ECE4634
Digital Communications
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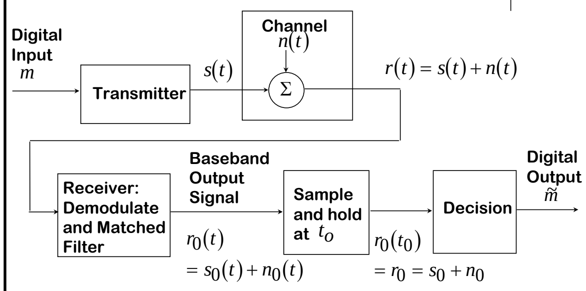
Instructor: R. Michael Buehrer

Lecture #29: BERs for Binary Signaling – Part III

Overview

- The last two lectures we have shown examined the probability of bit error (i.e., BER) for binary signaling
- We started by determining the BER for an arbitrary filter
- Last class we derived the probability of bit error when a matched filter is used
- Today we will look one more time at BER and relate it to our signal space diagram
- Specifically we will show that the BER can be directly related to the distance between point in our signal space. This will be very useful for *M*-ary signaling

Block Diagram for Binary Communications System



General Result for Probability of Error for Binary Signaling

- Assuming either signal is equally likely:

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right)$$

  - s_{01} is the response of the receiver to $s_1(t)$ as input
  - s_{02} is the response of the receiver to $s_2(t)$ as input
  - σ_0^2 is the noise power in the receiver
- The error probability is a function of the "signal to noise ratio" at the receiver

Procedure for Deriving Error Probability of Binary Systems

- For a specific signal set and receiver:
  - Find s_{01}, the response of the receiver to signal s_1
  - Find s_{02}, the response of the receiver to signal s_2
  - Express $(s_{01}-s_{02})^2$ in terms of E_b
  - Find σ_0^2, the noise power at the receiver
  - Express σ_0^2 in terms of noise power spectral density N_o
- For matched filter, we simply need to find the difference energy E_d.
- Note: Pulse shaping does not effect the P_e calculation (if we assume no ISI) for the matched filter - but it does change the correlator or matched filter implementation
  - We will assume rectangular pulse shapes (this means that *correlator* is equal to an *integrator)*

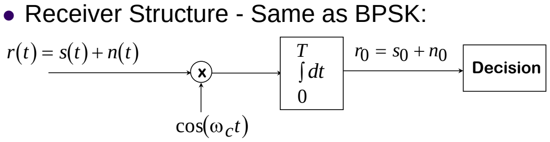
Example: BASK

- Transmitted Signals:

$$s_1(t) = A\cos(\omega_c t)\Big|_0^T$$

$$s_2(t) = 0\Big|_0^T$$

- Receiver Structure - Same as BPSK:



Example: BASK (continued)

- Now we can define the difference signal: $s_d(t) = s_1(t) - s_2(t)$
- Thus,

$$E_d = \int_0^T s_d^2(t)\,dt \qquad = A\cos(2\pi f_c t)\Big|_0^T$$

$$= \int_0^T \left(A\cos(2\pi f_c t)\right)^2 dt$$

$$= \frac{A^2 T}{2}$$

- and

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{A^2T}{4N_o}}\right)$$

Example: BASK (continued)

- Now we would like to represent the error rate in terms of the average energy per bit. The average energy per *bit* is:

$$E_b = \frac{1}{2}\int_0^T s_1^2(t)\,dt + \frac{1}{2}\int_0^T s_2^2(t)\,dt = \frac{A^2T}{4}$$

- Thus,

$$P_e = Q\left(\sqrt{\frac{A^2T}{4N_o}}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

Note that the performance is the same as baseband unipolar NRZ. This makes sense since the complex baseband equivalent of BASK is unipolar NRZ and we can analyze systems in complex baseband.

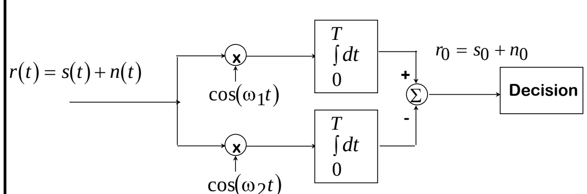
Example: BFSK

- Transmitted Signals:

$$s_1(t) = A\cos(\omega_1 t)\Big|_0^T$$

$$s_2(t) = A\cos(\omega_2 t)\Big|_0^T$$

- Receiver Structure - Same as BASK for two frequencies:



Example: BFSK (continued)

- Now we can define the difference signal:

$$s_d(t) = s_1(t) - s_2(t) = A\cos(\omega_1 t) - A\cos(\omega_2 t)\Big|_0^T$$

- Thus,

$$E_d = \int_0^T s_d^2(t)dt = \int_0^T \left(A\cos(\omega_1 t) - A\cos(\omega_2 t)\right)^2 dt = A^2 T$$

- and

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{A^2 T}{2N_o}}\right)$$

Probability of Error for BFSK

$$E_b = \frac{1}{2}\int_0^T s_1^2(t)dt + \frac{1}{2}\int_0^T s_2^2(t)dt = \frac{A^2 T}{2}$$

- Result:

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right) = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Note that this is the same as *Unipolar* signaling! This should make sense since BFSK uses Unipolar signaling to modulate two carriers.

BER Calculations using Distance Properties

- We have seen that the probability of bit error (with equally likely symbols) when using a matched filter can be written as

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_o}}\right)$$

where E_d is the *difference energy*

- We can interpret this result in terms of the distance between the symbols in a signal space $d_{i,j}$ representation as

$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

Proof

- Recall that for any signal made up of K basis functions:

$$s_i(t) = \sum_{k=1}^{K} c_{ik} f_k(t)$$

- Thus,

$$\begin{aligned}
E_d &= \int_0^T \left(s_1(t) - s_2(t)\right)^2 dt \\
&= \int_0^T \left(\sum_{i=1}^{K} c_{1i} f_i(t) - \sum_{i=1}^{K} c_{2i} f_i(t) \right)^2 dt \\
&= \int_0^T \left(\left[\sum_{i=1}^{K} c_{1i} f_i(t) \right] \left[\sum_{i=1}^{K} c_{1i} f_i(t) \right] - 2 \left[\sum_{i=1}^{K} c_{1i} f_i(t) \right] \left[\sum_{i=1}^{K} c_{2i} f_i(t) \right] + \left[\sum_{i=1}^{K} c_{2i} f_i(t) \right] \left[\sum_{i=1}^{K} c_{2i} f_i(t) \right] \right) dt
\end{aligned}$$

Proof (cont.)

$$\begin{aligned}
E_d &= \int_0^T \left(\left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{1i} c_{1j} f_i(t) f_j(t) \right] - 2 \left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{1i} c_{2j} f_i(t) f_j(t) \right] + \left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{2i} c_{2j} f_i(t) f_j(t) \right] \right) dt \\
&= \int_0^T \left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{1i} c_{1j} f_i(t) f_j(t) \right] dt - 2 \int_0^T \left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{1i} c_{2j} f_i(t) f_j(t) \right] dt + \int_0^T \left[\sum_{i=1}^{K} \sum_{j=1}^{K} c_{2i} c_{2j} f_i(t) f_j(t) \right] dt \\
&= \sum_{i=1}^{K} c_{1i}^2 - 2 \sum_{i=1}^{K} c_{1i} c_{2i} + \sum_{i=1}^{K} c_{2i}^2 \\
&= \sum_{i=1}^{K} \left(c_{1i} - c_{2i} \right)^2 \\
&= d_{12}^2
\end{aligned}$$

- Where we have used

$$\int_0^T f_i(t) f_j(t) dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

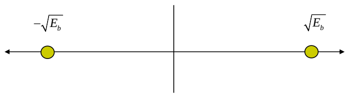
Proof (cont.)

- Thus,

$$\begin{aligned}
P_e &= Q\left(\sqrt{\frac{E_d}{2N_o}} \right) \\
&= Q\left(\sqrt{\frac{d_{i,j}^2}{2N_o}} \right) \\
&= Q\left(\frac{d_{i,j}}{\sqrt{2N_0}} \right)
\end{aligned}$$

Example: BPSK

- BPSK

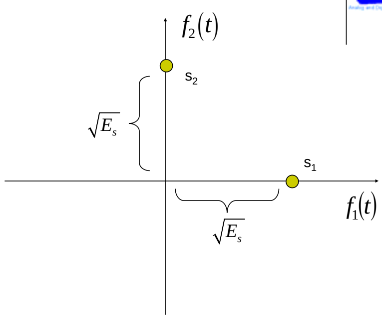


- The Euclidean distance between the points s_1 and s_2 is $2\sqrt{E_b}$
- In terms of E_b:

$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right) = Q\left(\frac{2\sqrt{E_b}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

Ex: BFSK

$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_o}}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$



Example: BASK

- We have solved this once, but let's use the distance properties

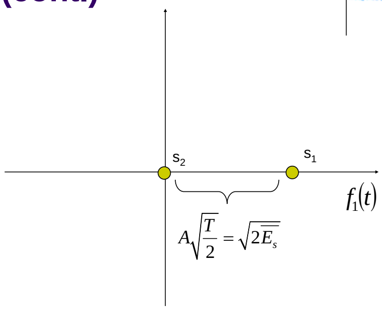
$$s_1(t) = A\cos(\omega_c t)\Big|_0^T$$
$$s_2(t) = 0\Big|_0^T$$

- A single basis function $f_1(t) = \sqrt{\frac{2}{T}}\cos(\omega_c t)\Big|_0^T$
- The points in signal space

$$s_1 = A\sqrt{\frac{T}{2}}$$
$$s_2 = 0$$

Example (cont.)

$$\overline{E_s} = \frac{1}{2}\left(0 + \frac{A^2T}{2}\right) = \frac{A^2T}{4}$$



Example (cont.)

- The probability of error is then

$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_o}}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

General Binary Modulation

- For any arbitrary binary modulation scheme

$$\mathbf{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

- Arbitrarily choosing

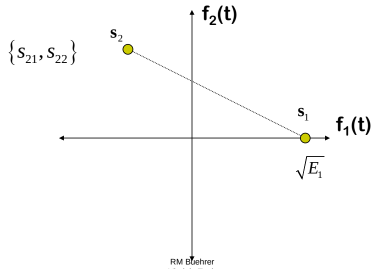
$$f_1(t) = \frac{s_1(t)}{\sqrt{E_b}} \quad 0 \le t \le T$$

- We have

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

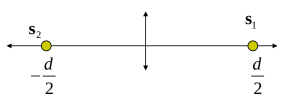
Constellation Diagram

- The resulting signal space diagram



Constellation Diagram

- Rotating to a single axis :



$$\begin{aligned} d &= \sqrt{\left(\sqrt{E_1} - s_{21}\right)^2 + \left(0 - s_{22}\right)^2} \\ &= \sqrt{E_1 - 2\sqrt{E_1}s_{21} + s_{21}{}^2 + s_{22}{}^2} \\ &= \sqrt{E_1 - 2\sqrt{E_1}s_{21} + E_2} \end{aligned}$$

Probability of Error

- The resulting probability of error is then

$$\begin{aligned} P_s(e) &= Q\left(\sqrt{\frac{2(d/2)^2}{N_o}}\right) \\ &= Q\left(\sqrt{\frac{\left(E_1 - 2\sqrt{E_1}s_{21} + E_2\right)}{2N_o}}\right) \end{aligned}$$

- If the symbols have equal energy

$$\begin{aligned} P_s(e) &= Q\left(\sqrt{\frac{\left(E_b - \sqrt{E_b}s_{21}\right)}{N_o}}\right) \\ &= Q\left(\sqrt{\frac{E_b(1-\rho_{21})}{N_o}}\right) \end{aligned}$$

$$\rho_{12} = \frac{1}{\sqrt{E_1E_2}}\int_0^T s_1(t)s_2(t)dt$$

Examples

- BPSK → $\rho_{12} = -1$

$$P_s(e) = Q\left(\sqrt{\frac{E_b\left(1-(-1)\right)}{N_o}}\right)$$
$$= Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

Thus, for binary modulation, BPSK provides the best possible performance

- BFSK, BASK → $\rho_{12} = 0$

$$P_s(e) = Q\left(\sqrt{\frac{E_b\left(1-0\right)}{N_o}}\right)$$
$$= Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

Most Important Results

- Polar Signaling
  - Baseband Polar NRZ, BPSK

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Unipolar Signaling
  - Baseband Unipolar NRZ, Coherent ASK, Coherent FSK

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

  - In general, Polar Signaling has a 3 dB advantage over Unipolar

BER Curves for BPSK, ASK and FSK

- ASK & FSK are approximately 3dB worse than BPSK