

ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer
 Lecture #24: The AWGN
 Channel and
 The Receiver



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Introduction

- Until this point we have concentrated on describing transmission schemes to convey information from one location to another
- Baseband Transmitter
 - Transmit pulses whose amplitude, delay or width conveys the information
 - Pulse shaping for improved bandwidth
- We have also discussed *bandpass* transmission techniques
 - Conveying the information by modulating the phase, frequency or amplitude of a sinusoidal carrier
 - For linear modulation schemes (e.g., PSK, ASK) this can be viewed as simply multiplying a pulse stream by a sinusoidal carrier
- Today we introduce two concepts
 - The Channel
 - The Receiver
- What to read – Sections 8.10-8.11, 9.1-9.3

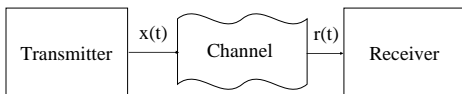
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The Channel

- The channel is the medium through which the system communicates information
- In general $r(t) = x(t) \otimes h(t, \tau) + n(t)$ where $h(t, \tau)$ is the time-varying channel impulse response, $n(t)$ is thermal noise and \otimes is the convolution operation. (channel = filter)



- However, for now let us only consider the attenuation C and noise added by the channel

$$r(t) = Cx(t) + n(t)$$

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Thermal Noise



- All objects with physical temperature T_p greater than 0 Kelvin generate electrical noise.
- This noise power is given by

$$P_n = kT_n B$$

Diagram showing the equation $P_n = kT_n B$ with callouts: k is Botzman's Constant $1.38 \times 10^{-23} \text{ J/K}$, T_n is Noise temperature of device, and B is Bandwidth of device.

$$kT_n = \text{Noise Spectral Density}$$

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AWGN



- Noise in the system contributes an additional random voltage on top of the received signal. Thus, it is **additive**.
- The noise is uncorrelated from one sample to the next (i.e., $R_x(\tau) = \delta(\tau)$). A delta function in the correlation function means that the PSD is a constant, thus it is **white**.
- Noise in the system originates from components in the receiver. Since there are many discrete components all contributing some small amount, the sum tends to a Gaussian process (Central Limit Theorem). Thus, noise samples have a *Gaussian* probability distribution.
- **Additive White Gaussian Noise (AWGN)**

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AWGN Channels



- The term 'AWGN Channel' is something of a misnomer.
- The channel doesn't necessarily add noise (at least not all of the noise). It attenuates the signal to such a degree that the internal noise of the receiver (as well as the noise observed by the antenna in wireless systems) is comparable to that of the received signal.
- Usually, we assume that the received signal has normalized average received power, while the noise has some power σ^2 where σ is the standard deviation of the thermal noise.
- In "AWGN Channels" we assume that the only distortion to the signal is the AWGN. Normalizing ($C = 1$):

$$r(t) = x(t) + n(t)$$

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AWGN – Gaussian RP



- Let us take a closer look at the noise process $n(t)$. It is the result of summation of many random signals generated in the receiver and observed by the antenna (or picked up by the wire connecting the receiver and transmitter).
- The central limit theorem suggests that we can approximate the summation as a *Gaussian random process*.
- The important things to understand is that
 - Samples of a Gaussian random process are Gaussian random variables
 - If a GRP is input to a linear time-invariant system, the output is also a GRP

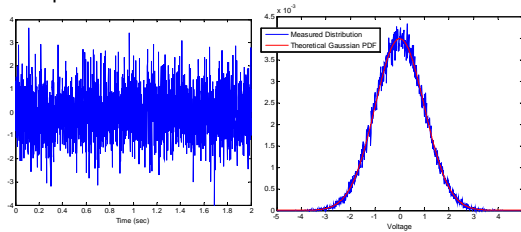
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Example



- Example Gaussian Random process and measured probability density function of samples taken at 1ksps



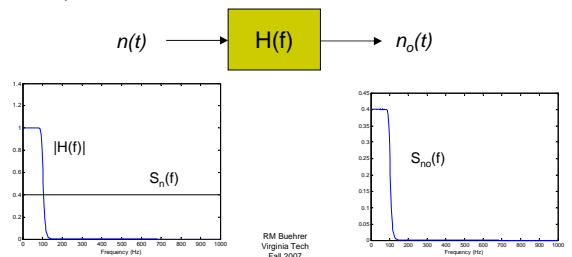
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Example – Filter



- Now consider the output of a filter with the following frequency response when white Gaussian noise is input

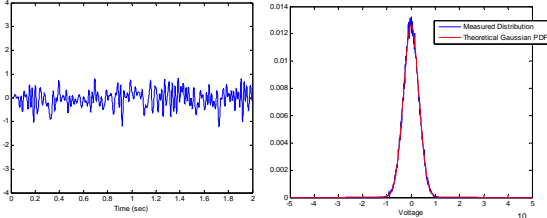


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Example - Output Noise



- Output noise power is reduced due to filtering, but samples are still Gaussian.
- Note that Gaussian distribution has smaller variance (power is equal to variance)



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AWGN – White RP



- A 'white' random process is one whose power spectral density (PSD) is a constant:

$$S_n(f) = K \quad \forall f$$

- The autocorrelation of any random process is simply the inverse Fourier Transform of the PSD

$$\begin{aligned} R_n(\tau) &= F^{-1}\{S_n(f)\} \\ &= K\delta(\tau) \end{aligned}$$

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Output Autocorrelation



- Let us return to a linear time-invariant filter.
- If the input random process is white, the output power spectral density follows the filter frequency response
- What effect does this have on the autocorrelation function?
- Recall that the output PSD is related to the input PSD as

$$S_Y(f) = S_X(f) |H(f)|^2$$

- If a white process is applied to an ideal Low Pass Filter:

$$\begin{aligned} S_Y(f) &= K \left| \text{rect}\left(\frac{f}{2B}\right) \right|^2 \\ &= K \text{rect}\left(\frac{f}{2B}\right) \end{aligned}$$

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Output Autocorrelation



- The resulting autocorrelation function is then

$$\begin{aligned} R_Y(\tau) &= F^{-1}\{S_Y(f)\} \\ &= F^{-1}\left\{K \operatorname{rect}\left(\frac{f}{2B}\right)\right\} \\ &= 2KB \operatorname{sinc}(2B\tau) \end{aligned}$$

- This is termed *band-limited white noise*

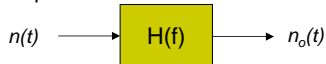
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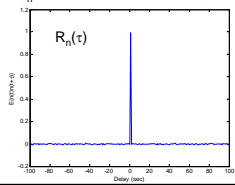
Example – Filter Revisited



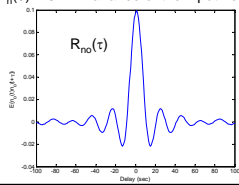
- Let's return to the filter example when white Gaussian noise is input



Consecutive samples are uncorrelated
 $R_n(0) = 1 = \text{variance of the input noise}$



Consecutive samples are now correlated
 $R_{n_o}(0) = 0.1 = \text{variance of the input noise}$



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Important Note



- While we often consider white noise with a flat (i.e., constant) power spectral density, in actuality this type of process is not realistic

- A process with a truly constant PSD has infinite power

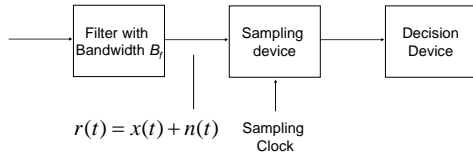
$$\begin{aligned} P_{tot} &= \int_{-\infty}^{\infty} S_n(f) df \\ &= \int_{-\infty}^{\infty} K df \\ &= \infty \end{aligned}$$

- However, noise which is white over the bandwidth B of the receiver (i.e., band-limited white noise) does not have infinite power and is realistic.
- If such a process is sampled at $T_s = 1/B$ consecutive samples have zero correlation ($R_n(\tau) = K \operatorname{sinc}(\tau B)$)

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Receiver



$$r(t) = x(t) + n(t)$$

- For now we assume that $B_f \gg B$ (i.e., filter bandwidth is much greater than the signal bandwidth) so that the signal is not distorted by the filter
- Later we will show that this isn't the best thing to do

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Receiver



- Assuming a fairly benign channel, at the receiver we observe the transmitted signal plus noise

$$r(t) = x(t) + n(t)$$

- Both the transmitted signal $x(t)$ and the noise $n(t)$ are *a priori* unknown.
- The job of the receiver is to determine the most likely transmitted signal based on the observed received signal
- Note: Typically the received signal $r(t)$ is characterized by its *signal-to-noise ratio* or *SNR*

$$\frac{S}{N} = \frac{\text{desired signal power}}{\text{noise power or variance}} = \frac{E\{x^2(t)\}}{E\{n^2(t)\}} = \frac{A^2}{\sigma^2}$$

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Receiver



- Example: Assume that PAM is used to transmit binary data

$$x(t) = \sum_{n=-\infty}^{\infty} d_n p(t - nT_s)$$

$$d_n = \begin{cases} A & b=1 \\ 0 & b=0 \end{cases}$$

Note that this is equivalent to a unipolar non-return to zero line code.

- At the receiver we observe $r(t) = x(t) + n(t)$
- Where $n(t)$ is a white Gaussian random process whose samples are independent Gaussian random variables with zero mean and variance σ^2
 - Note that the variance is a function of the receiver noise bandwidth and the noise power spectral density

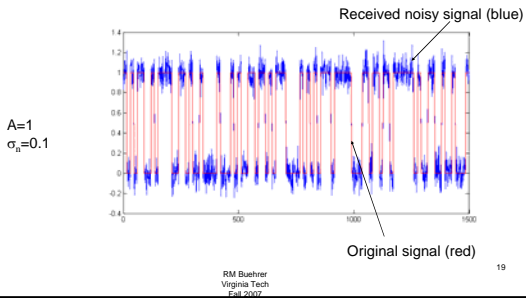
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Receiver (cont.)



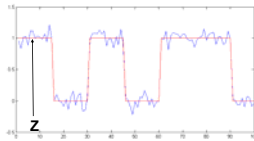
- Example plot from the output of the filter:



Make a decision



- To determine the received 'symbol' we could simply sample at any point in the symbol interval and compare with a threshold (0.5)



$$\hat{b} = \begin{cases} 1 & Z \geq 0.5 \\ 0 & Z < 0.5 \end{cases}$$

Error Probability



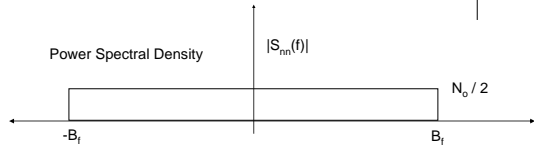
- The sample value Z is a conditional GRV.
 - Assuming $b=1$, Z is normal with mean A and standard deviation σ $\rightarrow Z = N(A, \sigma)$
 - The probability of error is then the probability that $Z < A/2$ given that $b=1$.

$$\begin{aligned} P_e(b=1) &= \Pr(Z < A/2 | b=1) \\ &= \Pr(N < -A/2) \\ &= \int_{-\infty}^{-A/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_{A/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_{A/2\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= Q\left(\frac{A}{2\sigma}\right) \end{aligned}$$

The same analysis assuming that $b=0$ will produce the same result

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Noise Power



- Let $B_f = 10R_s$ to avoid distorting the received signal

- Then $\sigma_n^2 = N_o B_f = 10R_s N_o$ $P_r = Q\left(\frac{A}{2\sigma}\right)$
 $= Q\left(\frac{A^2}{4\sigma^2}\right) = Q\left(\frac{A^2 T_s}{40N_o}\right)$

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Improving Probability of Error

- However, note that the noise value changes over the symbol duration while the amplitude value does not.
- Thus, we could improve our estimate by taking several samples and averaging the results.
- In the limit, we could simply integrate over the symbol duration to improve our estimate

$$y(t) = \int_{t-T}^t r(x) dx$$

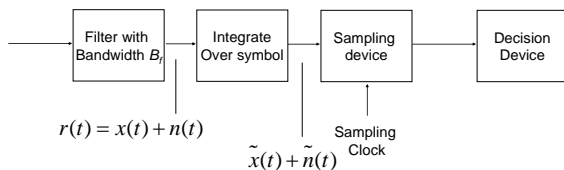
$$Z = y(T)$$

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Improved Receiver



- The integration is simply a form of low-pass filtering
- Thus, the filter and integration could be replaced by a single low pass filter with a square impulse response
 - Why?

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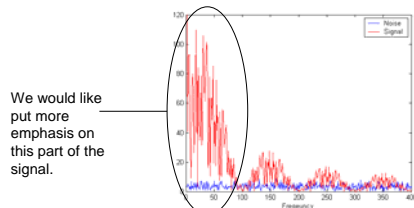
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Improving Probability of Error



- We can also see the benefit of low pass filtering in the frequency domain.
- The signal power decays rapidly in the frequency domain while the noise power does not.
- Thus, we can improve performance by low pass filtering the received signal since it will increase the signal to noise ratio
- Integrating is a form of low pass filtering



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Resulting probability of Error



- Again, assume that $b=1$ is sent.
- Since Z is the integral of a Gaussian Random Process it will be a GRV with mean

$$\begin{aligned}
 E\{Z\} &= E\left\{\int_0^{T_s} r(t) dt\right\} \\
 &= E\left\{\int_0^{T_s} \left(\underbrace{A p(t)}_{\text{constant}} + \underbrace{n(t)}_{\text{zero mean}}\right) dt\right\} \\
 &= AT_s
 \end{aligned}$$

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Variance of Z



- The variance of the new noise process is

$$\begin{aligned}
 \sigma_n^2 &= \int_{-\infty}^{\infty} S_{nn}(f) df \\
 &= \int_{-\infty}^{\infty} S_{nn}(f) |H(f)|^2 df \\
 &= \int_{-\infty}^{\infty} \frac{N_0}{2} \Pi\left(\frac{f}{2B_f}\right) |H(f)|^2 df
 \end{aligned}$$

- The integration is simply a filter with impulse response $h(t) = 1$ over 0 to T_s . Thus, $H(f)$ is

$$H(f) = e^{-j\pi f T_s} \text{sinc}(f T_s)$$

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Variance of Z (cont)



- Thus, the variance of the new noise process is

$$\sigma_n^2 = \frac{N_o}{2} \int_{-B}^B |T_s \text{sinc}(fT_s)|^2 df$$

- Now, since $B \gg 1/T_s$ we have

$$\begin{aligned} \sigma_n^2 &= \frac{N_o}{2} \int_{-B}^B |T_s \text{sinc}(fT_s)|^2 df \\ &\approx \frac{N_o}{2} T_s^2 \int_{-\infty}^{\infty} |\text{sinc}(fT_s)|^2 df \\ &= \frac{N_o}{2} T_s^2 \frac{1}{T_s} \\ &= \frac{N_o T_s}{2} \end{aligned}$$

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Probability of Error



- The decision statistic Z is then $N\left(AT_s, \sqrt{\frac{N_o T_s}{2}}\right)$

and the resulting probability of error is then

$$P_e (b=1) = Q\left(\sqrt{\frac{Z^2}{4\sigma_n^2}}\right) = Q\left(\sqrt{\frac{A^2 T_s}{2N_o}}\right)$$

Thus, we have improved performance dramatically by filtering the noise.

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Conclusions



- The simplest channel is one which simply adds noise to the received signal
 - However, this noise is unknown and thus complicates the receiver's job
- To improve the probability that the receiver makes the correct decision on the transmitted data, we want to increase the signal-to-noise ratio or SNR through filtering
 - Next lecture we will examine the *optimal filter* which maximizes the SNR

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