ECE4634
Digital Communications
Fall 2007

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Lecture #25:

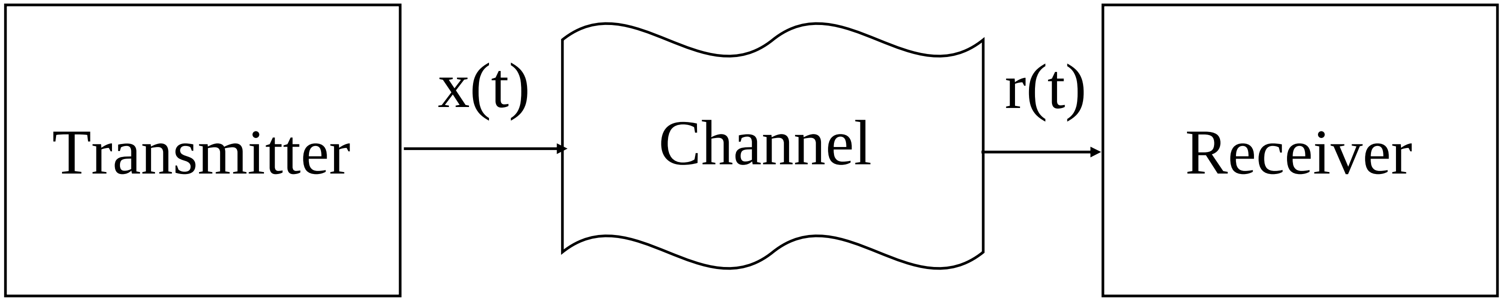
The Matched Filter

Introduction

- Until this point we have concentrated mainly on describing transmission schemes to convey information from one location to another
- Last class we discussed
  - the AWGN channel which introduces noise to the received signal
  - the receiver and found that filtering the received signal is necessary to reduce the impact of noise
- Today we will discuss the *optimal filter* in terms of SNR
  - We will focus on baseband signals but the results will also apply to bandpass signals
- What to read: Sections 10.1-10.2

The Channel

- The channel is the medium through which the system communicates information
- In general $r(t) = x(t) \otimes h(t,\tau) + n(t)$ where $h(t,\tau)$ is the time-varying channel impulse response, $\nu(t)$ is thermal noise and $\otimes$ is the convolution operation.



- However, for now let us only consider the attenuation *C* and noise added by the channel

$$r(t) = Cx(t) + n(t)$$

Optimal Filtering

- At this point we haven't given any thought to the 'optimal' choice of filters
  - In particular, a good choice of filters may help us reduce the effect noise at the front end of the receiver.
- Criteria for optimum filter design
  - Maximize SNR at output of filter
  - Signal at output:

$$\tilde{X}(f) = X(f)H(f)$$

  - Noise at output:

$$S_{\tilde{n}}(f) = S_n(f)\left|H(f)\right|^2$$

Filter transfer function

Pulse Spectrum

Noise PSD

Optimal Filtering (cont.)

- What is the SNR at sampling instant *T*?

$$\tilde{x}(T) = \int_{-\infty}^{\infty} \tilde{X}(f) e^{j2\pi fT} df$$

$$\overline{\tilde{n}_o^2(t)} = R_{\tilde{n}\tilde{n}}(0) = \int_{-\infty}^{\infty} \left|H(f)\right|^2 S_{nn}(f) df$$

$$\left(\frac{S}{N}\right)_{out} = \frac{\left|\tilde{x}(T)\right|^2}{\overline{\tilde{n}_o^2(t)}}$$

Optimal Filtering (cont.)

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- What choice of filter maximizes SNR at sampling instant *T* ?

$$\left(\frac{S}{N}\right)_{out} = \frac{\left|\tilde{x}(T)\right|^2}{\overline{\tilde{n}_o^2(t)}} = \frac{\left|\int_{-\infty}^{\infty} \tilde{X}(f)e^{j2\pi fT}df\right|^2}{\int_{-\infty}^{\infty} \left|H(f)\right|^2 S_{nn}(f)df}$$

$$= \frac{\left|\int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi fT}df\right|^2}{\int_{-\infty}^{\infty} \left|H(f)\right|^2 S_{nn}(f)df}$$

Need to choose *H(f)* to maximize SNR

Optimal Filtering (cont.)

- Schwarz's Inequality

$$\left| \int_{-\infty}^{\infty} A(f)B(f)df \right|^2 \leq \int_{-\infty}^{\infty} |A(f)|^2 df \int_{-\infty}^{\infty} |B(f)|^2 df$$

where the equality is attained when $A(f) = KB^*(f)$ for an arbitrary real constant *K*

- Let

$$A(f) = H(f)\sqrt{S_{nn}(f)}$$

$$B(f) = \frac{X(f)e^{j2\pi fT}}{\sqrt{S_{nn}(f)}}$$

Optimal Filtering (cont.)

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$$\left(\frac{S}{N}\right)_{out} \le \frac{\int_{-\infty}^{\infty} |H(f)|^2 S_n(f)df \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df}{\int_{-\infty}^{\infty} |H(f)|^2 S_n(f)df}$$

$$\le \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$

The equality is obtained when *A(f)=KB*(f)* or

$$H(f)\sqrt{S_{nn}(f)} = K\frac{X^*(f)e^{-j2\pi fT}}{\sqrt{S_{nn}(f)}}$$

$$\boxed{H(f) = K\frac{X^*(f)}{S_{nn}(f)}e^{-j2\pi fT}}$$

Optimal Filtering (cont.)

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- The choice of filter which gives the maximum signal to noise ratio at time *T* is:

$$H(f) = K \frac{X^*(f)}{S_n(f)} e^{-j2\pi fT}$$

  - K is an arbitrary constant
  - $X(f)$ is the pulse shape of the signal
  - $S_n(f)$ is the power spectral density of the noise
  - $e^{-j2\pi fT}$ inserts a delay to maximize output at time *T*

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Optimal Filtering (continued)

- Interpretation of Result:

$$H(f) = K\frac{X^*(f)}{S_n(f)}e^{-j2\pi fT}$$

- Frequency response is proportional to $X(f)$
  - Emphasize frequencies which contain large signal components
- Frequency response is inversely proportional to $S_n(f)$
  - Deemphasize frequencies which contain large noise components

Additive White Gaussian Noise

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- Important special case:
  - $S_n(f) = N_0/2$ where N_o is a constant
  - Additive White Gaussian Noise (AWGN)
  - Equal noise power at all frequencies
  - Samples uncorrelated in time domain:

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

  - Good approximation to receiver noise in many cases

Matched Filtering

- Let $S_n(f) = N_0/2$
- Then the optimal filter response becomes:

$$H(f) = \frac{2KX^*(f)}{N_0} e^{-j2\pi fT} = CX^*(f)e^{-j2\pi fT}$$

- In time domain:

$$h(t) = Cx(T-t)$$

- Filter is “matched” to signal
  - impulse response is just a time-reversed version of signal pulse

Maximum SNR

- The optimal filter, i.e., the filter which maximizes SNR is the matched filter
- The resulting maximum SNR is when the equality is met:

$$\left(\frac{S}{N}\right)_{out} \leq \int_{-\infty}^{\infty} \frac{\left|X(f)\right|^2}{S_n(f)} df$$

$$\left(\frac{S}{N}\right)_{\max} = \int_{-\infty}^{\infty} \frac{\left|X(f)\right|^2}{S_n(f)} df$$

Maximum SNR

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- For white noise:

$$\left(\frac{S}{N}\right)_{\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df$$

$$= \int_{-\infty}^{\infty} \frac{|X(f)|^2}{N_o/2} df$$

$$= \frac{2}{N_o} \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{2}{N_o} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \frac{2E_s}{N_o}$$

The Matched Filter

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- Note that the maximum SNR *does not depend on the pulse shape*. As long as the receiver filter is *matched* to the transmit pulse shape the output SNR is maximized and equal to

$$\frac{2E_s}{N_o}$$

- Thus, the pulse shape (theoretically) does not have an impact on the performance. We can modify the pulse shape to manipulate bandwidth at no penalty to performance.

Square Pulses

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- For square pulses $p(t) = rect\left(\frac{t}{T}\right)$

- If the noise is white $S_{nn}(f) = N_0/2$

  then the impulse response of the matched filter is

$$h(t) = Cx(T - t) = C\,rect\left(\frac{T - t}{T}\right) = C\,rect\left(\frac{t}{T}\right)$$

Thus, for square pulses the optimal filter is simply integration!

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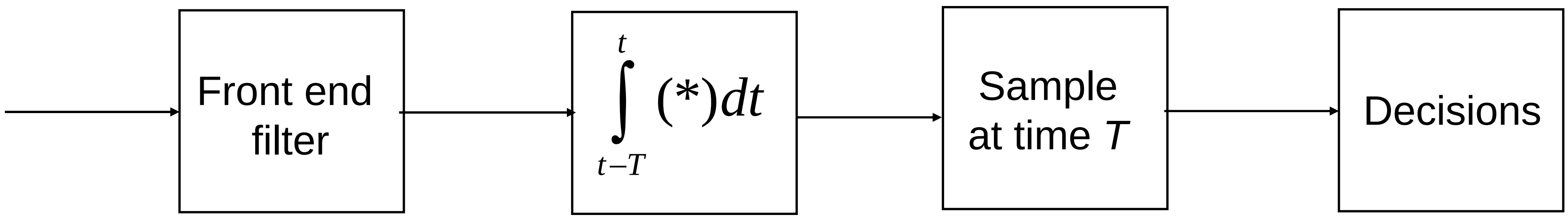
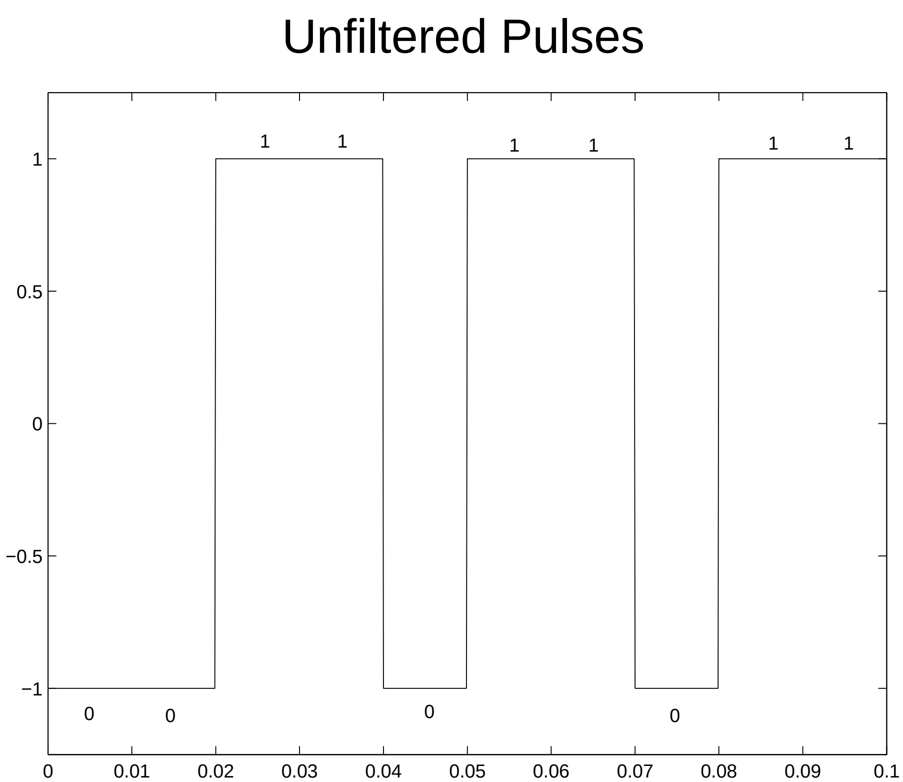
Square Pulses (cont.)

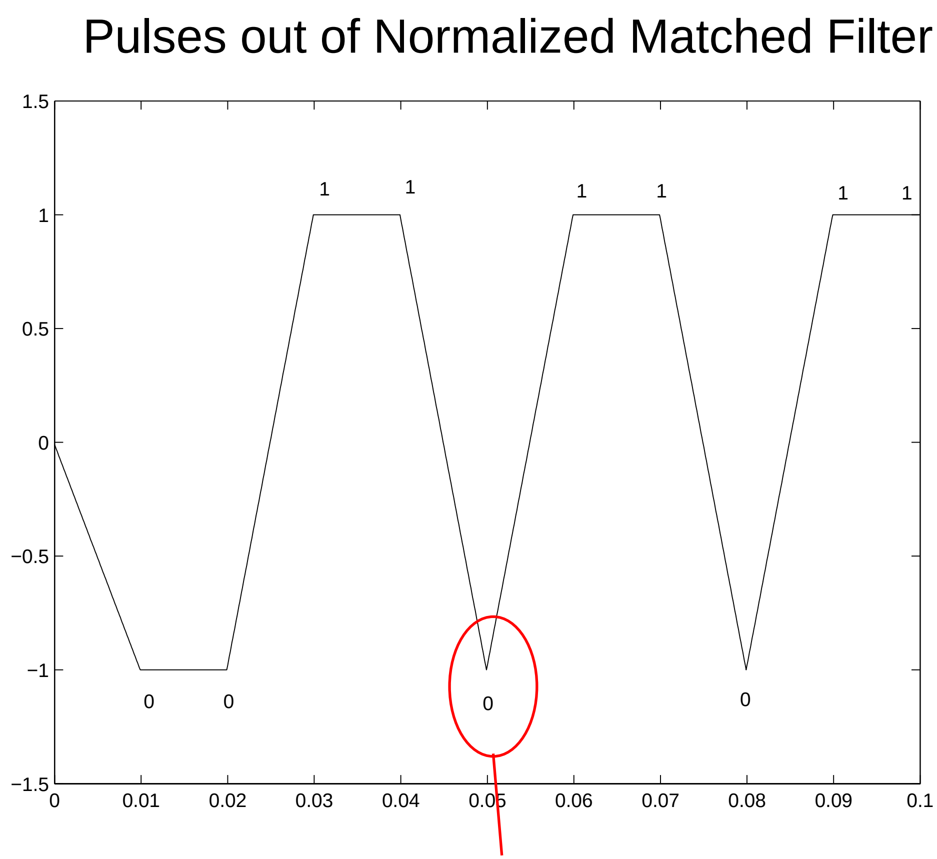
- The optimal receiver (matched filter) for square pulses is:



Matched Filter – Square Pulses

- One consequence – more sensitivity to timing





Need to sample at end of symbol to obtain maximum SNR

Intersymbol Interference

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- Recall that filtering can cause intersymbol interference
- To avoid ISI we must satisfy Nyquist's criterion at the *output* of the matched filter
- One way to accomplish this is to use *square root* raised cosine pulses and filters
- Square pulses also accomplish this but are impractical

Conclusions

- The simplest channel is one which simply adds noise to the received signal
  - However, this noise is unknown and thus complicates the receivers job
- To improve the probability that the receiver makes the correct decision on the transmitted data, we want to maximize the signal-to-noise ratio or SNR
  - This is accomplished with a *matched filter*