

# ECE4634

# Digital Communications

# Fall 2007

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Lecture #23: Random Processes



Analog and Digital Communications



# Why are Random Variables and Processes important?



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- Random Variables and Random Processes let us talk about quantities and signals which are unknown in advance
- The data sent through a communication system is modeled as random (otherwise no information is being sent!)
- The noise, interference, and fading introduced by the channel can all be modeled as random processes
- Even the measure of performance (Probability of Bit Error) is expressed in terms of a probability.
- What to read
  - Sections 8.6-8.9



# Course Objectives

- **Design digital communication systems, given constraints on data rate, bandwidth, power, fidelity, and complexity;**
- **Analyze the performance of a digital communication link when additive noise is present in terms of the signal-to-noise ratio and bit error rate;**
- Compute the power and bandwidth requirements of modern communication systems, including those employing ASK, PSK, FSK, and QAM modulation formats;
- Design a scalar quantizer for a given source with a required fidelity and determine the resulting data rate;
- **Determine the auto-correlation function of a line code and determine its power spectral density;**
- **Determine the power spectral density of bandpass digital modulation formats.**

# Random Processes



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- A random variable has a single value.
- We are concerned with signals which change with time.
- Random variables model unknown events.
- Random processes model unknown signals.
- **Definition:** A random process is an indexed set of functions of some parameter (usually time) that has certain statistical properties.
- A random process can also be thought of as an indexed set of random variables.
- If  $X(t)$  is a random process then  $X(1)$ ,  $X(1.5)$ , and  $X(37.5)$  are all random variables for specific times  $t$



# Random Processes

- A specific instance of a random process is termed a *sample function*
- The value of a random process  $X(t_1)$  is a random variable.
- Thus, a random process is an indexed set of random variables that have a specific cross-correlation and distribution that are determined by the underlying function
- We deal with *ensemble averages* and *time averages*
  - An ensemble average is the expected value of all possible sample functions each sampled at a specific time  $t_o$
  - A time average is the mean of a specific sample function over all time

# Description of a Random Process



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- To completely describe a random process we require an  $N$ -dimensional pdf

$$f(X(t_1), X(t_2), X(t_3) \dots X(t_N))$$

where  $N \rightarrow \infty$

- Most random processes of interest can be described more simply, however.



# Time vs. Ensemble Averages

- Time Average – hold random variable constant and average over all time

$$\langle X(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) dt$$

- Ensemble Average – hold time constant and average over all values of the random variable

$$\overline{X(t)} = \int_{-\infty}^{\infty} x f_X(x) dx$$

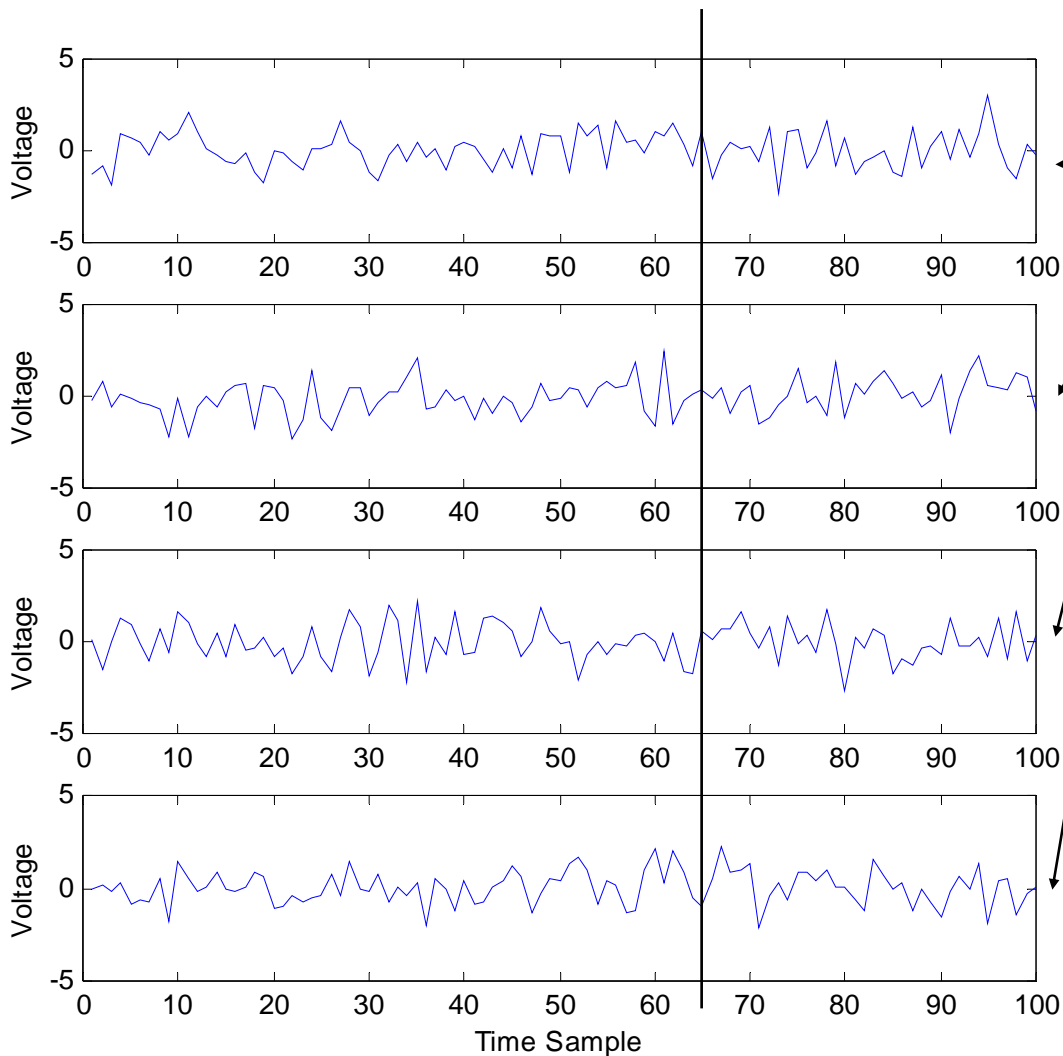
$$x = X(t_o)$$

# Example 23.1 : Gaussian Random Process



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Value at  $t=65$  is Gaussian RV



Four sample functions

- Thermal noise is Gaussian Random Process
- The value at any time sample is a Gaussian Random Variable



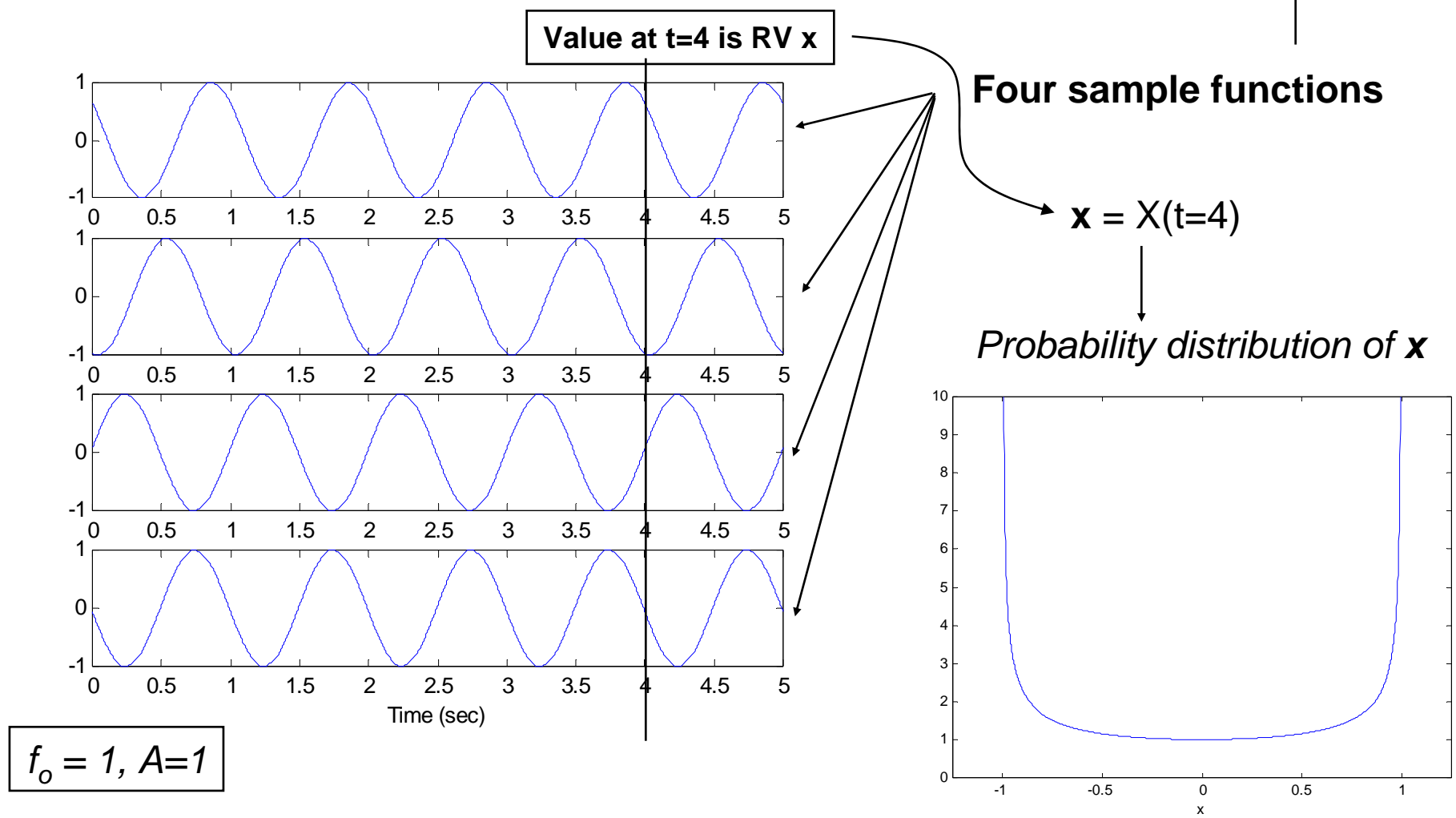
## Example 23.2

$$x(t) = A \sin(\omega_o t + \theta)$$

- Let  $A$  and  $\omega_o$  be known.
- $\theta$  is a random variable uniformly distributed on  $[0, 2\pi)$
- $x_1(t) = A \sin(\omega_o t + \pi/5)$  is a sample function
- The value at any time  $t_1$  is a random variable with distribution

$$f(x) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - x^2}} & |x| \leq A \\ 0 & \text{else} \end{cases}$$

# Example 23.2 (cont.)



# Stationary Random Processes



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- A stationary random process has statistical properties which do not change with time (i.e., all joint pdfs do not change)

$$f\left(X(t_1), X(t_2), \dots, X(t_N)\right) = f\left(X(t_1 + t_o), X(t_2 + t_o), \dots, X(t_N + t_o)\right)$$

- First order –  $f_X(x_1)$  where  $x_1=x(t_1)$  does not depend on the value of  $t_1$
- Second order –  $f_{X_1X_2}(x_1, x_2)$  where  $x_1=x(t_1)$   $x_2=x(t_2)$  doesn't depend the values of  $t_1$  and  $t_2$  but only the difference  $\tau = t_1 - t_2$
- A wide sense stationary (WSS) process has a mean and autocorrelation function which do not change with time (this is usually sufficient)

$$1. \quad E[x(t_1)] = \bar{X}$$

$$2. \quad E[x(t_1)x(t_2)] = E[x(t)x(t + \tau)] \\ = R_X(\tau)$$



# Ergodic Random Processes

- A random process is ergodic if the time average always converges to the statistical average.
  - i.e., we can use time averages of a sample function to estimate the ensemble averages
  - In real life we can not obtain a sufficient number of sample functions, so we rely on time averages of a single sample function.
- Unless specified, we will assume that all random processes are WSS and ergodic.
- Note that all ergodic processes are stationary, but not all stationary processes are ergodic.



## Example 23.3

$$x(t) = A \sin(\omega_o t + \theta_o) \quad \theta_o \text{ is uniform R.V. on } [0, 2\pi)$$

- First let us examine the ensemble averages:

$$\begin{aligned} \overline{x} &= \int_0^{2\pi} \frac{1}{2\pi} A \sin(\omega_o t + \theta) d\theta \\ &= 0 \end{aligned}$$

$$\begin{aligned} \overline{x^2} &= \int_0^{2\pi} \frac{1}{2\pi} A^2 \sin^2(\omega_o t + \theta) d\theta \\ &= \frac{A^2}{2} \end{aligned}$$

Note that we are finding the expectation over the random variable  $\theta$ .



## Example 23.3 (cont.)

- Now let us examine the time averages:

$$\begin{aligned}\langle x(t) \rangle &= \frac{1}{T_o} \int_0^{T_o} A \sin(\omega_o t + \theta) dt \\ &= 0\end{aligned}$$

Note that we are finding the expectation over *time*.

$$\begin{aligned}\langle x^2(t) \rangle &= \frac{1}{T_o} \int_0^{T_o} A^2 \sin^2(\omega_o t + \theta) dt \\ &= \frac{A^2}{2}\end{aligned}$$

- Thus, we say that this random process is *ergodic*

# Description of Random Processes



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- Knowing the pdf of individual samples of the random process is not sufficient. We also need to know how individual samples are related to each other (ideally the joint pdf).
- Two tools which are simpler but still give valuable information are available to describe this relationship:
  - Autocorrelation function
  - Power spectral density function

# Autocorrelation



- Autocorrelation measures how a random process changes with time.
- Intuitively,  $X(1)$  and  $X(1.1)$  will be more strongly related than  $X(1)$  and  $X(100000)$  (although it is possible to construct counterexamples). The autocorrelation function quantifies this.
- Definition:

$$\begin{aligned} R_X(t_1, t_2) &= E[x(t_1)x(t_2)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2) dx_1 dx_2 \end{aligned}$$



# Autocorrelation

- A *Wide-sense stationary (WSS)* process is one in which the first and second ensemble averages are stationary or independent of time.
- For a WSS process, the autocorrelation does not depend on the exact values of  $t_1$  and  $t_2$ , rather it depends only on the difference  $\tau = t_1 - t_2$

$$\begin{aligned} R_X(t_1, t_2) &= R_X(\tau) \\ &= E[X(t)X(t + \tau)] \end{aligned}$$

- Note that Power =  $R_X(0)$

# Properties of WSS Processes



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$$R_X(0) = \overline{x^2(t)} \quad \text{i.e., } R_X(0) = \text{the power}$$

$$R_X(\tau) = R_X(-\tau) \quad R_X(\tau) \text{ is even}$$

$$R_X(0) \geq |R_X(\tau)| \quad \text{The correlation peaks at zero time offset}$$

# Spectra of Random Processes



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- We are interested in random processes whose properties are do not change with time
  - Stationary random processes
- If a random process is stationary, it must last forever (there is no time when its properties change so the probabilities cannot go to zero)
- A signal which lasts forever (i.e., infinite in time) is a *power signal*
- Thus, we can describe the spectral properties through the *power spectral density*



# Spectra of Random Processes

- The power spectral density of a random process  $S_X(f)$  can be determined from the auto-correlation function  $R_X(\tau)$  through the Fourier Transform.
- In other words they form a Fourier Transform pair:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

- The Fourier Transform of an individual sample function is not particularly helpful
- However if we average the magnitude squared of the Fourier Transform of all possible sample functions we approach the Power Spectral Density



# Filtering

- If we filter a random process  $X(t)$ , the output  $Y(t)$  is also a random process.
- The power spectral density of the resulting random process  $S_Y(f)$  can be determined as

$$S_Y(f) = S_X(f) |H(f)|^2$$

where  $H(f)$  is the transfer function of the filter.

# Gaussian Random Processes



- A GRP is a collection of  $N$  random variables with distribution

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} |\det \mathbf{C}|^{1/2}} e^{-(1/2)[(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})]}$$

- $\mathbf{X}$  is a  $N$ -dimensional Gaussian Random Variable,  $\mathbf{m}$  is the mean vector, and  $\mathbf{C}$  is the covariance matrix.
- Gaussian Random Processes have several special properties:
  - If a Gaussian random process is wide-sense stationary, then it is also stationary.
  - Any sample point from a Gaussian random process is a Gaussian random variable
  - If the input to a linear system is a Gaussian random process, then the output is also a Gaussian random process



# Complex Random Processes

- Complex random processes are useful for describing bandpass signals in complex baseband notation. A complex random process is represented as

$$g(t) = x(t) + j y(t)$$

where  $x(t)$  and  $y(t)$  are real random processes and  $j = \sqrt{-1}$

- Complex random processes are similar to real random processes with minor differences in definitions:

$$R_g(t_1, t_2) = \overline{g^*(t_1)g(t_2)}$$

- A WSS complex random process is one in which

$$R_g(t_1, t_2) = R_g(\tau)$$

# Summary



- Random variables can be used to form models of a communications system
- Random processes are used to model random *signals* which represent the waveforms in a communication system.
- Random processes are simply collections of indexed random variables
- Sampling or integrating/sampling random processes results in random variables. *We will typically work with random variables.*
- However, it is important for you to understand that these random variables come from random processes.
- Thus, it is important that you have a good understanding of random variables (and in more advanced study random processes) to fully grasp communications systems