ECE4634
Digital Communications
Fall 2007

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Lecture #28: Calculation of BERs for Binary Signaling – Part II

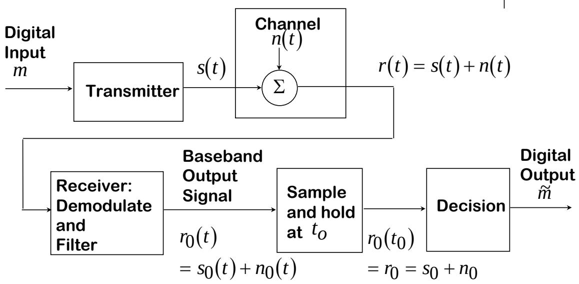
Overview

- In this lecture we continue our discussion of the BER of binary signaling in AWGN
- Specifically, we will take the result from the previous lecture and determine the impact of using a matched filter receiver.
- We will find that the BER for the matched filter is related to the average energy devoted per bit and the noise power spectral density
- What to read – 10.2 – 10.4

Bandwidth and Energy Efficiency

- Previously we have discussed is the amount of bandwidth that would be required to send a given signal
  - A system which sends a lot of data in a small bandwidth is said to have good *Bandwidth Efficiency*
  - Typically higher order (*M*-ary) modulation results in higher bandwidth efficiency
- Now we consider how the communication system performs in noise
  - A system which makes few errors when the received Signal-to-Noise Ratio (SNR) is small is said to have good *Energy Efficiency*
- It is possible to trade bandwidth efficiency for energy efficiency in system design

Block Diagram for Binary Communications System



Probability of Error

$$P_e = \Pr\left[\text{error}|s_1 \text{ sent}\right]\Pr\left[s_1 \text{ sent}\right] + \Pr\left[\text{error}|s_2 \text{ sent}\right]\Pr\left[s_2 \text{ sent}\right]$$

$$P_e = \frac{1}{2}\int_{-(V_T - s_{01})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda + \frac{1}{2}\int_{(V_T - s_{02})/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda$$

- Or

$$P_e = \frac{1}{2} Q\left(\frac{-V_T + s_{01}}{\sigma_0}\right) + \frac{1}{2} Q\left(\frac{V_T - s_{02}}{\sigma_0}\right)$$

General Result for Binary Signaling

- We found the optimal threshold to be

$$V_T = \frac{s_{01} + s_{02}}{2}$$

- Thus, we have:

$$P_e = Q\left(\frac{s_{01} - s_{02}}{2\sigma_0}\right) = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right)$$

Interpretation of the result

- The term $(s_{01} - s_{02})^2$ corresponds to signal power
  - Increasing signal power decreases bit error rate
- The term σ_0^2 represents noise power
  - Increasing noise power increases bit error rate
- The argument of the *Q*-function may be thought of as a signal to noise ratio.
  - Usually this is expressed in terms of E_b/N_o, where E_b is the average energy per bit of transmitted data.
- The *Q*-function decreases very rapidly as its argument increases.

Matched Filters

- To this point in our BER calculations we have not made any assumptions about the low-pass filter used
- We know that *matched filters* provide the largest SNR, so we would like to examine the performance when using matched filter receivers.
- From our previous work we know that for a matched filter:

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{P_n(f)} df$$

- Now, for white noise: $\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_o/2} df$

Matched Filters (cont.)

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_o/2} df$$

$$= \frac{2}{N_o} \int_{-\infty}^{\infty} |S(f)|^2 df$$

$$= \frac{2}{N_o} \int_{-\infty}^{\infty} s^2(t) dt$$

$$= \frac{2E_s}{N_o}$$

→ SNR out of Matched filter

Probability of Error for Matched Filter

- Now, returning to the probability of error expression:

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right)$$

- We can write the term

$$(s_{01}-s_{02}) = (s_{01}(t_o) - s_{02}(t_o)) = s_{do}(t_o)$$

where $s_{do}(t_o)$ is the filter response to the difference signal $s_d(t) = s_1(t) - s_2(t)$

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Probability of Error for Matched Filters (cont.)

- The BER can be minimized by maximizing the argument within the Q-function. This is identical to the matched filter problem. Specifically, if the filter is matched to $s_d(t)$, the SNR at the filter output is :

$$\frac{(s_{do}(t_o))^2}{\sigma_0^2} = \frac{2E_d}{N_o}$$

where E_d is the energy of the difference signal:

$$E_d = \int_0^T s_d^2(t)dt = \int_0^T (s_1(t) - s_2(t))^2\, dt$$

- s_{01} is the filter response to $s_1(t)$
- s_{02} is the filter response to $s_2(t)$
- s_{01}-s_{02} is the filter response to $s_1(t)$ - $s_2(t)$ since filters are linear
- E_d is termed the *difference energy*

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Probability of Error for Matched Filters (cont.)

- Now, the matched filter SNR is:

$$\left(\frac{S}{N}\right)_{out} = \frac{[s_{do}(t_o)]^2}{\sigma_o^2} = \frac{[s_{01}-s_{02}]^2}{\sigma_o^2} = \frac{2E_d}{N_o}$$

- Now, since the BER is

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right)$$

we have

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right)$$

← Probability of Error for matched filter

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Procedure for Deriving Error Probability of Binary Systems

- For a specific signal set and receiver:
  - Find s_{01}, the response of the receiver to signal s_1
  - Find s_{02}, the response of the receiver to signal s_2
  - Express $(s_{01} - s_{02})^2$ in terms of E_b
  - Find σ_0^2, the noise power at the receiver
  - Express σ_0^2 in terms of noise power spectral density N_o
- For matched filter, we simply need to find the difference energy E_d.
- Note: Pulse shaping does not effect the P_e calculation (if we assume no ISI) for the **matched filter** - but it does change the correlator or matched filter implementation
  - We will assume rectangular pulse shapes (this means that *correlator* is equal to an *integrator)*

Example: Baseband Unipolar Signaling

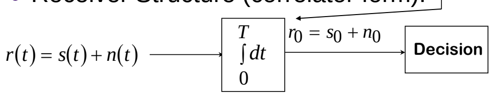
- Transmitted Signals:

$$s_1(t) = A\Big|_0^T$$

$$s_2(t) = 0\Big|_0^T$$

Note: square pulses at transmitter, thus square pulses at correlator

- Receiver Structure (correlator form):



Example: Baseband Unipolar Signaling (continued)

- Now we can define the difference signal: $s_d(t) = s_1(t) - s_2(t) = A\Big|_0^T$
- Thus,

$$E_d = \int_0^T s_d^2(t)dt = \int_0^T (A)^2\, dt = A^2 T$$

- and

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{A^2 T}{2N_o}}\right)$$

Example: Baseband Unipolar Signaling (continued)

- Now we would like to represent the error rate in terms of the average energy per bit. Since this is a binary modulation scheme the average energy per bit is same as the average *energy per symbol*. The average energy per *bit* is:

$$E_b = \frac{1}{2}\int_0^T s_1^2(t)dt + \frac{1}{2}\int_0^T s_2^2(t)dt = \frac{A^2T}{2}$$

- Thus, $P_e = Q\left(\sqrt{\frac{A^2T}{2N_o}}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$

Example: Baseband Polar Signaling

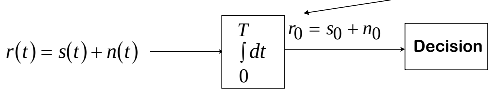
- Transmitted Signals:

$$s_1(t) = A|_0^T$$

$$s_2(t) = -A|_0^T$$

Note: square pulses at transmitter, thus square pulses at correlator

- Receiver Structure (correlator form):



Example: Baseband polar Signaling (continued)

- Now we can define the difference signal: $s_d(t) = s_1(t) - s_2(t) = 2A|_0^T$
- Thus,

$$E_d = \int_0^T s_d^2(t)dt = \int_0^T (2A)^2\, dt = 4A^2T$$

- and $P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_o}}\right)$

Example: Baseband polar Signaling (continued)

- Now we would like to represent the error rate in terms of the average energy per bit. The average energy per *bit* is:

$$E_b = \frac{1}{2}\int_0^T s_1^2(t)dt + \frac{1}{2}\int_0^T s_2^2(t)dt = A^2T$$

- Thus, $P_e = Q\left(\sqrt{\frac{2A^2T}{N_o}}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$

Note that the E_b/N_o required to achieve the same probability of error is 1/2 that required of unipolar signaling. This is a 3dB advantage in *energy efficiency*.

Example: BPSK

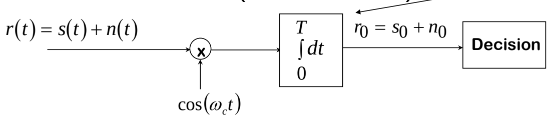
- Transmitted Signals:

$$s_1(t) = A\cos(\omega_c t)\Big|_0^T$$

$$s_2(t) = -A\cos(\omega_c t)\Big|_0^T$$

Note: square pulses at transmitter, thus square pulses at correlator (matched filter)

- Receiver Structure (correlator form):



Example: BPSK (continued)

- Now we can define the difference signal $s_d(t) = s_1(t) - s_2(t)$

$$= 2A\cos(2\pi f_c t)\Big|_0^T$$

- Thus,

$$E_d = \int_0^T s_d^2(t)dt$$

$$= \int_0^T \left(2A\cos(2\pi f_c t)\right)^2 dt$$

$$= 2A^2T$$

- and

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{A^2T}{N_o}}\right)$$

Example: BPSK (continued)

- Now we would like to represent the error rate in terms of the average energy per bit. The average energy per *bit* is:

$$E_b = \frac{1}{2}\int_0^T s_1^2(t)\,dt + \frac{1}{2}\int_0^T s_2^2(t)\,dt = \frac{A^2T}{2}$$

- Thus, $P_e = Q\left(\sqrt{\frac{A^2T}{N_o}}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$

Note that the performance is the same as baseband bipolar NRZ. This makes sense since the complex baseband equivalent of BPSK is bipolar NRZ and we can analyze systems in complex baseband.

Most Important Results

- Polar Signaling
  - Baseband Polar NRZ, BPSK

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

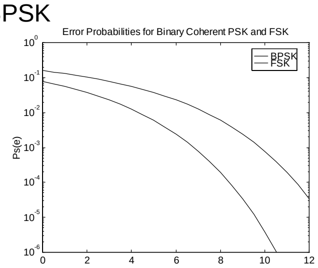
- Unipolar Signaling
  - Baseband Unipolar NRZ, Coherent ASK, Coherent FSK

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

  - In general, Polar Signaling has a 3 dB advantage over Unipolar

BER Curves for BPSK and FSK

- ASK & FSK are approximately 3dB worse than BPSK



Summary

- Today we have examined the probability of error for the matched filter with binary signaling.
- The performance is written in terms of the energy devoted *per bit* (regardless of bit rate) and the noise power spectral density
- In the coming lectures we will examine the performance of *M-ary signaling*