/2 \cdot 1/2 = 1/4$$

Example 22.6: Binomial Distribution



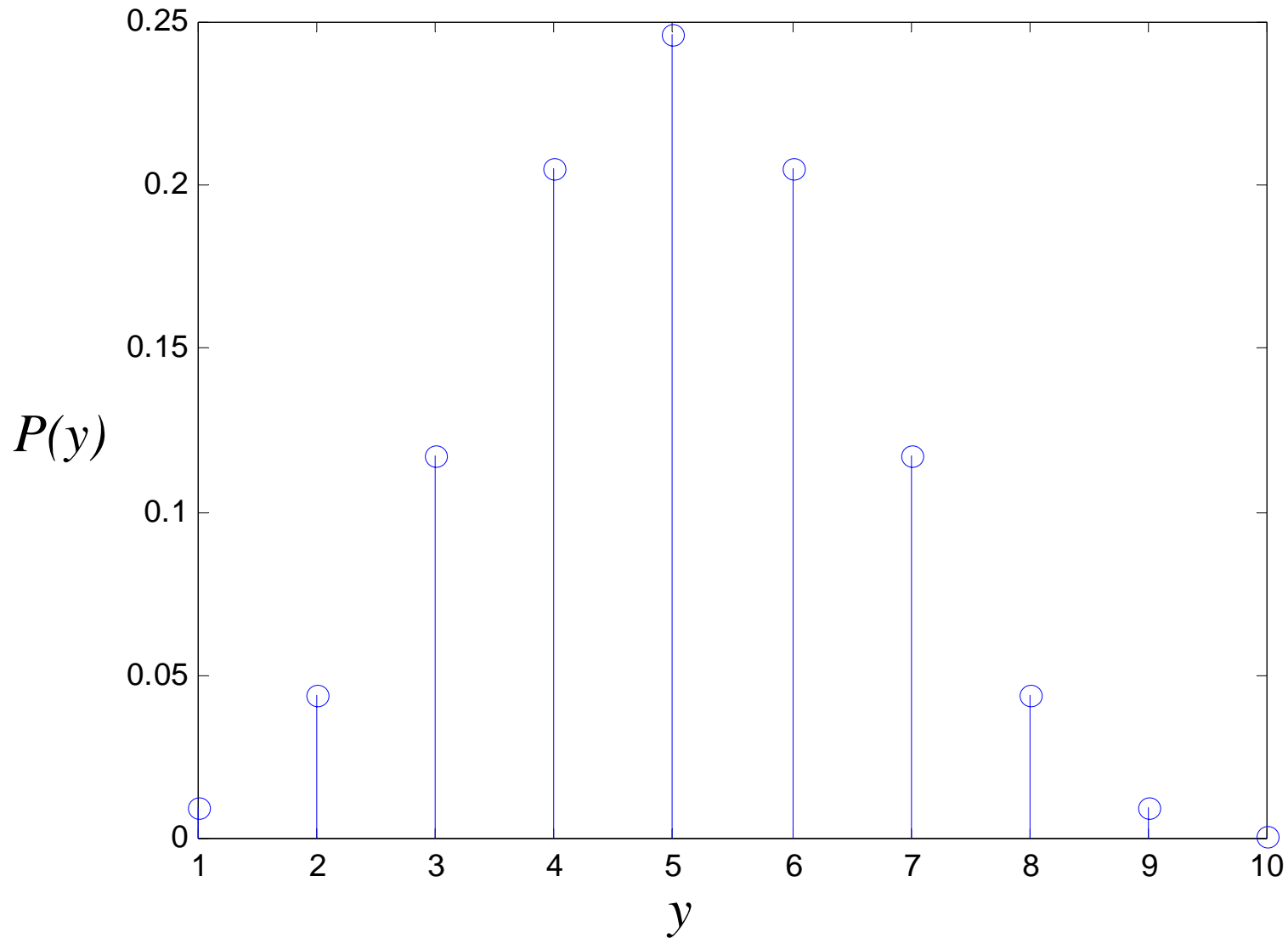
- Let $Y = \sum_{i=1}^n X_i$ where $\{X_i, i = 1, \dots, n\}$ are independent

binary RVs with:
$$P_X(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \end{cases}$$

- Then $P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}$, $\binom{n}{y} = \frac{n!}{y!(n-y)!}$

- Mean: $m_x = n \cdot p$
- Variance: $\sigma_x^2 = n \cdot p \cdot (1-p)$

Example 22.6 : $n=10, p=1/2$





Example 22.6 (continued)

- Suppose that we transmit a 31 bit sequence with error correction capable of correcting up to 3 errors.
- If the probability of a bit error is $p=0.001$, what is the probability that the codeword is received in error?

$$\begin{aligned} P(\text{codeword error}) &= 1 - P(\text{correct codeword}) \\ &= 1 - \sum_{i=0}^3 \binom{31}{i} (0.999)^{31-i} (0.001)^i \approx 3 \times 10^{-8} \end{aligned}$$

- If no error correction is used, the error probability is:

$$\begin{aligned} 1 - \Pr[\text{No Errors}] &= 1 - (1 - P_e)^{31} \\ 1 - (1 - 0.001)^{31} &= 0.0305 \end{aligned}$$

Central Limit Theorem



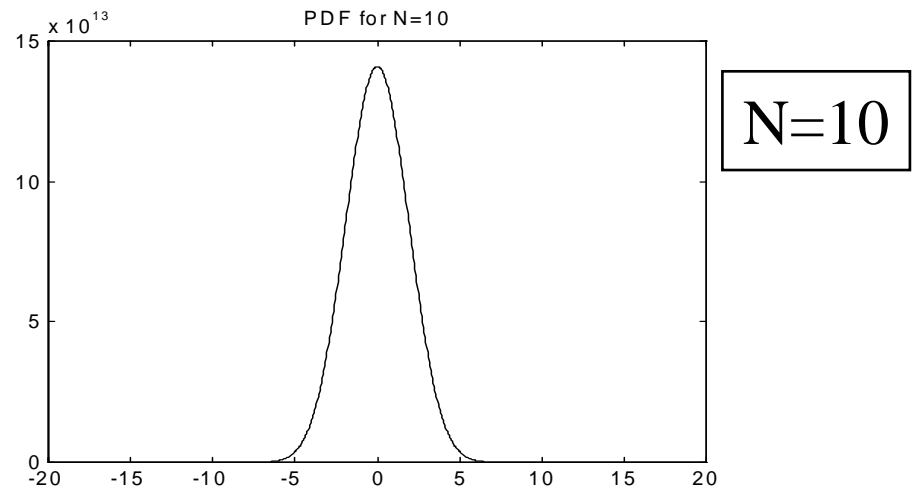
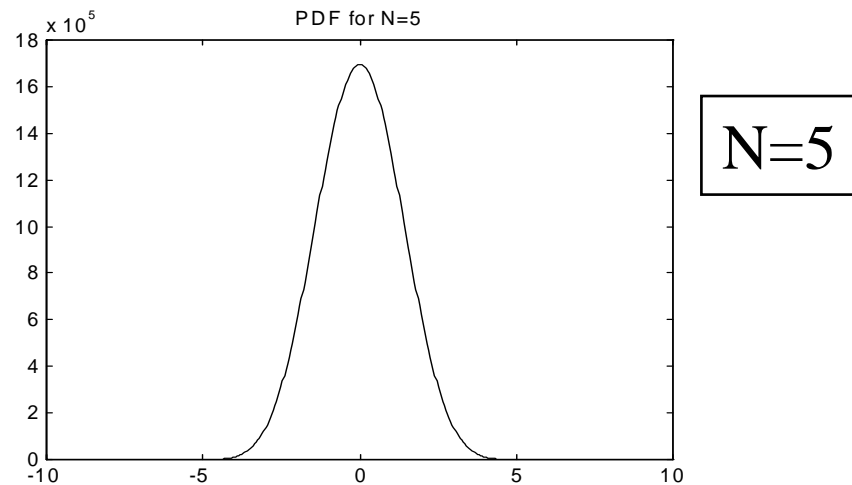
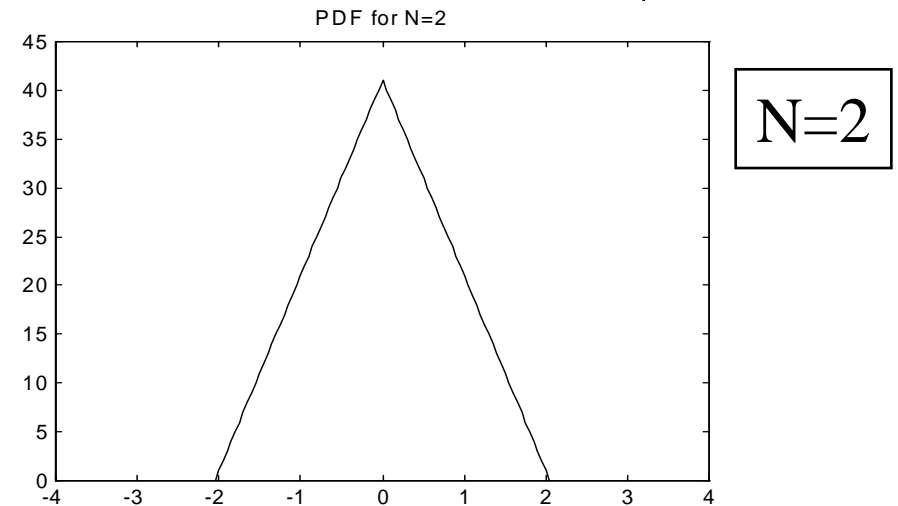
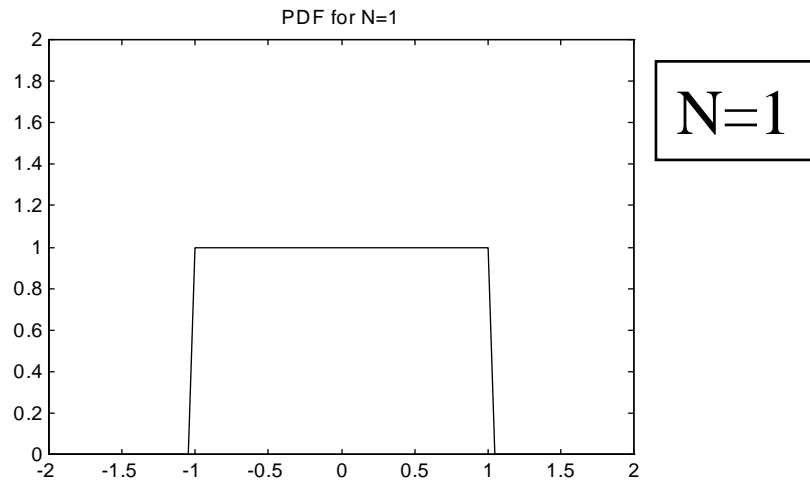
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- Let X_1, X_2, \dots, X_N be a set of independent random variables with identical pdfs
- Let:
$$Y = \sum_{i=1}^N X_i$$
- Then as $N \rightarrow \infty$, the distribution of Y will tend towards a Gaussian distribution
- In practice, $N = 10$ is usually enough to see this effect
- Thermal noise results from the random movement of many electrons - it is well modeled by a Gaussian distribution

Example of Central Limit Theorem: Sum of uniform random variables



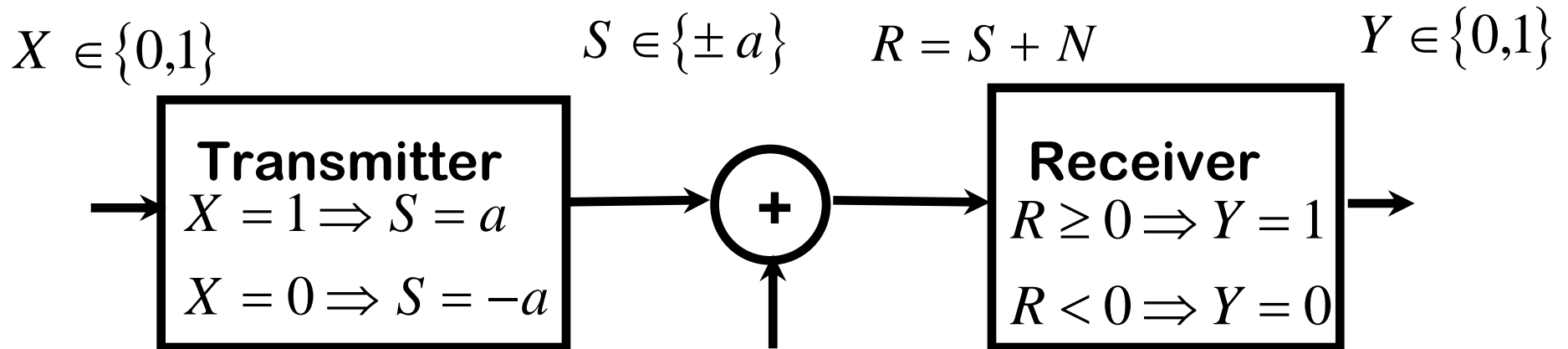
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Example 22.7: A Communication System with Gaussian Noise



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$$N \sim \mathcal{N}(0, \sigma_n^2)$$

Noise

Gaussian Random Variable
(also called *Normal Distribution*)
mean = 0, variance = σ_n^2

Description of Communications System Model



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- $X \in \{0,1\}$
 - binary random variable representing one bit of input data
 - either value is equally likely

$$p_x(x) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \end{cases}$$

- $S \in \{-a,+a\}$
 - binary random variable representing transmitted signal level

$$X = 1 \Rightarrow S = a, X = 0 \Rightarrow S = -a$$

$$p_s(x) = \begin{cases} \frac{1}{2} & x=a \\ \frac{1}{2} & x=-a \end{cases}$$

- $N \sim \mathcal{N}(0, \sigma_n^2)$
 - Gaussian random variable representing noise
 - Mean = 0, Variance = σ_n^2

Description of Communications System Model (continued)



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- $R = S + N$ $R \in \{\mathfrak{R}\}$
 - random variable representing the received signal
 - sum of signal plus noise components

- $Y \in \{0,1\}$
 - random variable representing bit decision by receiver

$$R \geq 0 \Rightarrow Y = 1$$

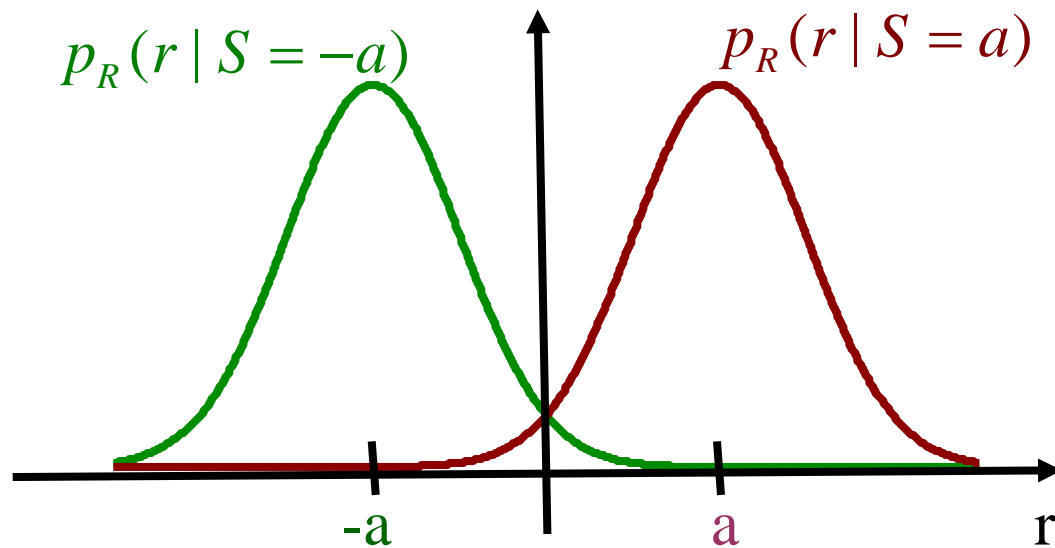
$$R < 0 \Rightarrow Y = 0$$

Description of Communications System Model (continued)



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- $R = S + N$ $N \sim \mathcal{N}(0, \sigma_n^2) \rightarrow \begin{cases} R | (S = a) \sim \mathcal{N}(a, \sigma_n^2) \\ R | (S = -a) \sim \mathcal{N}(-a, \sigma_n^2) \end{cases}$
 - random variable representing the received signal
 - sum of signal plus noise components
 - random variable representing bit decision by receiver



Bit Error Calculation



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$$\begin{array}{l} Y \in \{0,1\} \\ R \geq 0 \Rightarrow Y = 1 \\ R < 0 \Rightarrow Y = 0 \end{array}$$

$$\begin{aligned} P_e &= \Pr[Y = 1|X = 0] \Pr[X = 0] + \Pr[Y = 0|X = 1] \Pr[X = 1] \\ &= 0.5 \Pr[Y = 1|X = 0] + 0.5 \Pr[Y = 0|X = 1] \\ &= \Pr[Y = 1|X = 0] \quad (\text{Because of symmetry}) \\ &= \Pr[R \geq 0|X = 0] \\ &= \Pr[S + N \geq 0|S = -a] \\ &= \Pr[-a + N \geq 0] \\ &= \Pr[N \geq a] \\ P_e &= \Pr[N \geq a] = \int_a^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x^2}{2\sigma_n^2}} dx = Q\left(\frac{a}{\sigma_n}\right) \end{aligned}$$

The Q -function

- The Q -function is a standard form for expressing error probabilities without a closed form:

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

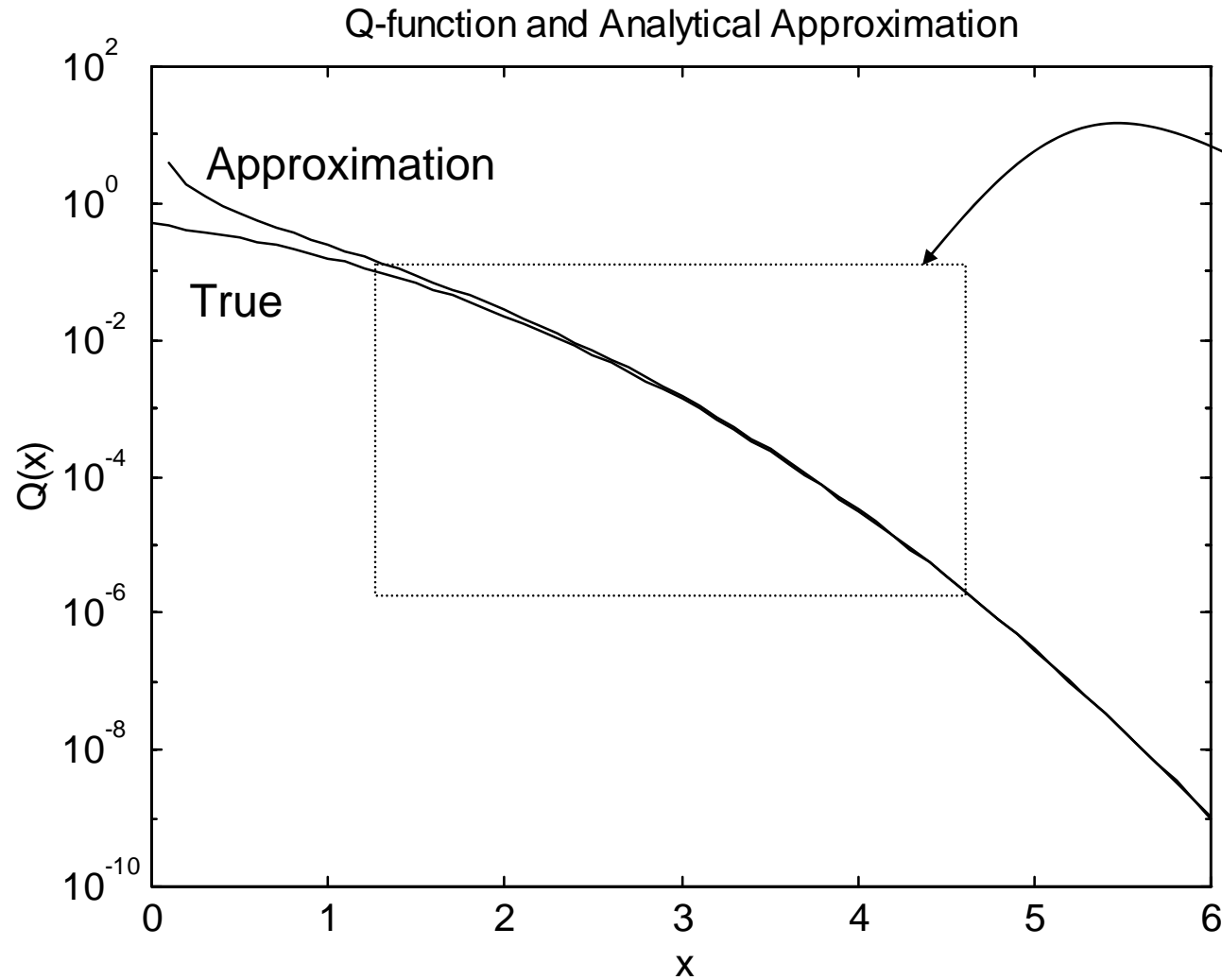
- Numerical Calculation of Q -function:

$$Q(x) = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \left[1 - \frac{1}{x^2} + \frac{1 \cdot 3}{x^4} + \dots + \frac{(-1)^n \cdot 1 \cdot 3 \cdot \dots \cdot (2n-1)}{x^{2n}} \right]$$
$$\approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, \text{ for } x \geq 3$$

The Q-function and its Approximation



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Communications systems typically interested in this area

Summary



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- Deterministic signals may be handled through linear system theory
- Random signals require understanding of probability and random processes
- Probability theory follows from a few basic assumptions
- Random variables may be described by:
 - A name
 - A set of values the variable can take
 - A probability distribution
- We will use random variables heavily in discussing the performance of digital communication systems