ECE4634
Digital Communications
Fall 2007

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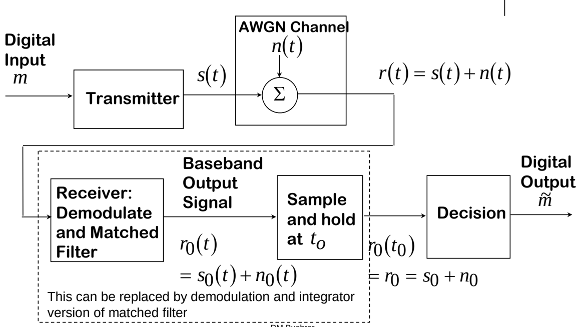
Lecture #27: Performance of Binary Signaling in Noise

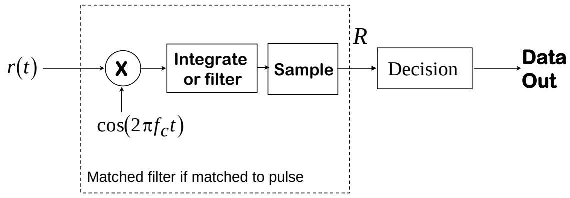
Introduction

- We have now discussed receiver structures for various modulation schemes.
- Our focus has been on demodulation and filtering
- We now focus on the final decision (called detection) and BER
- We will start with binary modulation schemes and move to *M*-ary modulation
- What to read – Sections 10.2-10.3

Block Diagram for Binary Communications System



Receiver for BPSK



Binary Communication System Model: Transmitter

- Binary signal generated by transmitter

$$s(t) = \begin{cases} s_1(t)\big|_0^T, & m=1 \\ s_2(t)\big|_0^T, & m=0 \end{cases}$$

- Where $s_1(t)$ and $s_2(t)$ might have different:
  - phases
  - amplitudes
  - frequencies

Binary Communications System Model: Received Signal

- Received Signal

$$r(t) = s(t) + n(t)$$

- AWGN:
  - $P_n(f) = N_0/2$ where N_0 is a constant
  - Additive White Gaussian Noise (AWGN)
  - Equal noise power at all frequencies
  - Samples uncorrelated in time domain:

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

  - Good approximation to receiver noise in many cases

Binary Communications System Model: Baseband Received Signal

- After demodulation and (matched) filtering we have a baseband signal:

  $$r_0(t) = s_0(t) + n_0(t)$$

- Baseband noise signal:

  $$n_0(t)$$

- Baseband signal

  $$s_0(t) = \begin{cases} s_{01}(t), & m = 1 \\ s_{02}(t), & m = 0 \end{cases}$$

Binary Communications System Model: Sampled Received Signal

- After sampling, we have the 'decision variable' or 'decision statistic' used for decisions

  $$r_0 = s_0 + n_0$$

- Sampled noise:

  $$n_0$$

- Sampled signal:
  - $$s_0 = \begin{cases} s_{01}, & m = 1 \\ s_{02}, & m = 0 \end{cases}$$

Receiver Decision

- Let V_T be a threshold voltage for deciding *m*
  - Usually, $V_T = \frac{s_{01} + s_{02}}{2}$ for equally likely signals
- We make a decision according to the rule
  - $$\tilde{m} = \begin{cases} 1, & r_0 \geq V_T \\ 0, & r_0 < V_T \end{cases}$$

Later we will generalize the threshold to decision regions for *M*-ary modulation

Probability of Error

- We are interested in the probability of making a wrong decision. Let:

$$\Pr[\text{error}|s_1 \text{ sent}] = \int_{-\infty}^{V_T} f(r_0|s_1)\,dr_0$$

$$\Pr[\text{error}|s_2 \text{ sent}] = \int_{V_T}^{\infty} f(r_0|s_2)\,dr_0$$

$f(r_o|s_i)$ = conditional pdf of r_o given that s_i was sent.

- Average probability of error

$$P_e = \Pr[\text{error}|s_1 \text{ sent}]\Pr[s_1 \text{ sent}] + \Pr[\text{error}|s_2 \text{ sent}]\Pr[s_2 \text{ sent}]$$

  - If either signal is equally likely:

$$P_e = 0.5\cdot\Pr[\text{error}|s_1 \text{ sent}] + 0.5\cdot\Pr[\text{error}|s_2 \text{ sent}]$$

Problem Statement

- Our goal is to
  - Find an expression for probability of error
  - Choose a value of V_T to minimize error probability
- We already solved the first part of this problem earlier in the semester, when we studied the application of random variables to communications systems

PDF of decision statistic

- The noise $n(t)$ is a Gaussian random process
- After filtering, $n_0(t)$ is a Gaussian random process
- After sampling, n_0 is a Gaussian random variable
  - Mean of n_0 : $E[n_0] = 0$
  - Variance of n_0 : $E[n_0^2] = \overline{n_0^2(t)} = \sigma_0^2$
- Since the signal is deterministic, s_{01} and s_{02} are constants:

$$E[r_0|m=1] = s_{01}$$

$$E[r_0|m=0] = s_{02}$$

PDF of decision statistic

- The decision statistic given s_0 was sent:

$$r_0 = s_{01} + n_0$$

- The decision statistic given s_1 was sent:

$$r_0 = s_{02} + n_0$$

- In either case, r_0 is a Gaussian random variable with mean = s_{01} or s_{02} and standard deviation σ_0

PDF of Test Statistic (continued)

- If m=1 (s_{01} is sent):

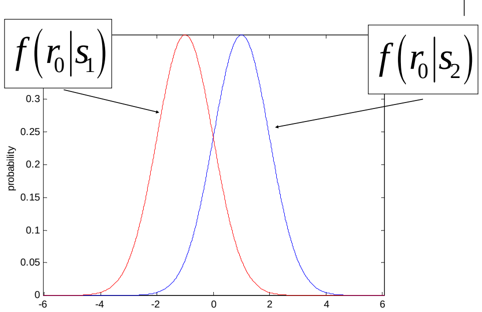
$$f\left(r_0 \middle| s_1\right) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{01})^2}{2\sigma_0^2}}$$

- If m=0 (s_{02} is sent)

$$f\left(r_0 \middle| s_2\right) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{02})^2}{2\sigma_0^2}}$$

Conditional probability distributions

Example 27.1: s_{01} = -1, s_{02} = +1



Probability of Error

- Assuming equally likely signaling (i.e., either transmit signal is equally likely)

$$P_e = \frac{1}{2}\int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{01})^2}{2\sigma_0^2}} dr_0$$
$$+\frac{1}{2}\int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{02})^2}{2\sigma_0^2}} dr_0$$

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Change of variables

- Let $\lambda = -(r_0 - s_{01})/\sigma_0$ in first integral
- Let $\lambda = (r_0 - s_{02})/\sigma_0$ in second integral
- We now have:

$$P_e = \frac{1}{2}\int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda + \frac{1}{2}\int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda$$

- Or $P_e = \frac{1}{2}Q\left(\frac{-V_T + s_{01}}{\sigma_0}\right) + \frac{1}{2}Q\left(\frac{V_T - s_{02}}{\sigma_0}\right)$

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Optimal (i.e., minimum BER) Choice for Threshold

$$P_e = \frac{1}{2}\int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda + \frac{1}{2}\int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda$$

$$\frac{dP_e}{dV_T} = \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}} = 0$$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}}$$

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Optimal Choice for Threshold (continued)

$$e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} = e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}}$$

$$(V_T - s_{01})^2 = (V_T - s_{02})^2$$

$$(V_T^2 - 2V_T s_{01} + s_{01}^2) = (V_T^2 - 2V_T s_{02} + s_{02}^2)$$

$$2V_T (s_{02} - s_{01}) = s_{02}^2 - s_{01}^2$$

$$\boxed{V_T = \frac{s_{01} + s_{02}}{2}}$$

General Result for Binary Signaling

- If we let

$$V_T = \frac{s_{01} + s_{02}}{2}$$

- Then we have:

$$P_e = Q\left(\frac{s_{01} - s_{02}}{2\sigma_0}\right) = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right)$$

- We will interpret this result next time

Summary

- Today we began looking at the performance of binary signaling in AWGN
- The optimal threshold (in terms of minimizing BER) is midpoint between the two expected outputs (provided that the symbols are equally likely)
- The error probability can be written using the *Q*-function.
  - The argument of the *Q*-function is the ratio of the signal to noise power at the output of the low pass filter