ECE4634
Digital Communications
Fall 2007

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Lecture #23: Random Processes

Why are Random Variables and Processes important?

- Random Variables and Random Processes let us talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random (otherwise no information is being sent!)
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability.
- What to read
  - Sections 8.6-8.9

Course Objectives

- **Design digital communication systems, given constraints on data rate, bandwidth, power, fidelity, and complexity;**
- **Analyze the performance of a digital communication link when additive noise is present in terms of the signal-to-noise ratio and bit error rate;**
- Compute the power and bandwidth requirements of modern communication systems, including those employing ASK, PSK, FSK, and QAM modulation formats;
- Design a scalar quantizer for a given source with a required fidelity and determine the resulting data rate;
- **Determine the auto-correlation function of a line code and determine its power spectral density;**
- **Determine the power spectral density of bandpass digital modulation formats.**

Random Processes

- A random variable has a single value.
- We are concerned with signals which change with time.
- Random variables model unknown events.
- Random processes model unknown signals.
- ***Definition:*** A random process is an indexed set of functions of some parameter (usually time) that has certain statistical properties.
- A random process can also be thought of as an indexed set of random variables.
- If $X(t)$ is a random process then $X(1)$, $X(1.5)$, and $X(37.5)$ are all random variables for specific times t

Random Processes

- A specific instance of a random process is termed a *sample function*
- The value of a random process $X(t_1)$ is a random variable.
- Thus, a random process is an indexed set of random variables that have a specific cross-correlation and distribution that are determined by the underlying function
- We deal with *ensemble* averages and *time averages*
  - An ensemble average is the expected value of all possible sample functions each sampled at a specific time t_o
  - A time average is the mean of a specific sample function over all time

Description of a Random Process

- To completely describe a random process we require an *N*-dimensional pdf

$$f\left(X\left(t_1\right), X\left(t_2\right), X\left(t_3\right) \ldots X\left(t_N\right)\right)$$

where $N \to \infty$

- Most random processes of interest can be described more simply, however.

Time vs. Ensemble Averages

- Time Average – hold random variable constant and average over all time

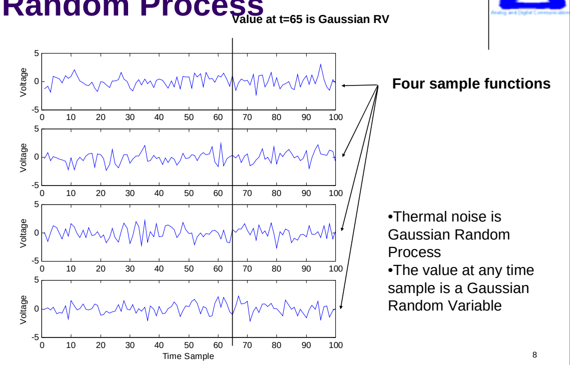
$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\,dt$$

- Ensemble Average – hold time constant and average over all values of the random variable

$$\overline{X(t)} = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$x = X(t_o)$$

Example 23.1 : Gaussian Random Process



- •Thermal noise is Gaussian Random Process
- •The value at any time sample is a Gaussian Random Variable

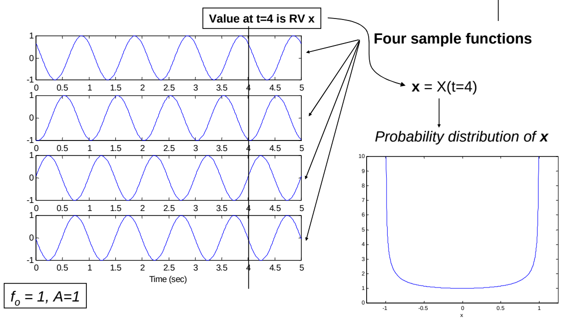
Example 23.2

$$x(t) = A\sin(\omega_o t + \theta)$$

- Let A and ω_o be known.
- θ is a random variable uniformly distributed on $[0,2\pi)$
- $x_1(t)=A\sin(\omega_o t + \pi/5)$ is a sample function
- The value at any time t_1 is a random variable with distribution

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - x^2}} & |x| \le A \\ 0 & else \end{cases}$$

Example 23.2 (cont.)



Stationary Random Processes

- A stationary random process has statistical properties which do not change with time (i.e., all joint pdfs do not change)

$$f\left(X\left(t_1\right), X\left(t_2\right), \ldots X\left(t_N\right)\right) = f\left(X\left(t_1+t_o\right), X\left(t_2+t_o\right), \ldots X\left(t_N+t_o\right)\right)$$

  - First order – $f_X(x_1)$ *where* $x_1 = x(t_1)$ does not depend on the value of t_1
  - *Second order* – $f_{X1X2}(x_1, x_2)$ *where* $x_1 = x(t_1)$ $x_2 = x(t_2)$ doesn't depend the values of t_1 *and* t_2 *but only the difference* $\tau = t_1 - t_2$
- A wide sense stationary (WSS) process has a mean and autocorrelation function which do not change with time (this is usually sufficient)

$$1.\quad E\left[x(t_1)\right] = \overline{X}$$

$$2.\quad E\left[x(t_1)x(t_2)\right] = E\left[x(t)x(t+\tau)\right] = R_X(\tau)$$

Ergodic Random Processes

- A random process is ergodic if the time average always converges to the statistical average.
  - i.e., we can use time averages of a sample function to estimate the ensemble averages
  - In real life we can not obtain a sufficient number of sample functions, so we rely on time averages of a single sample function.
- Unless specified, we will assume that all random processes are WSS and ergodic.
- Note that all ergodic processes are stationary, but not all stationary processes are ergodic.

Example 23.3

$x(t) = A\sin(\omega_o t + \theta_o)$ $\quad \theta_o$ *is uniform R.V. on [0,2π)*

- First let us examine the ensemble averages:

$$\overline{x} = \int_0^{2\pi} \frac{1}{2\pi} A\sin(\omega_o t + \theta) d\theta$$
$$= 0$$

$$\overline{x^2} = \int_0^{2\pi} \frac{1}{2\pi} A^2 \sin^2(\omega_o t + \theta) d\theta$$
$$= \frac{A^2}{2}$$

Note that we are finding the expectation over the random variable θ.

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Example 23.3 (cont.)

- Now let us examine the time averages:

$$\langle x(t) \rangle = \frac{1}{T_o} \int_0^{T_o} A\sin(\omega_o t + \theta) dt$$
$$= 0$$
$$\langle x^2(t) \rangle = \frac{1}{T_o} \int_0^{T_o} A^2 \sin^2(\omega_o t + \theta) dt$$
$$= \frac{A^2}{2}$$

Note that we are finding the expectation over *time*.

- Thus, we say that this random process is *ergodic*

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Description of Random Processes

- Knowing the pdf of individual samples of the random process is not sufficient. We also need to know how individual samples are related to each other (ideally the joint pdf).
- Two tools which are simpler but still give valuable information are available to describe this relationship:
  - Autocorrelation function
  - Power spectral density function

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Autocorrelation

- Autocorrelation measures how a random process changes with time.
- Intuitively, *X(1)* and *X(1.1)* will be more strongly related than *X(1)* and *X(100000)* (although it is possible to construct counterexamples). The autocorrelation function quantifies this.
- Definition:

$$R_X(t_1, t_2) = E[x(t_1)x(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2)\,dx_1 dx_2$$

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Autocorrelation

- A *Wide-sense stationary (WSS)* process is one in which the first and second ensemble averages are stationary or independent of time.
- For a WSS process, the autocorrelation does not depend on the exact values of t_1 and t_2, rather it depends only on the difference $\tau = t_1 - t_2$

$$R_X(t_1, t_2) = R_X(\tau)$$
$$= E\left[X(t)X(t+\tau)\right]$$

- Note that Power $= R_X(0)$

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Properties of WSS Processes

$R_X(0) = \overline{x^2(t)}$ i.e., $R_X(0)$= the power

$R_X(\tau) = R_X(-\tau)$ $R_X(\tau)$ is even

$R_X(0) \geq |R_X(\tau)|$ The correlation peaks at zero time offset

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Spectra of Random Processes

- We are interested in random processes whose properties are do not change with time
  - Stationary random processes
- If a random process is stationary, it must last forever (there is no time when its properties change so the probabilities cannot go to zero)
- A signal which lasts forever (i.e., infinite in time) is a *power signal*
- Thus, we can describe the spectral properties through the *power spectral density*

Spectra of Random Processes

- The power spectral density of a random process $S_x(f)$ can be determined from the auto-correlation function $R_x(\tau)$ through the Fourier Transform.
- In other words they form a Fourier Transform pair:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} d\tau$$

- The Fourier Transform of an individual sample function is not particularly helpful
- However if we average the magnitude squared of the Fourier Transform of all possible sample functions we approach the Power Spectral Density

Filtering

- If we filter a random process *X(t)*, the output *Y(t)* is also a random process.
- The power spectral density of the resulting random process $S_Y(f)$ can be determined as

$$S_Y(f) = S_X(f)|H(f)|^2$$

where *H(f)* is the transfer function of the filter.

Gaussian Random Processes

- A GRP is a collection of *N* random variables with distribution

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\left|\det \mathbf{C}\right|^{1/2}} e^{-(1/2)[(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})]}$$

- **X** is a *N*-dimensional Gaussian Random Variable, **m** is the mean vector, and **C** is the covariance matrix.
- Gaussian Random Processes have several special properties:
  - If a Gaussian random process is wide-sense stationary, then it is also stationary.
  - Any sample point from a Gaussian random process is a Gaussian random variable
  - If the input to a linear system is a Gaussian random process, then the output is also a Gaussian random process

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Complex Random Processes

- Complex random processes are useful for describing bandpass signals in complex baseband notation. A complex random process is represented as

$$g(t) = x(t) + j\,y(t)$$

where x(t) and y(t) are real random processes and $j = \sqrt{-1}$

- Complex random processes are similar to real random processes with minor differences in definitions:

$$R_g(t_1, t_2) = \overline{g^*(t_1)g(t_2)}$$

- A WSS complex random process is one in which

$$R_g\left(t_1, t_2\right) = R_g\left(\tau\right)$$

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Summary

- Random variables can be used to form models of a communications system
- Random processes are used to model random *signals* which represent the waveforms in a communication system.
- Random processes are simply collections of indexed random variables
- Sampling or integrating/sampling random processes results in random variables. *We will typically work with random variables*.
- However, it is important for you to understand that these random variables come from random processes.
- Thus, it is important that you have a good understanding of random variables (and in more advanced study random processes) to fully grasp communications systems

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