ECE4634
Digital Communications
Fall 2007

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Lecture #24: The AWGN Channel and The Receiver

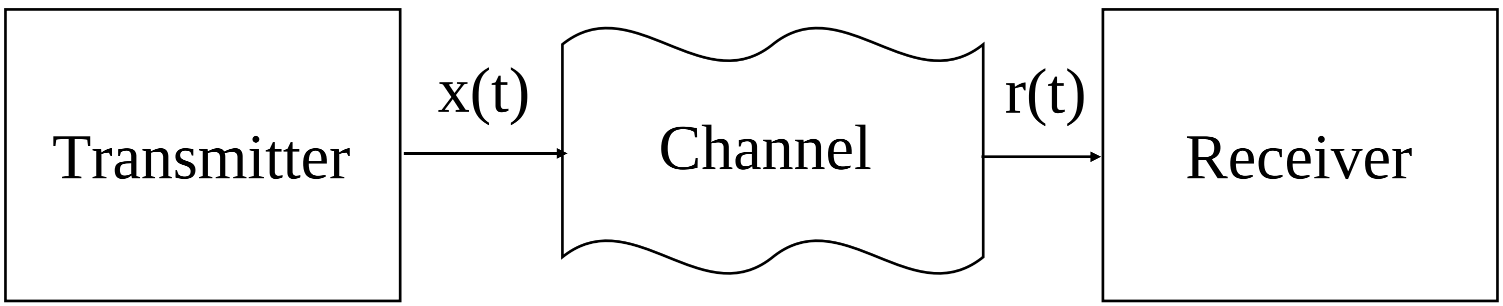
Analog and Digital Communications

Introduction

- Until this point we have concentrated on describing transmission schemes to convey information from one location to another
- Baseband Transmitter
  - Transmit pulses whose amplitude, delay or width conveys the information
  - Pulse shaping for improved bandwidth
- We have also discussed *bandpass* transmission techniques
  - Conveying the information by modulating the phase, frequency or amplitude of a sinusoidal carrier
  - For linear modulation schemes (e.g., PSK, ASK) this can be viewed as simply multiplying a pulse stream by a sinusoidal carrier
- Today we introduce two concepts
  - The Channel
  - The Receiver
- What to read – Sections 8.10-8.11, 9.1-9.3

The Channel

- The channel is the medium through which the system communicates information
- In general $r(t) = x(t) \otimes h(t,\tau) + n(t)$ where $h(t,\tau)$ is the time-varying channel impulse response, $n(t)$ is thermal noise and $\otimes$ is the convolution operation. (channel = filter)



- However, for now let us only consider the attenuation *C* and noise added by the channel

$$r(t) = Cx(t) + n(t)$$

Thermal Noise

- All objects with physical temperature T_p greater than 0 Kelvin generate electrical noise.
- This noise power is given by

$$P_n = kT_n B$$

Bandwidth of device

Botzman's Constant 1.38×10^{-23} J/K

Noise temperature of device

$$kT_n = \text{Noise Spectral Density}$$

AWGN

- Noise in the system contributes an additional random voltage on top of the received signal. Thus, it is **additive**.
- The noise is uncorrelated from one sample to the next (i.e., $R_x(\tau)=\delta(\tau)$). A delta function in the correlation function means that the PSD is a constant, thus it is **white.**
- Noise in the system originates from components in the receiver. Since there are many discrete components all contributing some small amount, the sum tends to a Gaussian process (Central Limit Theorem). Thus, noise samples have a *Gaussian* probability distribution.
- **A**dditive **W**hite **G**aussian **N**oise (AWGN)

AWGN Channels

- The term 'AWGN Channel' is something of a misnomer.
- The channel doesn't necessarily add noise (at least not all of the noise). It attenuates the signal to such a degree that the internal noise of the receiver (as well as the noise observed by the antenna in wireless systems) is comparable to that of the received signal.
- Usually, we assume that the received signal has normalized average received power, while the noise has some power σ^2 where σ is the standard deviation of the thermal noise.
- In "AWGN Channels" we assume that the only distortion to the signal is the AWGN. Normalizing (C = 1):

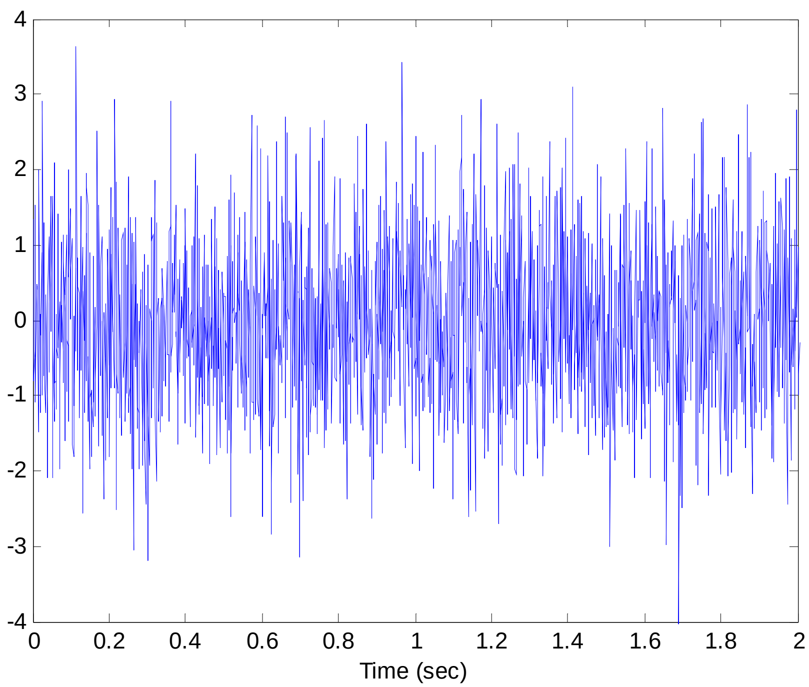
$$r(t) = x(t) + n(t)$$

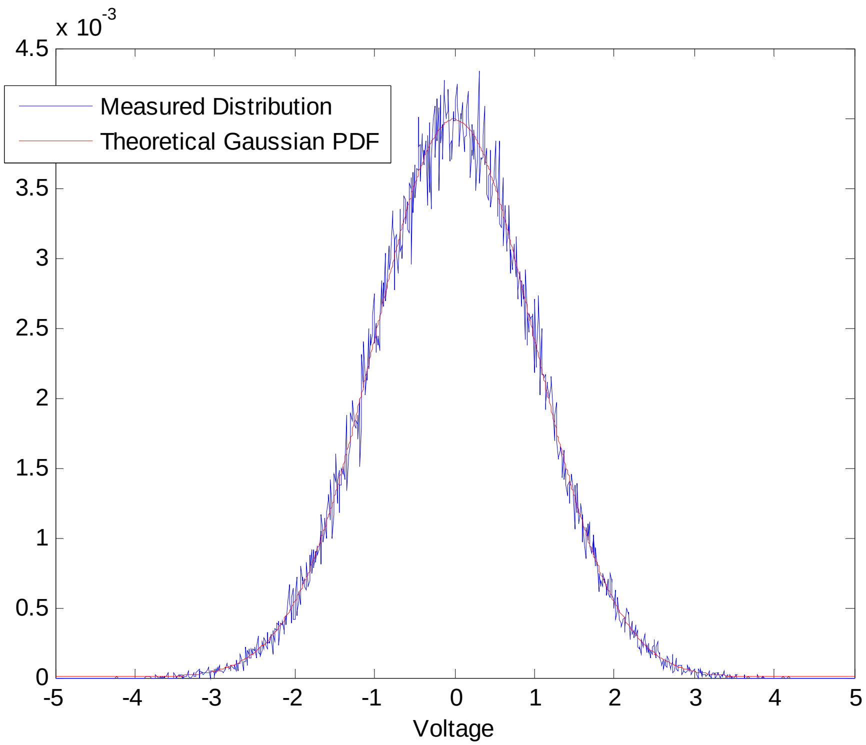
AWGN – Gaussian RP

- Let us take a closer look at the noise process *n(t).* It is the result of summation of many random signals generated in the receiver and observed by the antenna (or picked up by the wire connecting the receiver and transmitter).
- The central limit theorem suggests that we can approximate the summation as a *Gaussian random process.*
- The important things to understand is that
  - Samples of a Gaussian random process are Gaussian random variables
  - If a GRP is input to a linear time-invariant system, the output is also a GRP

Example

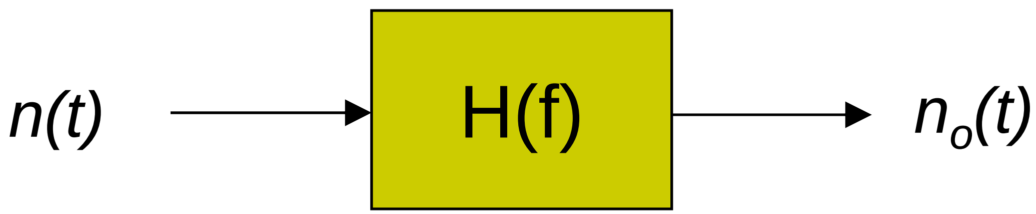
- Example Gaussian Random process and measured probability density function of samples taken at 1ksps

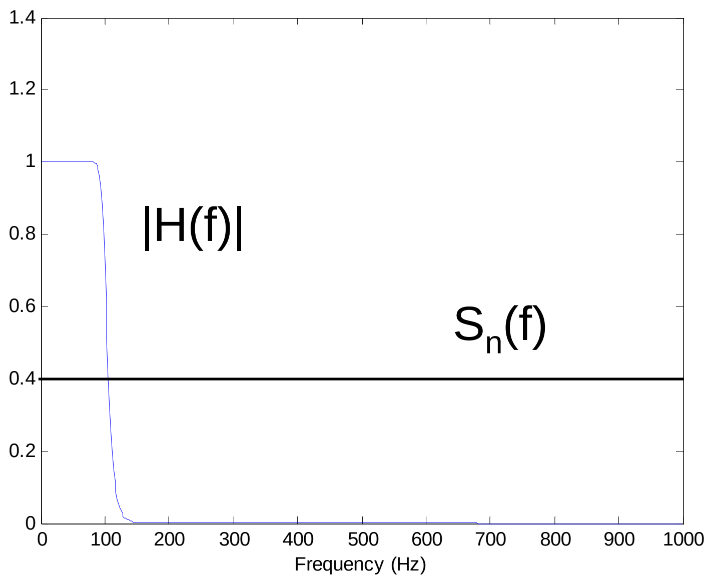


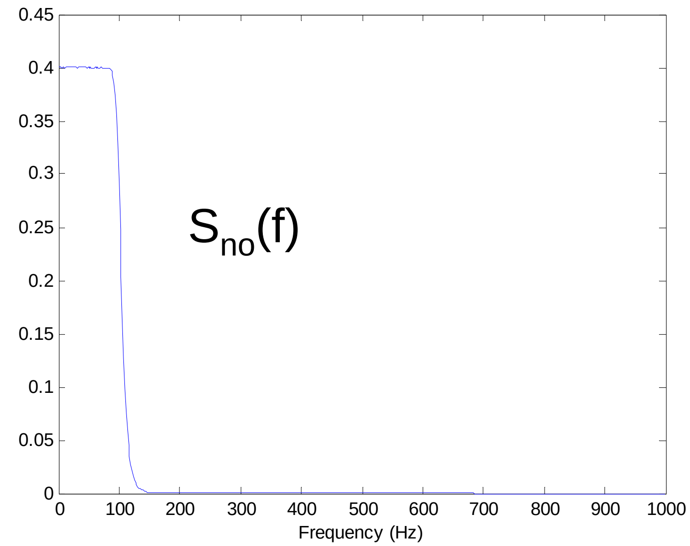


Example – Filter

- Now consider the output of a filter with the following frequency response when white Gaussian noise is input

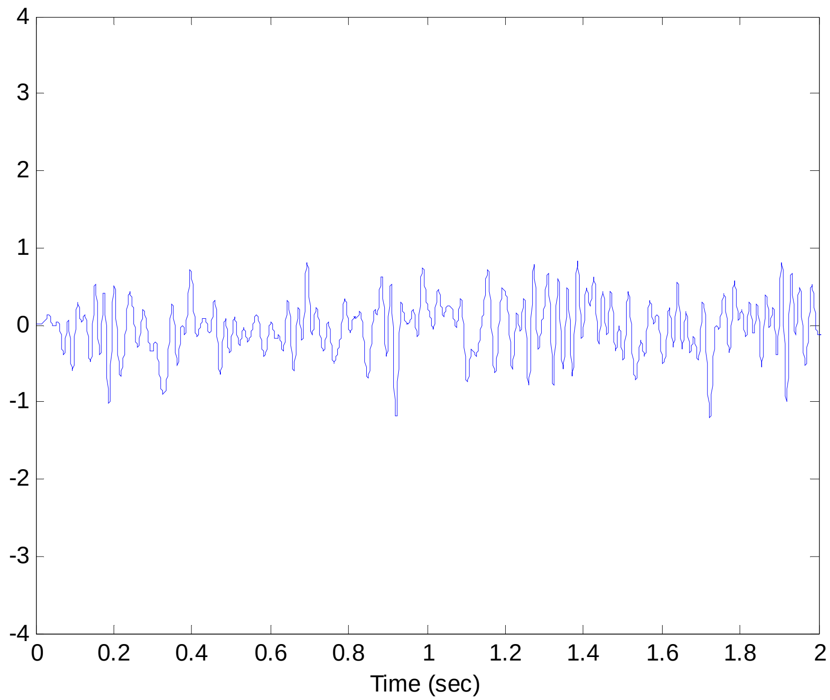


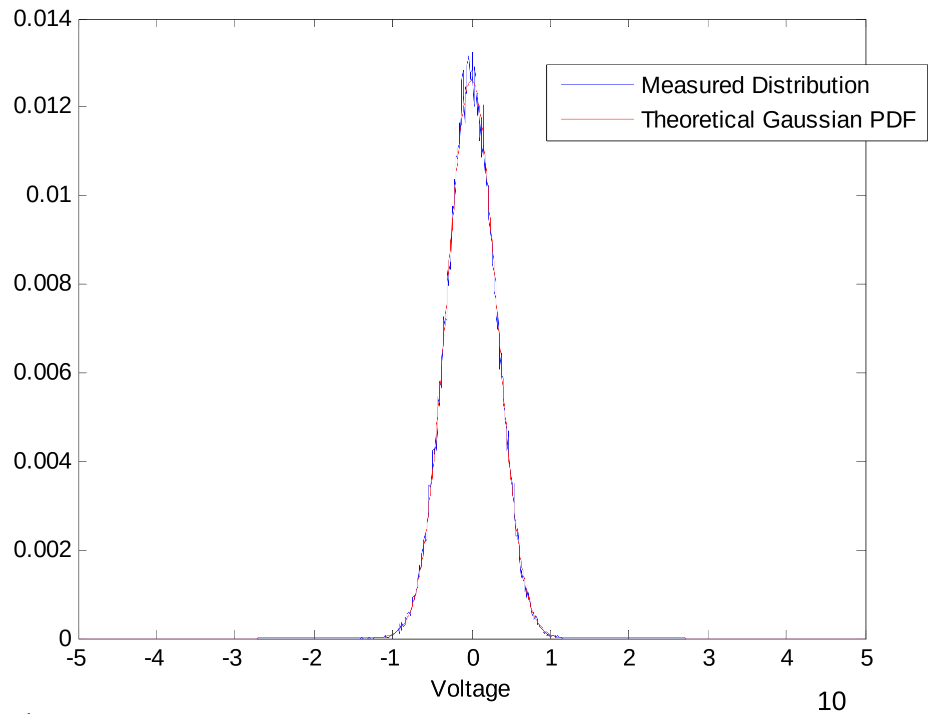




Example - Output Noise

- Output noise power is reduced due to filtering, but samples are still Gaussian.
- Note that Gaussian distribution has smaller variance (power is equal to variance)





AWGN – White RP

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- A 'white' random process is one whose power spectral density (PSD) is a constant:

$$S_n(f) = K \quad \forall f$$

- The autocorrelation of any random process is simply the inverse Fourier Transform of the PSD

$$R_n(\tau) = F^{-1}\{S_n(f)\}$$
$$= K\delta(\tau)$$

Output Autocorrelation

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- Let us return to a linear time-invariant filter.
- If the input random process is white, the output power spectral density follows the filter frequency response
- What effect does this have on the autocorrelation function?
- Recall that the output PSD is related to the input PSD as

$$S_Y(f) = S_X(f)\left|H(f)\right|^2$$

- If a white process is applied to an ideal Low Pass Filter:

$$S_Y(f) = K\left|rect\left(\frac{f}{2B}\right)\right|^2$$

$$= K\ rect\left(\frac{f}{2B}\right)$$

Output Autocorrelation

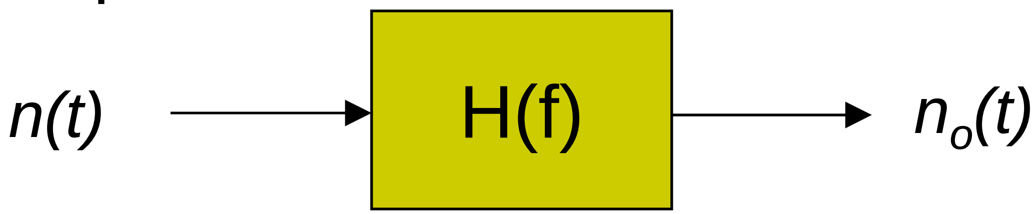
- The resulting autocorrelation function is then

$$R_Y(\tau) = F^{-1}\{S_Y(f)\}$$
$$= F^{-1}\left\{K\ rect\left(\frac{f}{2B}\right)\right\}$$
$$= 2KB\mathrm{sinc}(2\mathrm{B}\tau)$$

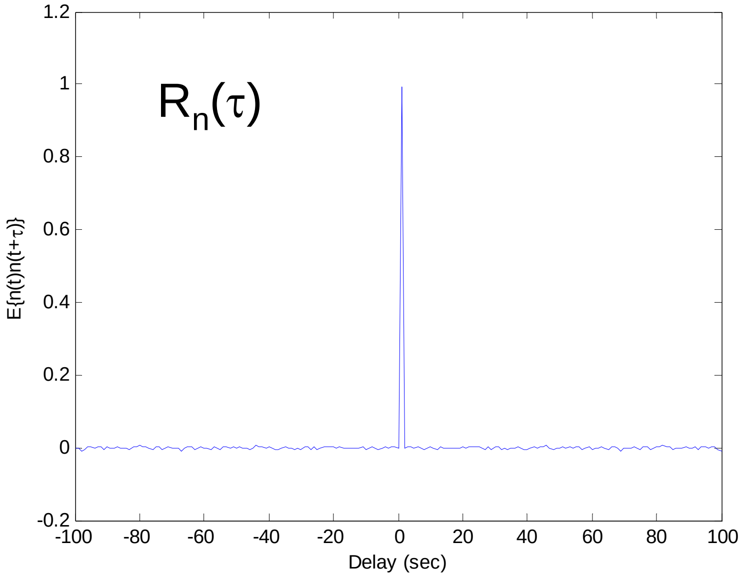
- This is termed *band-limited white noise*

Example – Filter Revisited

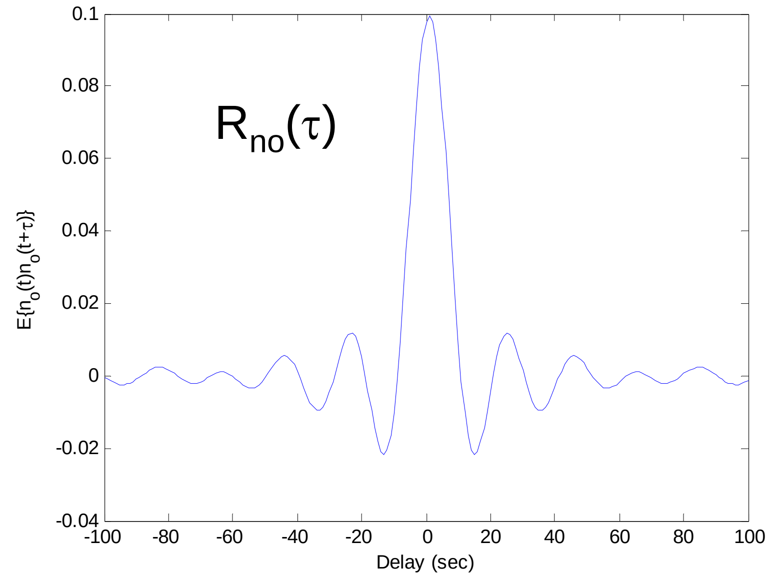
- Let's return to the filter example when white Gaussian noise is input



Consecutive samples are uncorrelated
$R_n(0)$ = 1 = variance of the input noise



Consecutive samples are now correlated
$R_n(0)$ = 0.1 = variance of the input noise



Important Note

- While we often consider white noise with a flat (i.e., constant) power spectral density, in actuality this type of process is not realistic
  - A process with a truly constant PSD has infinite power

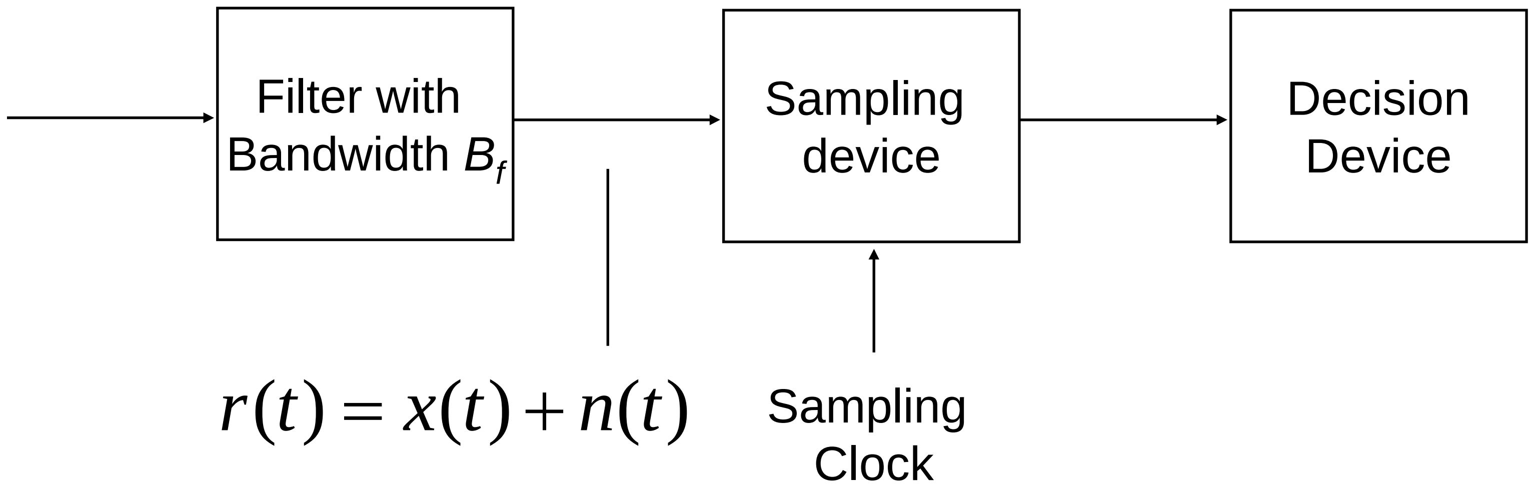
$$P_{tot} = \int_{-\infty}^{\infty} S_n(f)\,df$$

$$= \int_{-\infty}^{\infty} K\,df$$

$$= \infty$$

- However, noise which is white over the bandwidth *B* of the receiver (i.e., band-limited white noise) does not have infinite power and is realistic.
- If such a process is sampled at $T_s = 1/B$ consecutive samples have zero correlation ($R_n(\tau) = K\,\mathrm{sinc}(\tau B)$)

Receiver



$$r(t) = x(t) + n(t)$$

- For now we assume that $B_f >> B$ (i.e., filter bandwidth is much greater than the signal bandwidth) so that the signal is not distorted by the filter
- Later we will show that this isn't the best thing to do

Receiver

- Assuming a fairly benign channel, at the receiver we observe the transmitted signal plus noise

$$r(t) = x(t) + n(t)$$

- Both the transmitted signal x(t) and the noise *n(t)* are *apriori* unknown.
- The job of the receiver is to determine the most likely transmitted signal based on the observed received signal
- Note: Typically the received signal *r(t)* is characterized by its *signal-to-noise ratio or SNR*

$$\frac{S}{N} = \frac{\text{desired signal power}}{\text{noise power or variance}} = \frac{E\left\{x^2(t)\right\}}{E\left\{n^2(t)\right\}} = \frac{A^2}{\sigma^2}$$

Receiver

- Example: Assume that PAM is used to transmit binary data

$$x(t) = \sum_{n=-\infty}^{\infty} d_n p\left(t - nT_s\right)$$

$$d_n = \begin{cases} A & b = 1 \\ 0 & b = 0 \end{cases}$$

Note that this is equivalent to a unipolar non-return to zero line code.

- At the receiver we observe

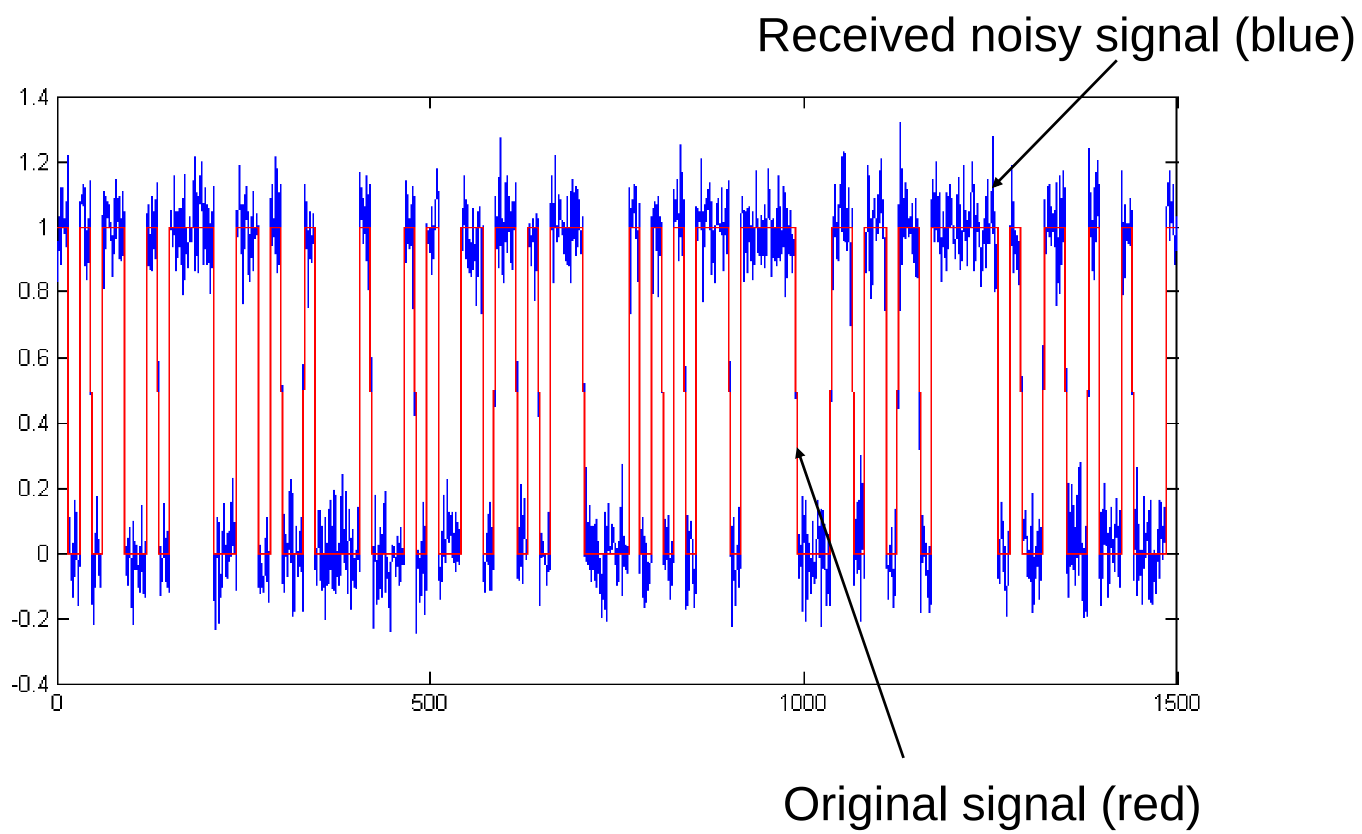
$$r(t) = x(t) + n(t)$$

- Where *n(t)* is a white Gaussian random process whose samples are independent Gaussian random variables with zero mean and variance σ^2
  - Note that the variance is a function of the receiver noise bandwidth and the noise power spectral density

Receiver (cont.)

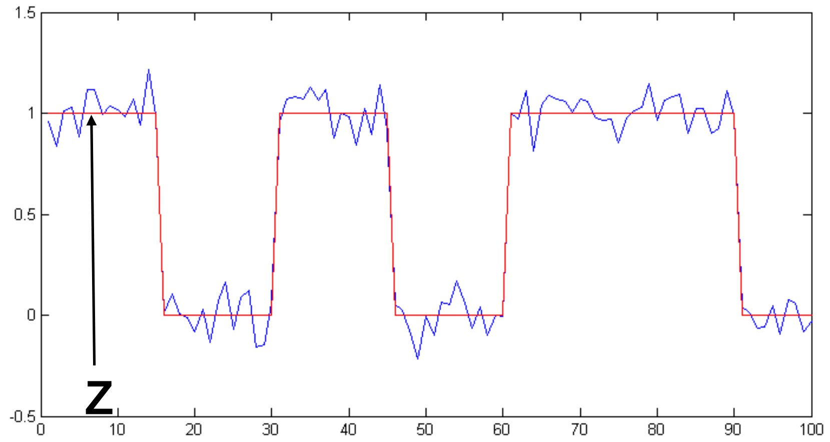
- Example plot from the output of the filter:

A=1
σ_n=0.1



Make a decision

- To determine the received 'symbol' we could simply sample at any point in the symbol interval and compare with a threshold (0.5)



$$\hat{b} = \begin{cases} 1 & Z \geq 0.5 \\ 0 & Z < 0.5 \end{cases}$$

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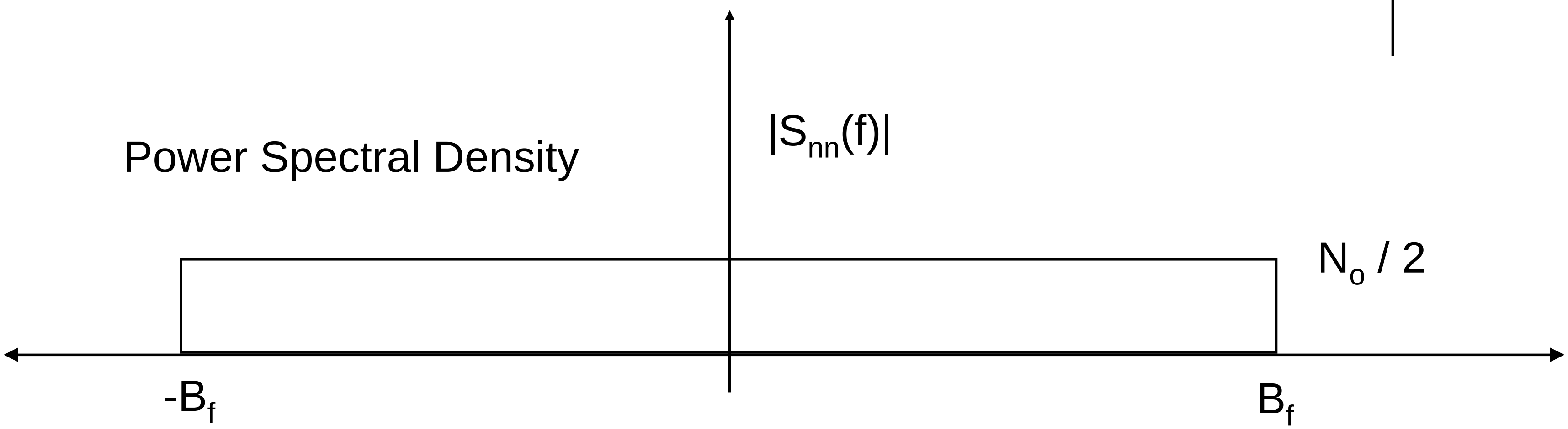
Error Probability

- The sample value Z is a conditional GRV.
  - Assuming b=1, Z is normal with mean A and standard deviation σ → $Z = N(A, \sigma)$
  - The probability of error is then the probability that $Z < A/2$ *given that b=1.*

$$
\begin{aligned}
P_e(b=1) &= \Pr\left(Z < A/2 \middle| b=1\right) \\
&= \Pr\left(N < -A/2\right) \\
&= \int_{-\infty}^{-A/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{A/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{A/2\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= Q\left(\frac{A}{2\sigma}\right)
\end{aligned}
$$

The same analysis assuming that b=0 will produce the same result

Noise Power



- Let $B_f = 10R_s$ to avoid distorting the received signal
- Then $\sigma_n^2 = N_oB_f = 10R_sN_o$

$$P_e = Q\left(\frac{A}{2\sigma}\right) = Q\left(\sqrt{\frac{A^2}{4\sigma_n^2}}\right) = Q\left(\sqrt{\frac{A^2T_s}{40N_o}}\right)$$

Improving Probability of Error

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- However, note that the noise value changes over the symbol duration while the amplitude value does not.
- Thus, we could improve our estimate by taking several samples and averaging the results.
- In the limit, we could simply integrate over the symbol duration to improve our estimate

$$y(t) = \int_{t-T}^{t} r(x)dx$$

$$Z = y(T)$$

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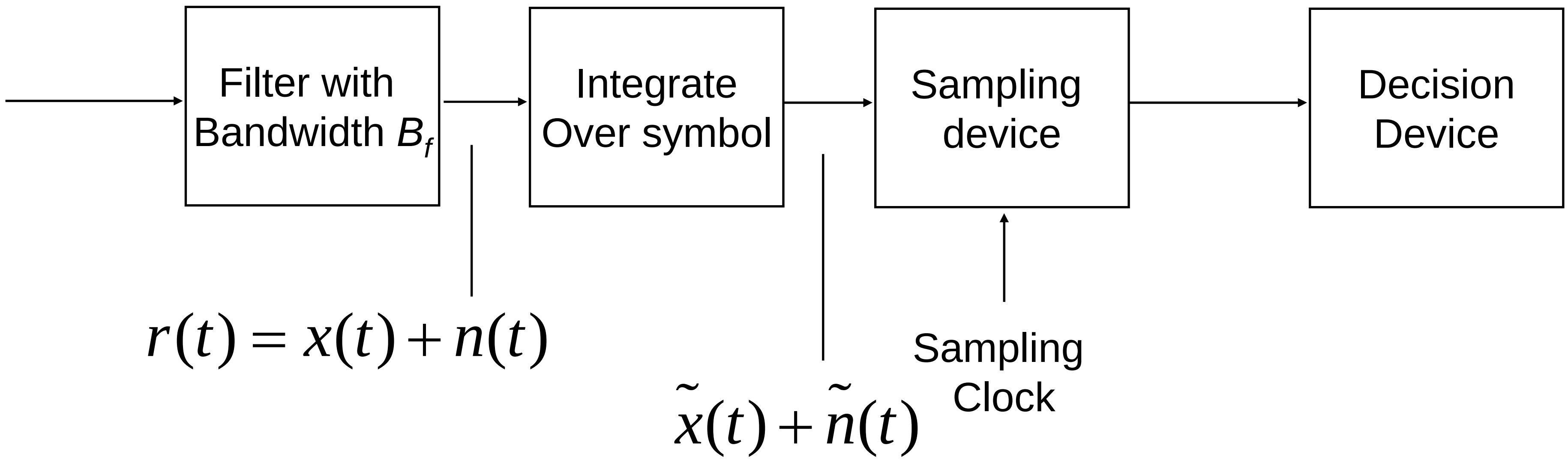
Improved Receiver



- The integration is simply a form of low-pass filtering
- Thus, the filter and integration could be replaced by a single low pass filter with a square impulse response
  - Why?

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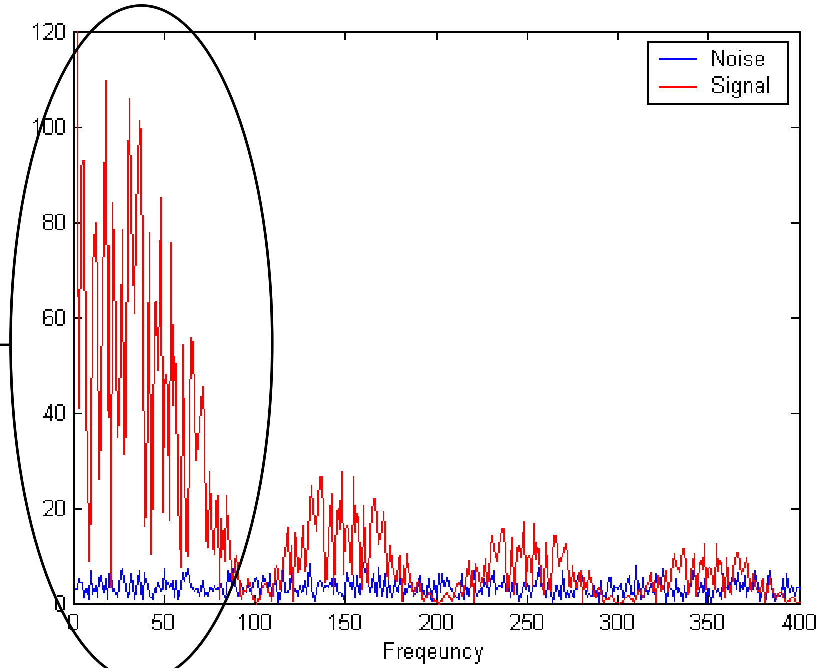
Improving Probability of Error

- We can also see the benefit of low pass filtering in the frequency domain.
- The signal power decays rapidly in the frequency domain while the noise power does not.
- Thus, we can improve performance by low pass filtering the received signal since it will increase the signal to noise ratio
- Integrating is a form of low pass filtering

We would like put more emphasis on this part of the signal.



Resulting probability of Error

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- Again, assume that *b*=1 is sent.
- Since *Z* is the integral of a Gaussian Random Process it will be a GRV with mean

$$
\begin{aligned}
E\{Z\} &= E\left\{\int_0^{T_s} r(t)dt\right\} \\
&= E\left\{\int_0^{T_s}\left(A\underbrace{p(t)}_{\text{constant}} + \underbrace{n(t)}_{\text{zero mean}}\right)dt\right\} \\
&= AT_s
\end{aligned}
$$

Variance of *Z*

- The variance of the new noise process is

$$\sigma_n^2 = \int_{-\infty}^{\infty} S_{\tilde{n}\tilde{n}}(f)df$$

$$= \int_{-\infty}^{\infty} S_{nn}(f)\left|H(f)\right|^2 df$$

$$= \int_{-\infty}^{\infty} \frac{N_o}{2}\Pi\left(\frac{f}{2B_f}\right)\left|H(f)\right|^2 df$$

- The integration is simply a filter with impulse response h(t) = 1 over 0 to T_s. Thus, H(f) is

$$H(f) = e^{-j\pi T} T_s \mathrm{sinc}\left(fT_s\right)$$

Variance of Z (cont)

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- Thus, the variance of the new noise process is

$$\sigma_n^2 = \frac{N_o}{2} \int_{-B_f}^{B_f} \left| T_s \mathrm{sinc}\left(f T_s \right) \right|^2 df$$

- Now, since B >> 1/T_s we have

$$\begin{aligned} \sigma_n^2 &= \frac{N_o}{2} \int_{-B}^{B} \left| T_s \mathrm{sinc}\left(f T_s \right) \right|^2 df \\ &\approx \frac{N_o}{2} T_s^{\,2} \int_{-\infty}^{\infty} \left| \mathrm{sinc}\left(f T_s \right) \right|^2 df \\ &= \frac{N_o}{2} T_s^{\,2} \frac{1}{T_s} \\ &= \frac{N_o T_s}{2} \end{aligned}$$

Probability of Error

- The decision statistic *Z* is then $N\left(AT_s, \sqrt{\frac{N_o T_s}{2}}\right)$

and the resulting probability of error is then

$$P_e(b=1) = Q\left(\sqrt{\frac{\overline{Z}^2}{4\sigma_{\tilde{n}}^2}}\right) = Q\left(\sqrt{\frac{A^2 T_s}{2N_o}}\right)$$

Thus, we have improved performance dramatically by filtering the noise.

Conclusions

- The simplest channel is one which simply adds noise to the received signal
  - However, this noise is unknown and thus complicates the receivers job
- To improve the probability that the receiver makes the correct decision on the transmitted data, we want to increase the signal-to-noise ratio or SNR through filtering
  - Next lecture we will examine the *optimal filter* which maximizes the SNR