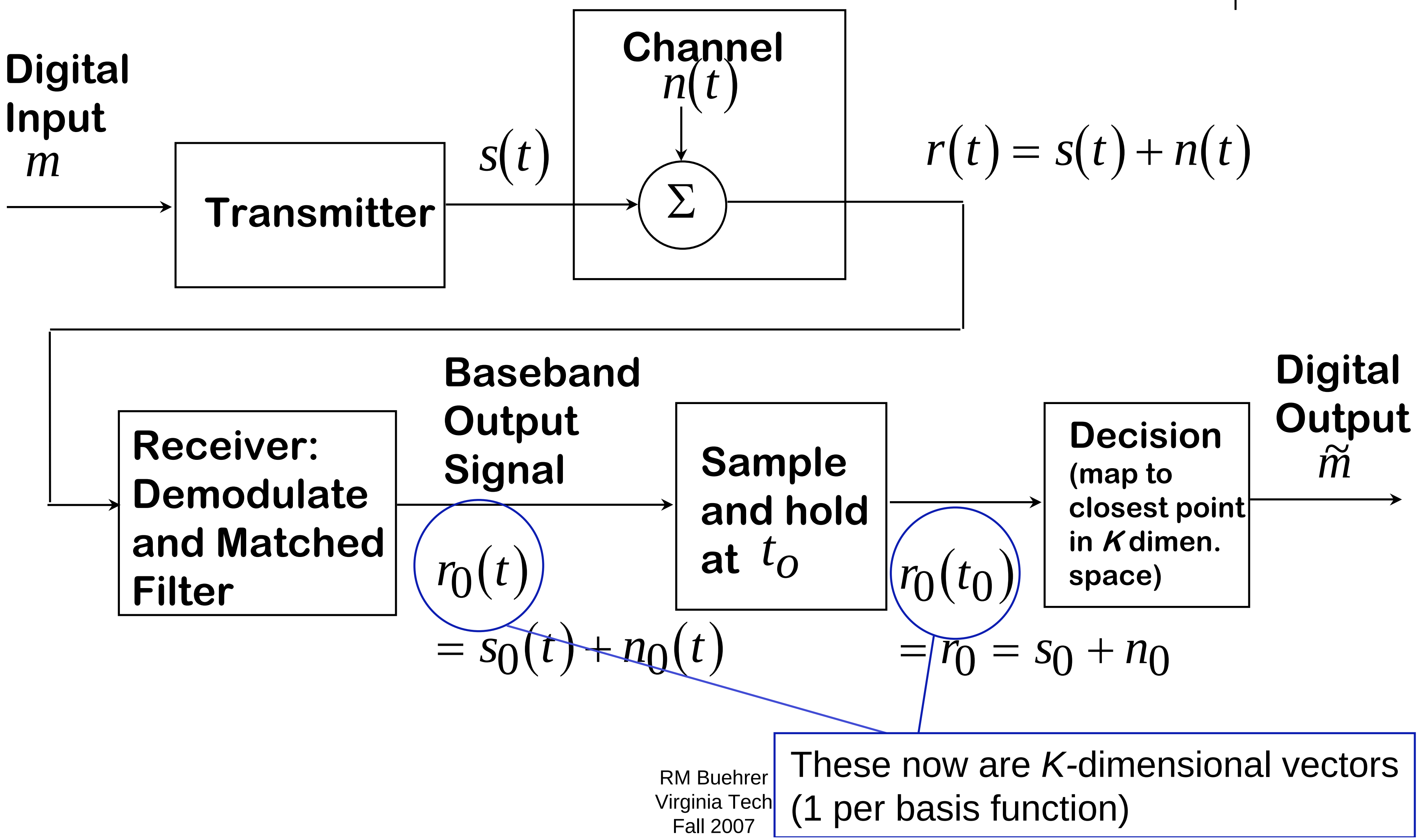
ECE4634 Digital Communications Fall 2007

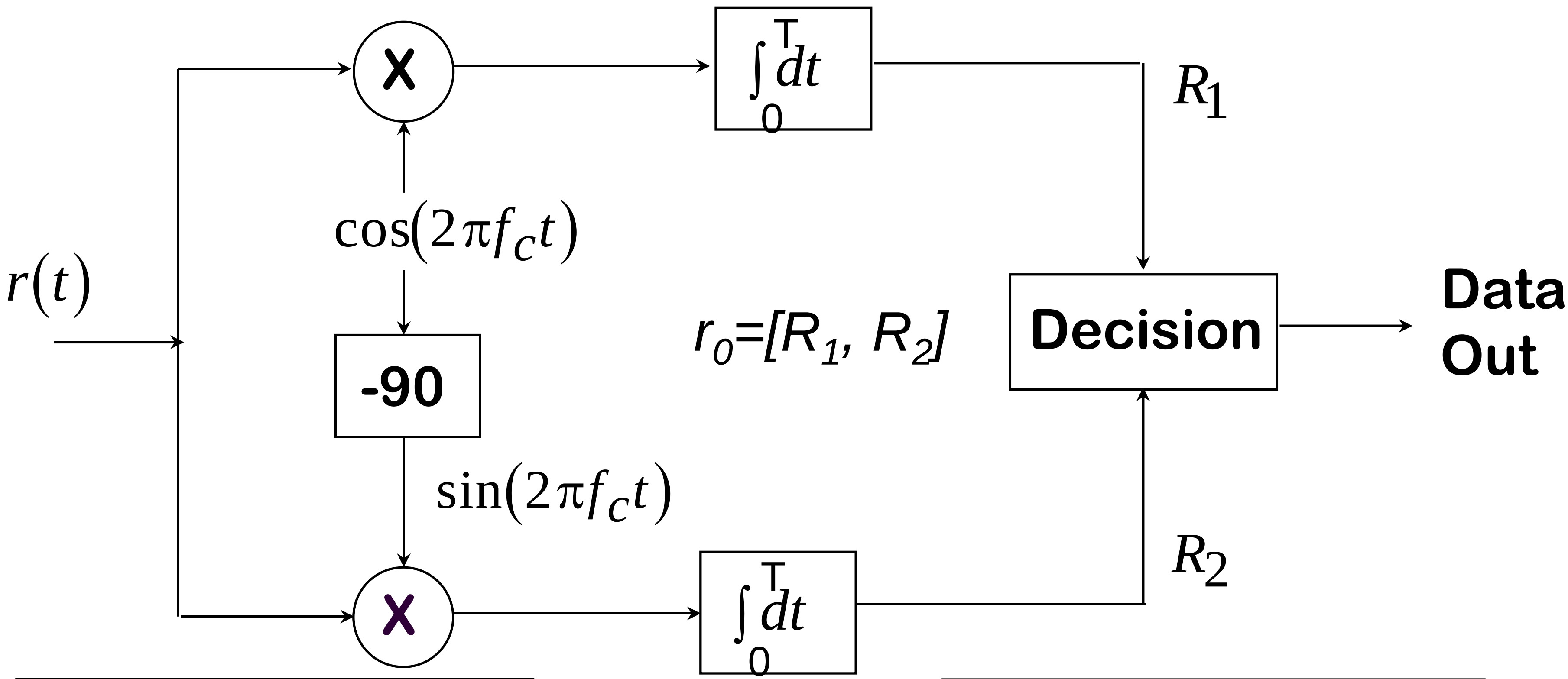
Instructor: R. Michael Buehrer

Lecture #30: BERs for *M*-ary Signaling

Block Diagram for *M*-ary Communications System



Ex: Receiver for *M*-ary PSK



Note: r_0 is a two-dim. vector. In the PSK/QAM cases we can interpret this as a complex number

$$r_0 = R_1 + jR_2$$

R_1 and R_2 tell us how In-Phase and Quadrature are modulated respectively

Binary Error Probability

- Recall that the error probability for binary signaling when using a matched filter is:

$$P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

- We can define the distance energy in terms of the distance between constellation points:

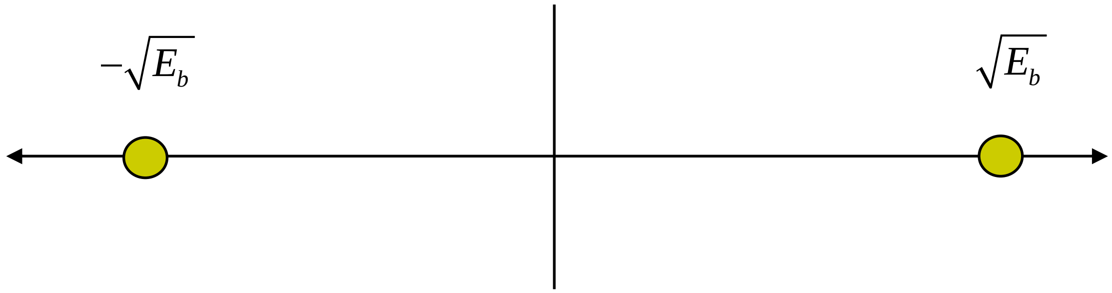
$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

- Where *d* is the distance between signal points in terms of E_b

Analog and Digital Communications

Example

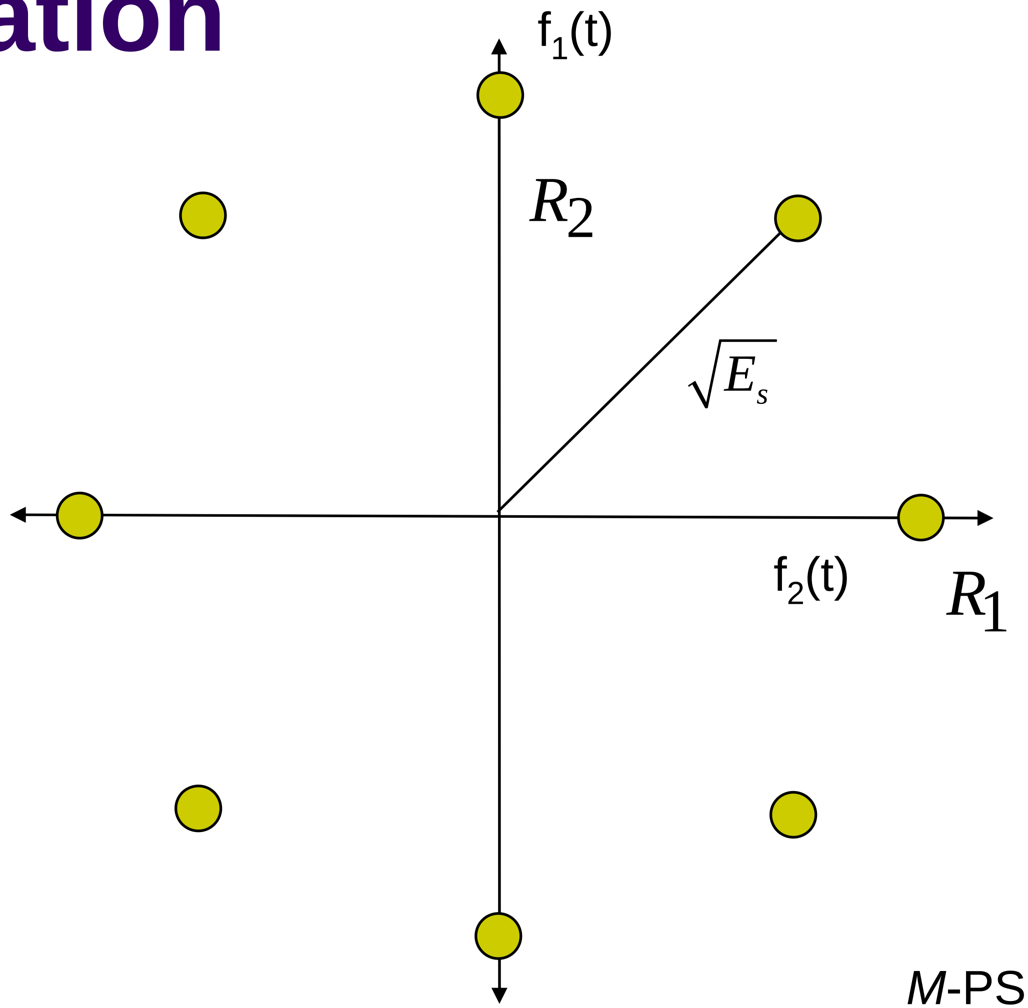
- BPSK



- The Euclidean distance between the points s_1 and s_2 is $2\sqrt{E_b}$
- In terms of E_b:

$$P_e = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right) = Q\left(\frac{2\sqrt{E_b}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

Error Probabilities for *M*-ary Modulation



$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi ft)$$

$$f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi ft)$$

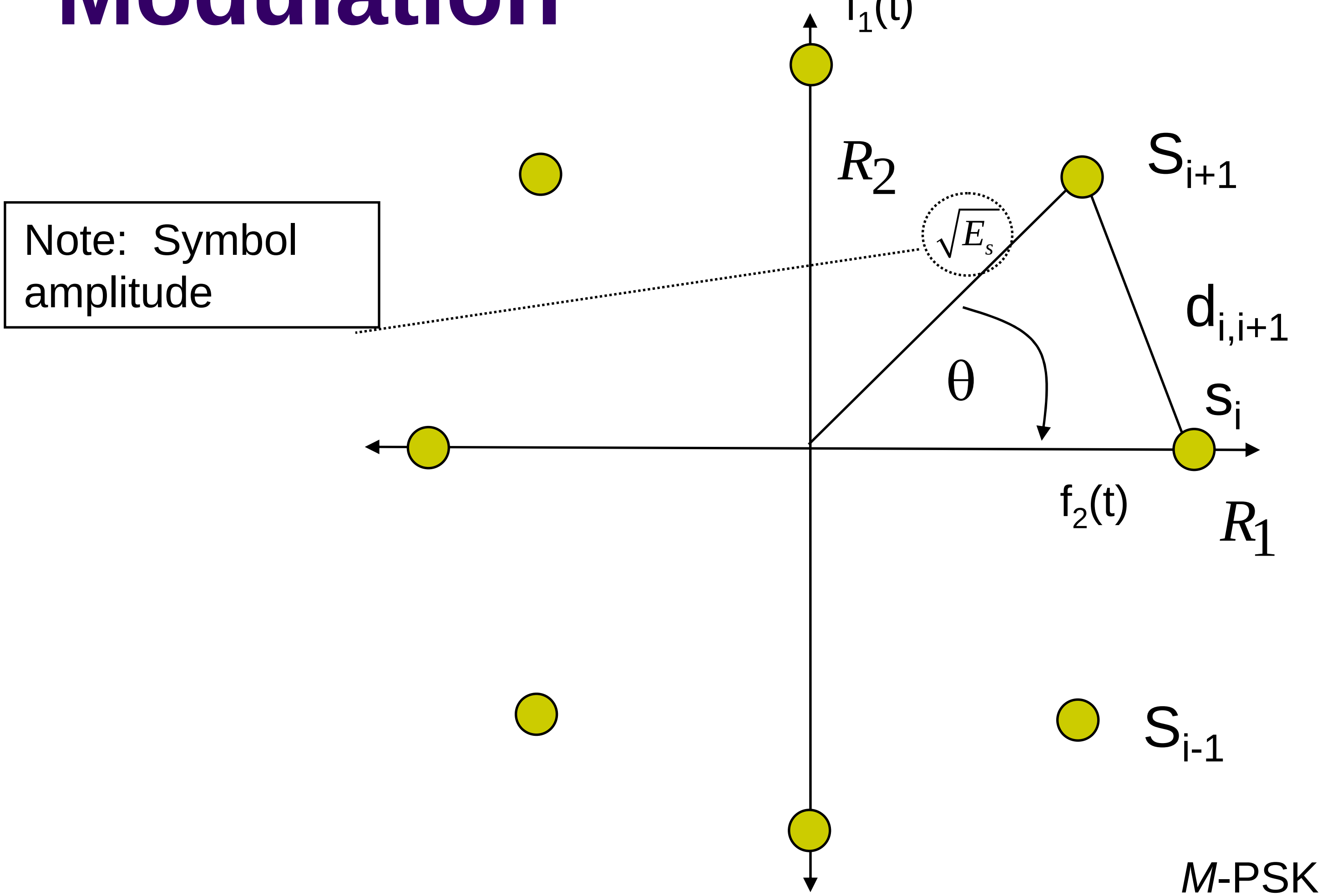
M-PSK

Error Probabilities for *M*-ary Modulation

- The general error probability for *M*-PSK is not straightforward to derive. However, we can derive a simple approximation assuming:
  - All symbols are equally likely and have identical error probabilities
  - The probability of symbol error is dominated by errors due to nearest neighbors

$$P_e \approx \Pr\left(\hat{s} = s_{i+1} \middle| s_i\right) + \Pr\left(\hat{s} = s_{i-1} \middle| s_i\right)$$

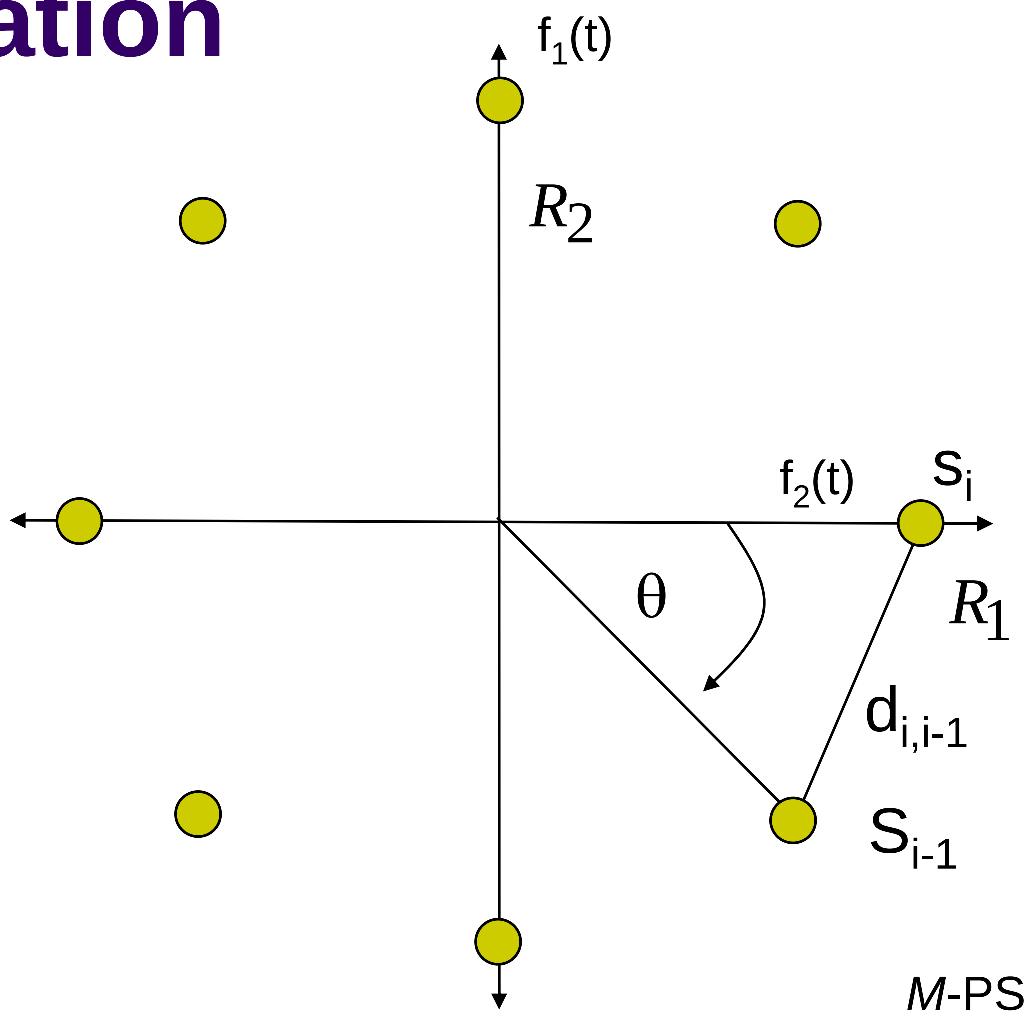
Error Probabilities for *M*-ary Modulation



$$f_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi ft)$$

$$f_2(t)=\sqrt{\frac{2}{T}}\sin(2\pi ft)$$

Error Probabilities for *M*-ary Modulation



$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi ft)$$

$$f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi ft)$$

M-PSK

Error Probabilities for *M*-ary Modulation

$$P_e \approx \Pr(s_{i+1} \mid s_i) + \Pr(s_{i-1} \mid s_i)$$

$$\approx Q\left(\frac{d_{i,i+1}}{\sqrt{2N_o}}\right) + Q\left(\frac{d_{i,i-1}}{\sqrt{2N_o}}\right)$$

$$\approx 2Q\left(\frac{d_{i,i+1}}{\sqrt{2N_o}}\right)$$

Note: $d_{i,i+1} = d_{i,i-1}$

Error Probabilities for *M*-ary Modulation (cont.)

Analog and Digital Communications

- Now, to find $d_{i,i+1}$ we use the law of cosines:

$$
\begin{aligned}
d^2 &= 2\left(\sqrt{E_s}\right)^2 - 2\left(\sqrt{E_s}\right)^2 \cos(\theta) \\
&= 2E_s\left(1-\cos(\theta)\right) \\
&= 4E_s \sin^2\left(\frac{\theta}{2}\right) \\
d &= 2\sqrt{E_s}\sin\left(\frac{\theta}{2}\right)
\end{aligned}
$$

Error Probabilities for *M*-ary Modulation (cont.)

Analog and Digital Communications

- The probability of error is then:

$$P_e \approx 2Q\left(\frac{d_{i,i+1}}{\sqrt{2N_o}}\right)$$

$$\approx 2Q\left(\frac{2\sqrt{E_s}\sin(\theta/2)}{\sqrt{2N_o}}\right)$$

$$\approx 2Q\left(\sqrt{\frac{\left(2\sqrt{E_s}\sin(\pi/M)\right)^2}{2N_o}}\right) \qquad \theta = 2\pi/M$$

$$\approx 2Q\left(\sqrt{\frac{2E_s\sin^2(\pi/M)}{N_o}}\right)$$

Error Probabilities for *M*-ary Modulation (cont.)

- The average energy per symbol is:

$$E_s$$

- While the average energy per bit is:

$$E_b = \frac{E_s}{\log_2(M)}$$

$$\boxed{E_b \log_2(M) = E_s}$$

*T*he probability of *symbol* error is then:

$$\boxed{P_e \approx 2Q\left(\sqrt{\frac{2E_b \log_2(M)\sin^2(\pi / M)}{N_o}}\right)}$$

Error Probabilities for *M*-ary Modulation (cont.)

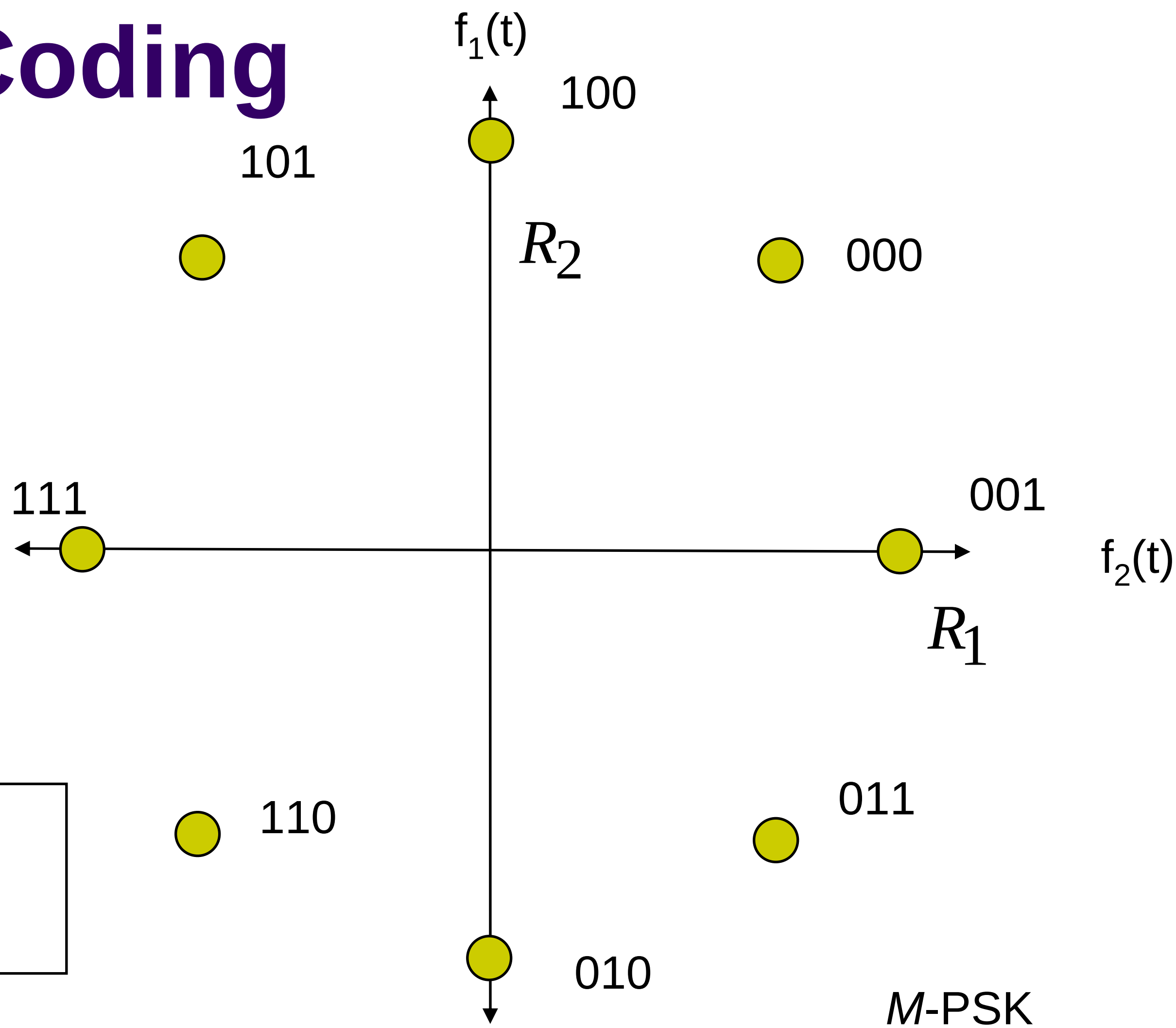
Analog and Digital Communications

- Note that this is the *symbol* error probability. The *bit* error probability, assuming that neighboring symbols differ by 1 bit (Gray Coding) is

$$P_b \approx \frac{P_s}{\log_2(M)} = \frac{2}{\log_2(M)} Q\left(\sqrt{\frac{2E_b \log_2(M)\sin^2(\pi / M)}{N_o}}\right)$$

- Note that this approximation is pessimistic for BPSK since we assumed 2 symbols could be mistaken for the desired symbol. There is only one other symbol in BPSK. Thus the error rate is double BPSK.

Gray Coding



All neighboring symbols differ only by one bit

M-PSK

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f t)$$

$$f_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f t)$$

Error Probabilities for *M*-ary Modulation (cont.)

Analog and Digital Communications

- Ex: QPSK (*M*=4)

$$P_b \approx \frac{2}{2} Q\left(\sqrt{\frac{2E_b 2\sin^2(\pi/4)}{N_o}}\right)$$

$$\approx Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

Same as BPSK!

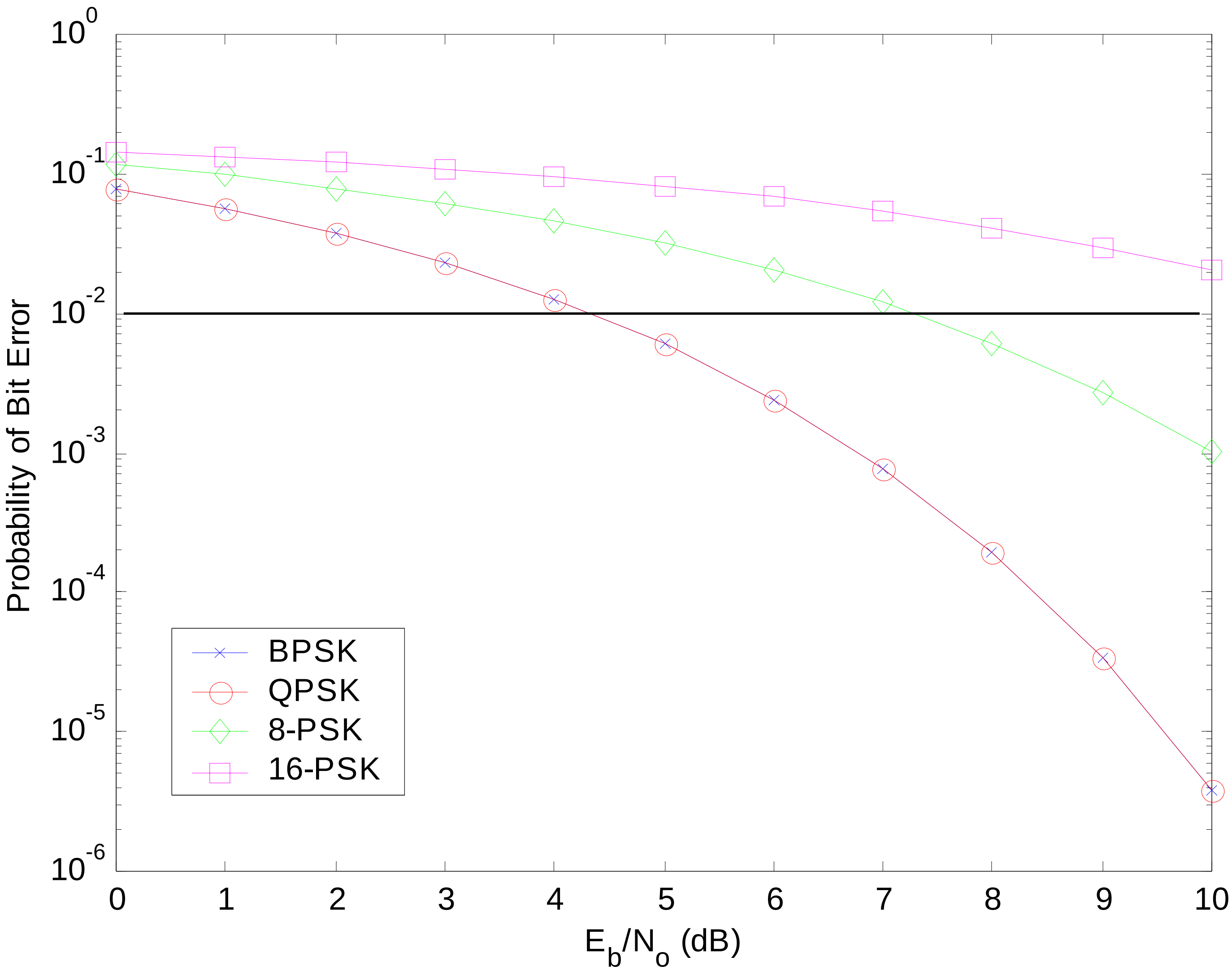
- Ex: 8-PSK

$$P_b \approx \frac{2}{3} Q\left(\sqrt{\frac{2E_b 3\sin^2(\pi/8)}{N_o}}\right)$$

$$\approx Q\left(\sqrt{\frac{0.88E_b}{N_o}}\right)$$

Performance Comparison for *M*-ary PSK

Analog and Digital Communications



Example: *M*-FSK

- Consider *M*-ary Frequency Shift Keying (*M*-FSK)
- M-FSK has *M* basis functions (one for each symbol) since each symbol is orthogonal
- The symbols can be written as

$$s_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + 2\pi i \Delta f t\right)$$

- If we choose Δf = 1/T, the correlation between symbols is zero thus we will need *M* basis functions (i.e., one for each symbol).

$$f_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{2\pi i}{T} t\right)$$

M- FSK (cont.)

Analog and Digital Communications

- Further, we can represent the symbols in signal space as *M*-dimensional vectors:

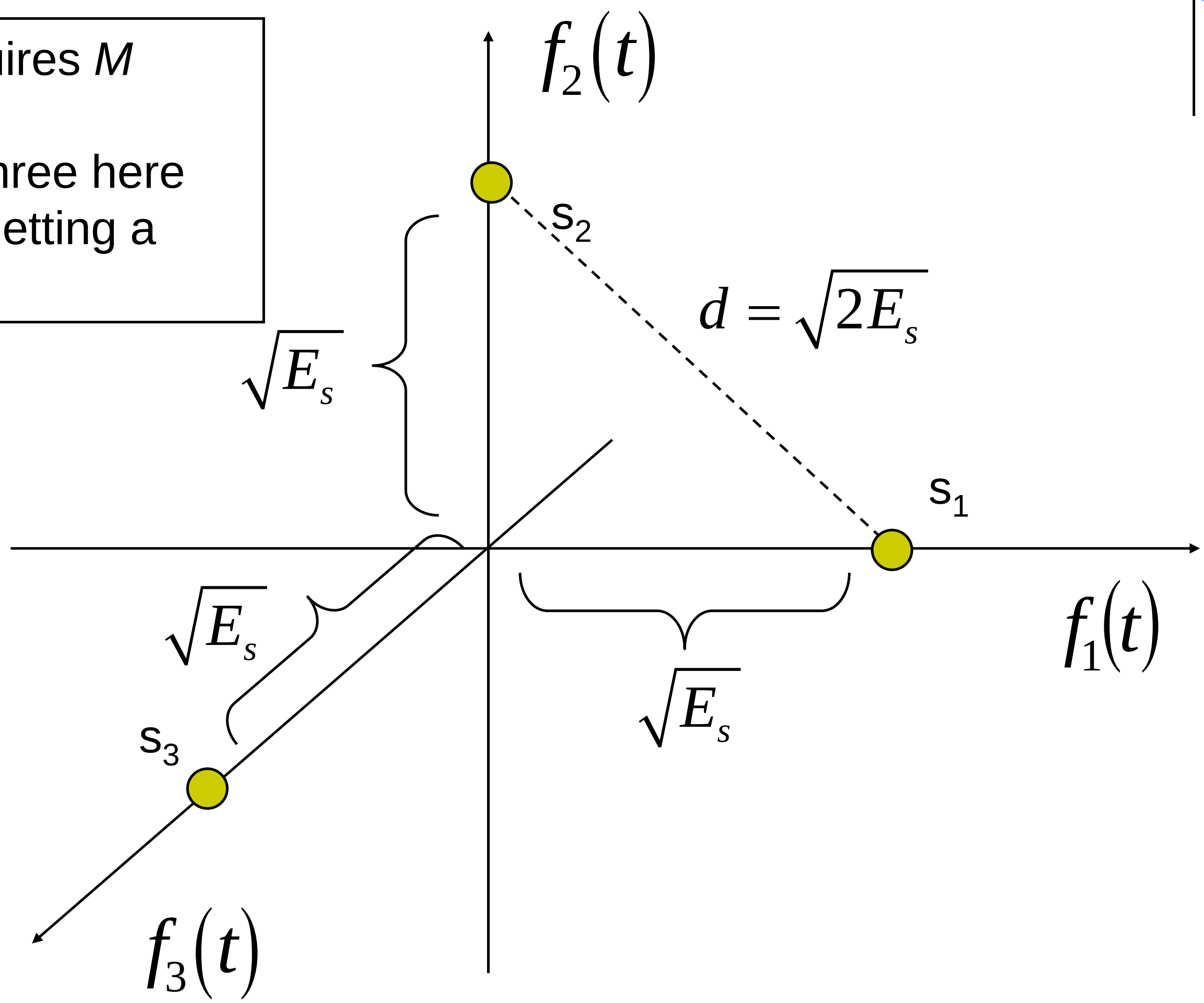
$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$\mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{E} & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$\mathbf{s}_3 = \begin{bmatrix} 0 & 0 & \sqrt{E} & 0 & \dots & 0 \end{bmatrix}$$
$$\vdots$$
$$\mathbf{s}_M = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \sqrt{E} \end{bmatrix}$$

$$\boxed{\overline{E}_s = 1}$$

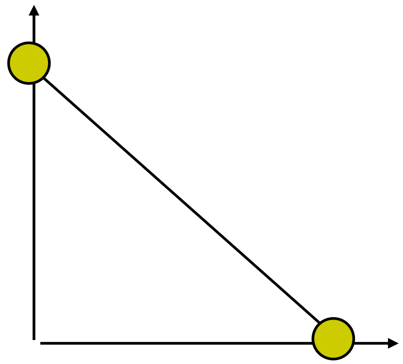
- Thus, the signal space diagram is a *M*-dimensional space with one symbol on each axis

Ex: *M*-FSK (cont.)

M-ary FSK requires *M* dimensions
We draw only three here so as to avoid getting a head-ache



Distance Between Symbols



$$d = \sqrt{2E_s}$$

- As we increase *M* the number dimensions increases and but the distance between points does not decrease
- In fact, in terms of energy per bit:

$$d = \sqrt{2E_b \log_2\left(M\right)}$$

Distance actually increases as *M* increases (in terms of energy per bit).

Probability of Symbol Error

Analog and Digital Communications

- Since there are *M-1* nearest neighbors, all at the same distance:

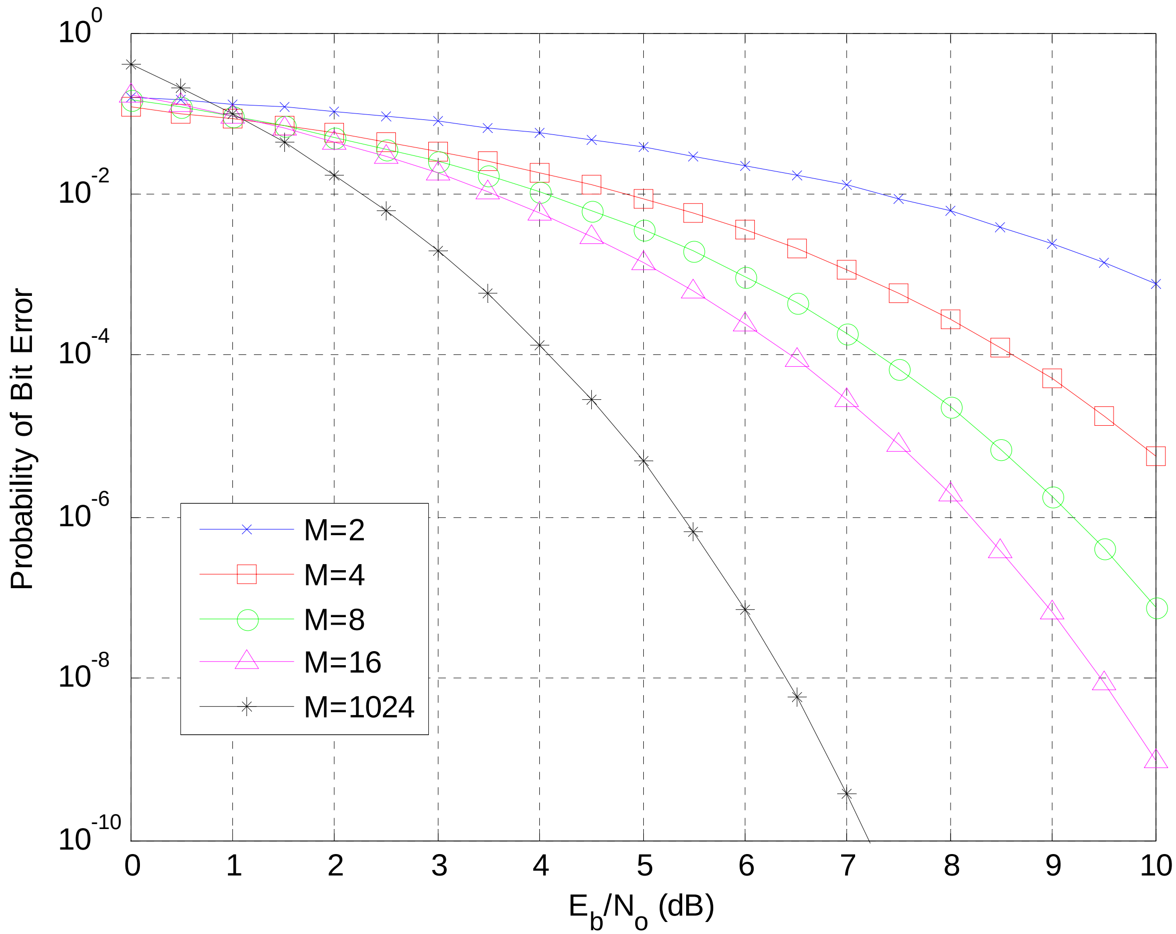
$$
\begin{aligned}
P_e &\approx \Pr(s_1 \mid s_i) + \Pr(s_2 \mid s_i) + \ldots \Pr(s_{i-1} \mid s_i) + \Pr(s_{i+1} \mid s_i) \ldots + \Pr(s_M \mid s_i) \\
&= Q\left(\frac{d_{i,1}}{\sqrt{2N_o}}\right) + Q\left(\frac{d_{i,2}}{\sqrt{2N_o}}\right) \ldots + Q\left(\frac{d_{i,i-1}}{\sqrt{2N_o}}\right) + Q\left(\frac{d_{i,i+1}}{\sqrt{2N_o}}\right) \ldots + Q\left(\frac{d_{i,M}}{\sqrt{2N_o}}\right) \\
&= (M-1)\, Q\left(\frac{d_{i,i+1}}{\sqrt{2N_o}}\right) \\
&= (M-1)\, Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_o}}\right) \\
&= (M-1)\, Q\left(\sqrt{\frac{\log_2(M) E_b}{N_o}}\right)
\end{aligned}
$$

BER

- The previous expression is the probability of *symbol error*
- We are more interested in the probability of *bit error*
- On average, one half of the bits are in error when we make a symbol error thus

$$P_b \approx \frac{(M-1)}{2} Q\left(\sqrt{\frac{\log_2(M)E_b}{N_o}}\right)$$

BER of *M*-FSK



- Performance improves as *M* gets larger
- However, bandwidth increases dramatically as *M* gets larger
- As *M* approaches infinity, M-FSK can approach the Shannon bound, albeit at infinite bandwidth

Error Probabilities for *M*-ary Modulation

Analog and Digital Communications

- Different quantities: symbol error rate, bit error rate
- *M*-ary PSK: $P_e \leq 2Q\left(\sqrt{2E_b \log_2 M / N_0}\sin(\pi/M)\right)$
  - Energy efficiency gets worse as *M* increases
  - Bandwidth efficiency improves as *M* increases
- We present some simple results without derivation
- QAM: Results are similar to *M*-ary PSK

$$P_e \approx 2Q\left(\sqrt{2\frac{E_b}{N_o}\eta_M}\right)$$

$$\eta_M = -4dB \quad M = 16$$
$$\eta_M = -6dB \quad M = 32$$

- *M*-ary FSK: $P_b \leq \frac{(M-1)}{2}Q\left(\sqrt{E_b \log_2 M / N_0}\right)$
  - Energy efficiency improves as *M* increases
  - Bandwidth efficiency gets worse as *M* increases

$B \sim 2MR_b$

Comparisons Between Digital Modulation Schemes

Modulation Scheme	Null-to-Null BW (square pulses)	E_b/N_o for $P_e=10^{-3}$
BPSK	$2R_b$	6.75dB
QPSK	R_b	6.75dB
8-PSK	$2/3\ R_b$	10.0dB
16-PSK	$1/2\ R_b$	14.25dB
16-QAM	$1/2\ R_b$	11.25dB
32-QAM	$2/5\ R_b$	13.25dB
BFSK	$3\ R_b$	9.75dB
8-FSK	$3\ R_b$	6.00dB

Summary

- Today we have examined the performance of *M*-ary modulation schemes
- Analysis is similar to binary case when examining nearest neighbors
- *M*-ary PSK/ASK → performance degrades as *M* increases
- *M*-ary FSK → performance improves as *M* increases