

ECE4634

Digital Communications

Fall 2007

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Lecture #27: Performance of
Binary Signaling in Noise



Analog and Digital Communications



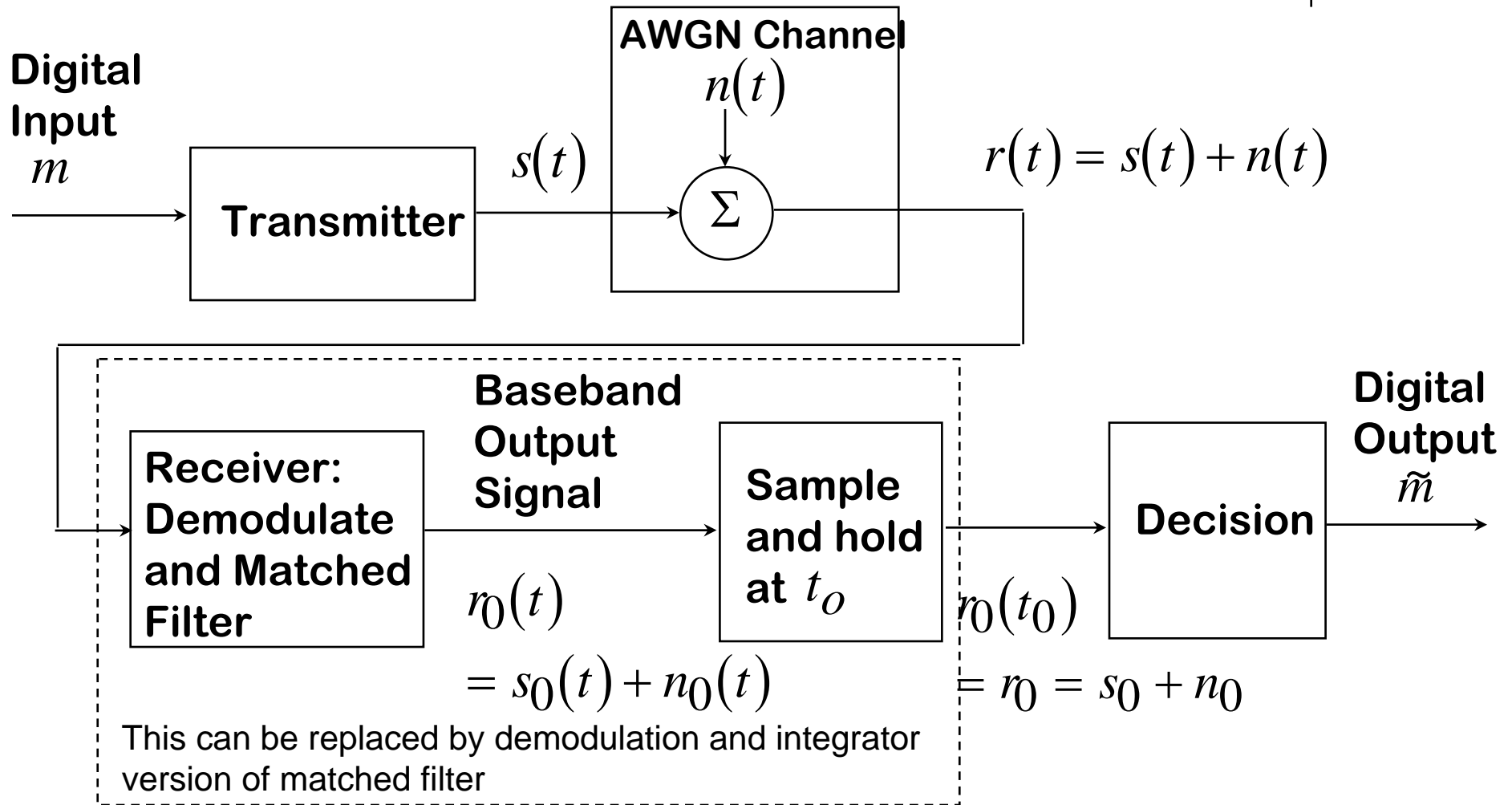
Introduction

- We have now discussed receiver structures for various modulation schemes.
- Our focus has been on demodulation and filtering
- We now focus on the final decision (called detection) and BER
- We will start with binary modulation schemes and move to M -ary modulation
- What to read – Sections 10.2-10.3

Block Diagram for Binary Communications System



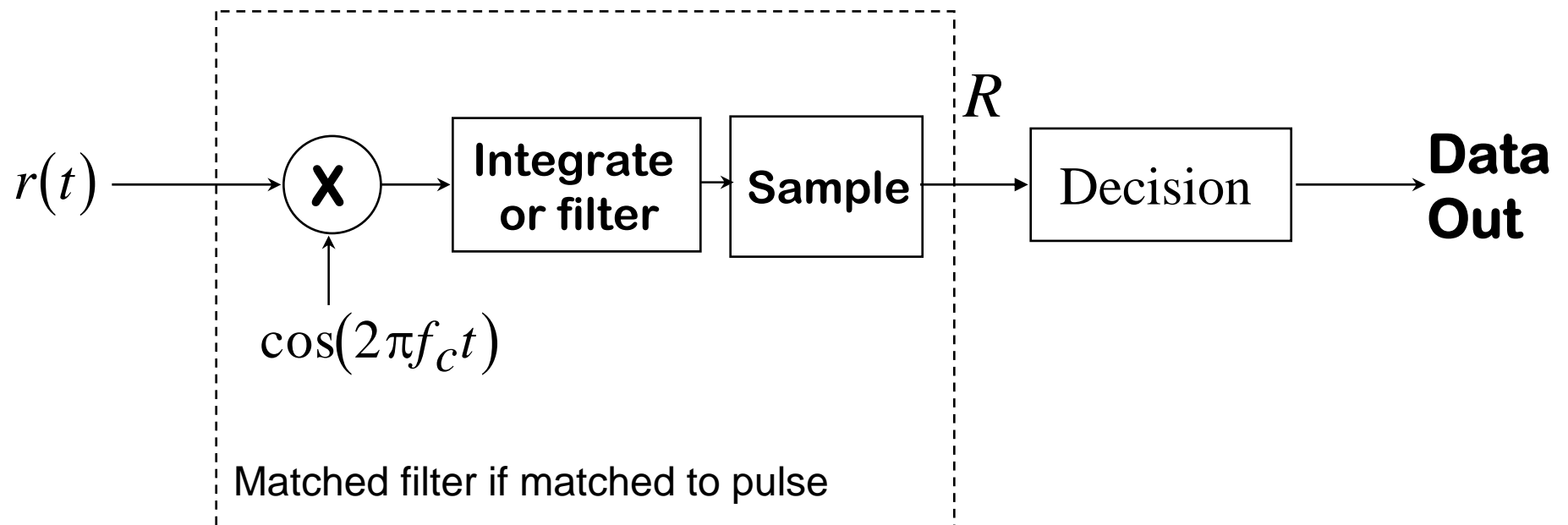
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Receiver for BPSK



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Binary Communication System Model: Transmitter



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- Binary signal generated by transmitter

$$s(t) = \begin{cases} s_1(t)|_0^T, & m = 1 \\ s_2(t)|_0^T, & m = 0 \end{cases}$$

- Where $s_1(t)$ and $s_2(t)$ might have different:
 - phases
 - amplitudes
 - frequencies

Binary Communications System Model: Received Signal



- Received Signal

$$r(t) = s(t) + n(t)$$

- AWGN:

- $P_n(f) = N_0/2$ where N_0 is a constant
- Additive White Gaussian Noise (AWGN)
- Equal noise power at all frequencies
- Samples uncorrelated in time domain:

$$R_n(\tau) = \frac{N_0}{2} \delta(\tau)$$

- Good approximation to receiver noise in many cases

Binary Communications System

Model: Baseband Received Signal



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- After demodulation and (matched) filtering we have a baseband signal:

$$r_0(t) = s_0(t) + n_0(t)$$

- Baseband noise signal:

$$n_0(t)$$

- Baseband signal

$$s_0(t) = \begin{cases} s_{01}(t), & m = 1 \\ s_{02}(t), & m = 0 \end{cases}$$

Binary Communications System Model: Sampled Received Signal



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- After sampling, we have the ‘decision variable’ or ‘decision statistic’ used for decisions

$$r_0 = s_0 + n_0$$

- Sampled noise:

$$n_0$$

- Sampled signal:

- $$s_0 = \begin{cases} s_{01}, & m = 1 \\ s_{02}, & m = 0 \end{cases}$$

Receiver Decision



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- Let V_T be a threshold voltage for deciding m
 - Usually, $V_T = \frac{s_{01} + s_{02}}{2}$ for equally likely signals

- We make a decision according to the rule

- $$\hat{m} = \begin{cases} 1, & r_0 \geq V_T \\ 0, & r_0 < V_T \end{cases}$$

Later we will generalize the threshold to decision regions for M -ary modulation

Probability of Error



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- We are interested in the probability of making a wrong decision. Let:

$$\Pr[\text{error}|s_1 \text{ sent}] = \int_{-\infty}^{V_T} f(r_0|s_1)dr_0$$

- $$\Pr[\text{error}|s_2 \text{ sent}] = \int_{\infty}^{V_T} f(r_0|s_2)dr_0$$

$f(r_o|s_i)$ =
conditional pdf
of r_o given that
 s_i was sent.

- Average probability of error

$$P_e = \Pr[\text{error}|s_1 \text{ sent}]\Pr[s_1 \text{ sent}] + \Pr[\text{error}|s_2 \text{ sent}]\Pr[s_2 \text{ sent}]$$

- If either signal is equally likely:

$$P_e = 0.5 \cdot \Pr[\text{error}|s_1 \text{ sent}] + 0.5 \cdot \Pr[\text{error}|s_2 \text{ sent}]$$

Problem Statement



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- Our goal is to
 - Find an expression for probability of error
 - Choose a value of V_T to minimize error probability
- We already solved the first part of this problem earlier in the semester, when we studied the application of random variables to communications systems

PDF of decision statistic



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- The noise $n(t)$ is a Gaussian random process
- After filtering, $n_0(t)$ is a Gaussian random process
- After sampling, n_0 is a Gaussian random variable
 - Mean of n_0 : $E[n_0] = 0$
 - Variance of n_0 : $E[n_0^2] = \overline{n_0^2(t)} = \sigma_0^2$
- Since the signal is deterministic, s_{01} and s_{02} are constants:

$$E[r_0|m = 1] = s_{01}$$

$$E[r_0|m = 0] = s_{02}$$



PDF of decision statistic

- The decision statistic given s_0 was sent:

$$r_0 = s_{01} + n_0$$

- The decision statistic given s_1 was sent:

$$r_0 = s_{02} + n_0$$

- In either case, r_0 is a Gaussian random variable with mean = s_{01} or s_{02} and standard deviation σ_0

PDF of Test Statistic (continued)



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- If $m=1$ (s_{01} is sent):

$$f(r_0|s_1) = \frac{1}{\sqrt{2\pi\sigma_0}} e^{-\frac{(r_0 - s_{01})^2}{2\sigma_0^2}}$$

- If $m=0$ (s_{02} is sent)

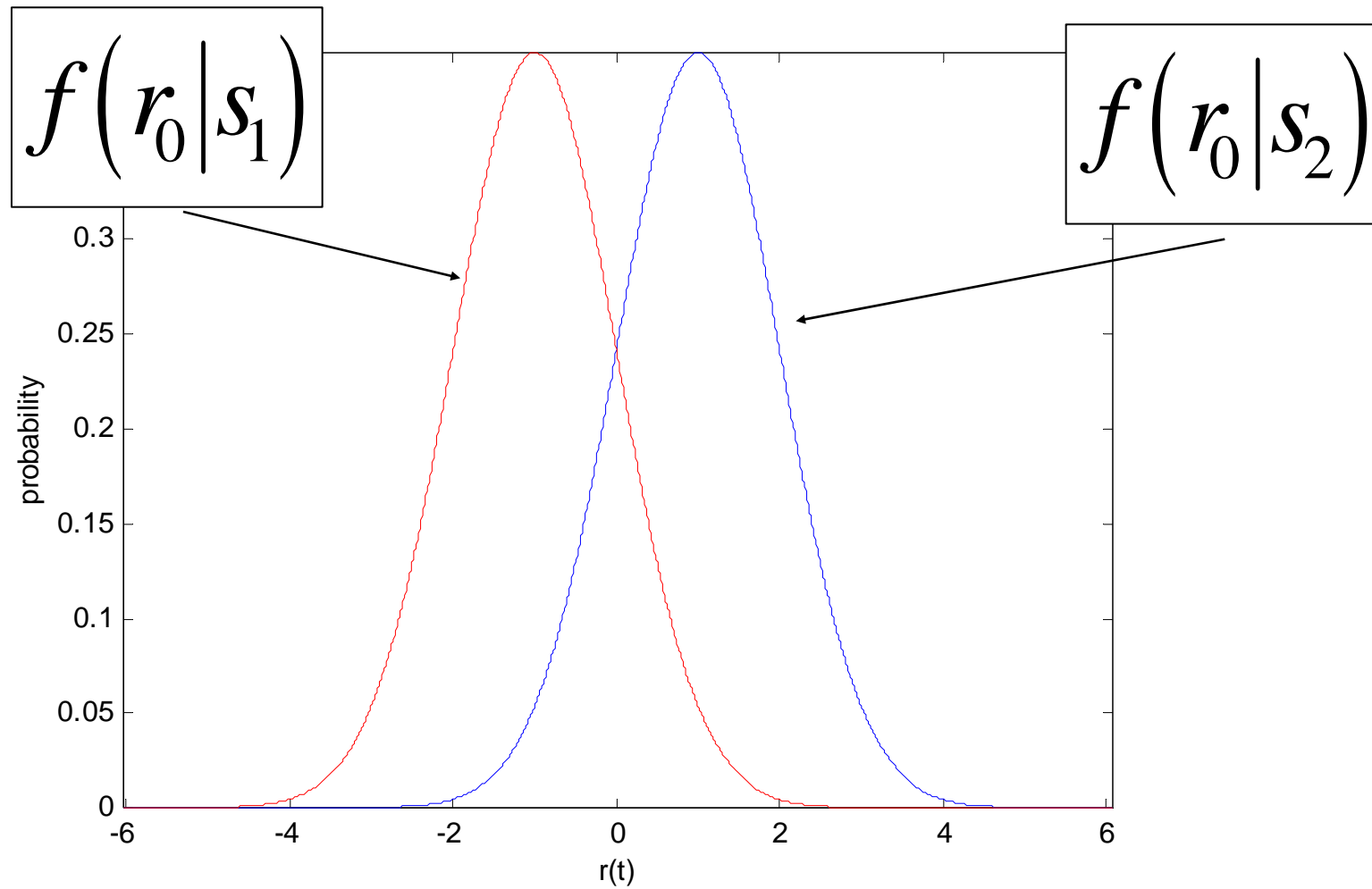
$$f(r_0|s_2) = \frac{1}{\sqrt{2\pi\sigma_0}} e^{-\frac{(r_0 - s_{02})^2}{2\sigma_0^2}}$$

*Conditional
probability
distributions*

Example 27.1: $s_{01} = -1$, $s_{02} = +1$



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Probability of Error



- Assuming equally likely signaling (i.e., either transmit signal is equally likely)

$$P_e = \frac{1}{2} \int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{01})^2}{2\sigma_0^2}} dr_0 + \frac{1}{2} \int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(r_0 - s_{02})^2}{2\sigma_0^2}} dr_0$$

Change of variables



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- Let $\lambda = -(r_0 - s_{01})/\sigma_0$ in first integral
- Let $\lambda = (r_0 - s_{02})/\sigma_0$ in second integral
- We now have:

$$P_e = \frac{1}{2} \int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda + \frac{1}{2} \int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda$$

- Or $P_e = \frac{1}{2} Q\left(\frac{-V_T + s_{01}}{\sigma_0}\right) + \frac{1}{2} Q\left(\frac{V_T - s_{02}}{\sigma_0}\right)$

Optimal (i.e., minimum BER) Choice for Threshold



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$$P_e = \frac{1}{2} \int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda + \frac{1}{2} \int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda$$

$$\frac{dP_e}{dV_T} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}} = 0$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}}$$

Optimal Choice for Threshold (continued)



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$$e^{-\frac{(V_T - s_{01})^2}{2\sigma_0^2}} = e^{-\frac{(V_T - s_{02})^2}{2\sigma_0^2}}$$

$$(V_T - s_{01})^2 = (V_T - s_{02})^2$$

$$(V_T^2 - 2V_T s_{01} + s_{01}^2) = (V_T^2 - 2V_T s_{02} + s_{02}^2)$$

$$2V_T (s_{02} - s_{01}) = s_{02}^2 - s_{01}^2$$

$$V_T = \frac{s_{01} + s_{02}}{2}$$

General Result for Binary Signaling



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- If we let

$$V_T = \frac{s_{01} + s_{02}}{2}$$

- Then we have:

$$P_e = Q\left(\frac{s_{01} - s_{02}}{2\sigma_0}\right) = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right)$$

- We will interpret this result next time



Summary

- Today we began looking at the performance of binary signaling in AWGN
- The optimal threshold (in terms of minimizing BER) is midpoint between the two expected outputs (provided that the symbols are equally likely)
- The error probability can be written using the Q-function.
 - The argument of the Q-function is the ratio of the signal to noise power at the output of the low pass filter