

ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer
Lecture #25:
The Matched Filter



1

Introduction



- Until this point we have concentrated mainly on describing transmission schemes to convey information from one location to another
- Last class we discussed
 - the AWGN channel which introduces noise to the received signal
 - the receiver and found that filtering the received signal is necessary to reduce the impact of noise
- Today we will discuss the *optimal filter* in terms of SNR
 - We will focus on baseband signals but the results will also apply to bandpass signals
- What to read: Sections 10.1-10.2

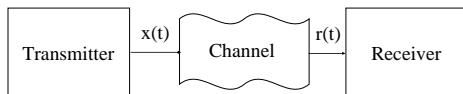
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2

The Channel



- The channel is the medium through which the system communicates information
- In general $r(t) = x(t) \otimes h(t, \tau) + n(t)$ where $h(t, \tau)$ is the time-varying channel impulse response, $n(t)$ is thermal noise and \otimes is the convolution operation.



- However, for now let us only consider the attenuation C and noise added by the channel

$$r(t) = Cx(t) + n(t)$$

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3

Optimal Filtering



- At this point we haven't given any thought to the 'optimal' choice of filters
 - In particular, a good choice of filters may help us reduce the effect noise at the front end of the receiver.
- Criteria for optimum filter design
 - Maximize SNR at output of filter
 - Signal at output:

$$\tilde{X}(f) = X(f)H(f)$$

Filter transfer function

Pulse Spectrum

- Noise at output:

$$S_n(f) = S_n(f)|H(f)|^2$$

Noise PSD

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4

Optimal Filtering (cont.)



- What is the SNR at sampling instant T ?

$$\tilde{x}(T) = \int_{-\infty}^{\infty} \tilde{X}(f)e^{j2\pi fT} df$$

$$\overline{n_o^2(t)} = R_{n_o}(0) = \int_{-\infty}^{\infty} |H(f)|^2 S_{n_o}(f) df$$

$$\left(\frac{S}{N}\right)_{out} = \frac{|\tilde{x}(T)|^2}{\overline{n_o^2(t)}}$$

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5

Optimal Filtering (cont.)



- What choice of filter maximizes SNR at sampling instant T ?

$$\left(\frac{S}{N}\right)_{out} = \frac{|\tilde{x}(T)|^2}{\overline{n_o^2(t)}} = \frac{\left| \int_{-\infty}^{\infty} \tilde{X}(f)e^{j2\pi fT} df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{n_o}(f) df}$$

$$= \frac{\left| \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi fT} df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{n_o}(f) df}$$

Need to choose $H(f)$ to maximize SNR

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6

Optimal Filtering (cont.)



- Schwarz's Inequality

$$\left| \int_{-\infty}^{\infty} A(f)B(f)df \right|^2 \leq \int_{-\infty}^{\infty} |A(f)|^2 df \int_{-\infty}^{\infty} |B(f)|^2 df$$

where the equality is attained when $A(f) = KB^*(f)$ for an arbitrary real constant K

- Let

$$A(f) = H(f)\sqrt{S_m(f)}$$

$$B(f) = \frac{X(f)e^{j2\pi fT}}{\sqrt{S_n(f)}}$$

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7

Optimal Filtering (cont.)



$$\left(\frac{S}{N} \right)_{out} \leq \frac{\int_{-\infty}^{\infty} |H(f)|^2 S_n(f) df \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df}{\int_{-\infty}^{\infty} |H(f)|^2 S_n(f) df}$$

$$\leq \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$

The equality is obtained when $A(f)=KB^*(f)$ or

$$H(f)\sqrt{S_m(f)} = K \frac{X^*(f)e^{-j2\pi fT}}{\sqrt{S_n(f)}}$$

$$H(f) = K \frac{X^*(f)}{S_n(f)} e^{-j2\pi fT}$$

8

Optimal Filtering (cont.)



- The choice of filter which gives the maximum signal to noise ratio at time T is:

$$H(f) = K \frac{X^*(f)}{S_n(f)} e^{-j2\pi fT}$$

- K is an arbitrary constant
- $X(f)$ is the pulse shape of the signal
- $S_n(f)$ is the power spectral density of the noise
- $e^{-j2\pi fT}$ inserts a delay to maximize output at time T

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9

Optimal Filtering (continued)



- Interpretation of Result:

$$H(f) = K \frac{X^*(f)}{S_n(f)} e^{-j2\pi fT}$$

- Frequency response is proportional to $X(f)$
 - Emphasize frequencies which contain large signal components
- Frequency response is inversely proportional to $S_n(f)$
 - Deemphasize frequencies which contain large noise components

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10

Additive White Gaussian Noise



- Important special case:

- $S_n(f) = N_0/2$ where N_0 is a constant
- Additive White Gaussian Noise (AWGN)
- Equal noise power at all frequencies
- Samples uncorrelated in time domain:

$$R_n(\tau) = \frac{N_0}{2} \delta(\tau)$$

- Good approximation to receiver noise in many cases

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11

Matched Filtering



- Let $S_n(f) = N_0/2$
- Then the optimal filter response becomes:

$$H(f) = \frac{2KX^*(f)}{N_0} e^{-j2\pi fT} = CX^*(f)e^{-j2\pi fT}$$

- In time domain:

$$h(t) = Cx(T-t)$$
- Filter is "matched" to signal
 - impulse response is just a time-reversed version of signal pulse

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12

Maximum SNR



- The optimal filter, i.e., the filter which maximizes SNR is the matched filter
- The resulting maximum SNR is when the equality is met:

$$\left(\frac{S}{N}\right)_{out} \leq \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$
$$\left(\frac{S}{N}\right)_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$

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13

Maximum SNR



- For white noise:

$$\left(\frac{S}{N}\right)_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$
$$= \int_{-\infty}^{\infty} \frac{|X(f)|^2}{N_o/2} df$$
$$= \frac{2}{N_o} \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{2}{N_o} \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$= \frac{2E_x}{N_o}$$

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14

The Matched Filter



- Note that the maximum SNR *does not depend on the pulse shape*. As long as the receiver filter is *matched* to the transmit pulse shape the output SNR is maximized and equal to

$$\frac{2E_x}{N_o}$$

- Thus, the pulse shape (theoretically) does not have an impact on the performance. We can modify the pulse shape to manipulate bandwidth at no penalty to performance.

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15

Square Pulses



- For square pulses $p(t) = \text{rect}\left(\frac{t}{T}\right)$
- If the noise is white $S_m(f) = N_0/2$ then the impulse response of the matched filter is

$$\begin{aligned} h(t) &= Cx(T-t) \\ &= C \text{rect}\left(\frac{T-t}{T}\right) \\ &= C \text{rect}\left(\frac{t}{T}\right) \end{aligned}$$

Thus, for square pulses the optimal filter is simply integration!

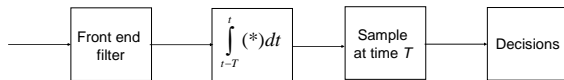
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16

Square Pulses (cont.)



- The optimal receiver (matched filter) for square pulses is:



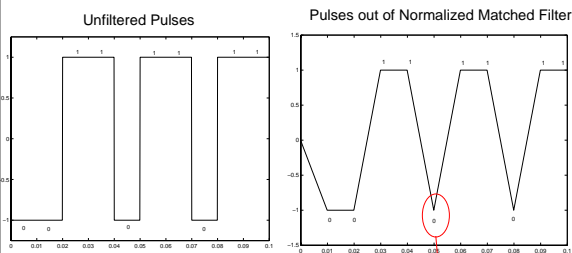
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17

Matched Filter – Square Pulses



- One consequence – more sensitivity to timing



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Need to sample at end of symbol to obtain maximum SNR

Intersymbol Interference



- Recall that filtering can cause intersymbol interference
- To avoid ISI we must satisfy Nyquist's criterion at the *output* of the matched filter
- One way to accomplish this is to use *square root* raised cosine pulses and filters
- Square pulses also accomplish this but are impractical

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19

Conclusions



- The simplest channel is one which simply adds noise to the received signal
 - However, this noise is unknown and thus complicates the receiver's job
- To improve the probability that the receiver makes the correct decision on the transmitted data, we want to maximize the signal-to-noise ratio or SNR
 - This is accomplished with a *matched filter*

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20
