ECE4634
Digital Communications
Fall 2007

Instructor: R. Michael Buehrer
Lecture #5: The Sampling Theorem

1

Overview

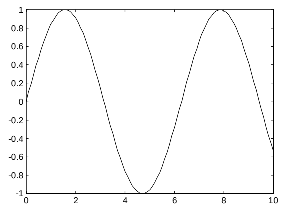
- We are studying *digital* communication systems
- Digital systems can be used to transmit either analog or digital information
- Analog information must be converted to digital format
  - This conversion includes *sampling* and some form of *quantization*
- Today we study the impact of sampling
- What to read – Section 5.1 in the text

Lecture Objectives

- The objectives of this lecture are to
  - Show that the sampling process can be done with (theoretically) no loss of information
  - Describe the conditions under which this occurs – i.e., to introduce the Sampling Theorem
  - Explain the practical significance of the sampling theorem

Digital Representation of Analog Signals

- Analog signals (e.g. voice,video) are continuous in time and amplitude:

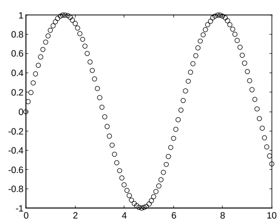


4

RM Buehrer
Virginia Tech
Fall 2007

Digital Representation of Analog Signals

- <u>Sampling</u> analog signals makes them discrete in *time*:

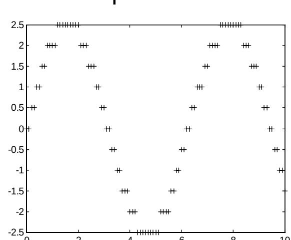


5

RM Buehrer
Virginia Tech
Fall 2007

Digital Representation of Analog Signals

- <u>Quantization</u> of sampled analog signals makes the samples discrete in amplitude:



•The number of discrete amplitude levels is directly related to the number of bits we are willing to use to represent each sample. Thus, we trade-off bit rate and fidelity

6

Virginia Tech
Fall 2007

Digital Representation of Analog Signals

- If done properly, sampling introduces no distortion into the signal
- Quantization does introduce distortion
  - There is a tradeoff between distortion and bandwidth requirements
  - More bits per sample → less distortion
  - Fewer bits per sample → lower bandwidth requirements
- We consider sampling today.
- We will discuss quantization shortly.

The Sampling Theorem

- We consider instantaneous sampling of a signal waveform ("ideal sampling" or "impulse sampling") which can be modeled as

$$w_s(t) = \underbrace{w(t)}_{\substack{\text{original}\\\text{signal}}} \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)}_{\substack{\text{impulse}\\\text{train}}}$$

$$= \sum_{n=-\infty}^{\infty} w(nT_s)\,\delta(t - nT_s)$$

- The train of impulse functions select sample values at regular intervals.
- How often do we have to sample to retrieve the original information? (*i.e.,* how small can T_s be?)

The Sampling Theorem (continued)

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s)$$

- **The train of impulse functions select sample values at regular intervals. Using a Fourier Series representation of the impulse train:**

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}, \omega_s = \frac{2\pi}{T_s}$$

- **Rewriting, we have:**

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

The Sampling Theorem (continued)

- Taking the Fourier Transform of signals:

$$W_s(f) = \frac{1}{T_s} W(f) * F\left\{ \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\}$$

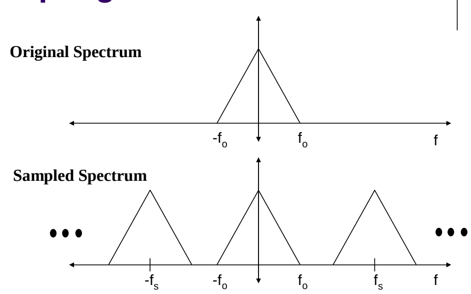
$$= \frac{1}{T_s} W(f) * \sum_{n=-\infty}^{\infty} F\left\{ e^{jn\omega_s t} \right\}$$

$$W_S(f) = \frac{1}{T_S} W(f) * \sum_{n=-\infty}^{\infty} \delta(f - nf_S),\ f_S = \frac{\omega_S}{2\pi}$$

$$\boxed{W_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} W(f - nf_s)}$$

Note: This also follows from the fact that the Fourier Transform of an impulse train is simply an impulse train.

Sampling Theorem



Sampling Theorem

- Let *w(t)* be a bandlimited signal with Fourier Transform:

$$W(f) = 0, \text{ for } |f| > B$$

- ***w(t)* can be perfectly reconstructed from uniformly spaced samples, provided those samples are taken at a rate** $f_s \geq 2B$
  - *2B* is called the Nyquist Rate
  - If $f_s < 2B$, aliasing results.
  - If the signal is not strictly bandlimited, then it must be passed through lowpass filter before sampling to practically limit its bandwidth

Recovering the Signal from Sampled Waveform

- Sampled signal: $W_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} W(f - nf_s)$
- Apply lowpass filter to recover original signal

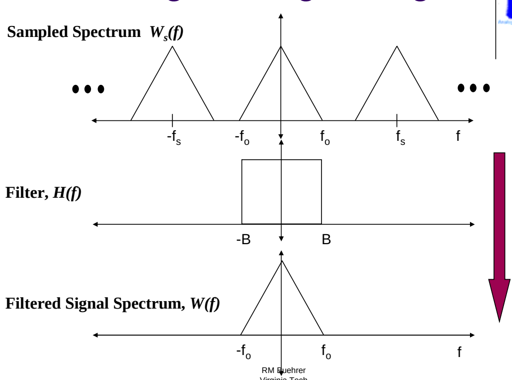
$W_s(f)$ → **Ideal Lowpass Filter w/bandwidth B** → $W(f)$

$$\begin{aligned} W(f) &= W_s(f)\Pi\left(\frac{f}{2B}\right) \\ &= \left(\frac{1}{T_s}\sum_{n=-\infty}^{\infty} W(f - nf_s)\right)\Pi\left(\frac{f}{2B}\right) \\ &= \frac{1}{T_s} W(f) \end{aligned}$$

13

Virginia Tech
Fall 2007

Recovering the Original Signal



14

RM Buehrer
Virginia Tech
Fall 2007

Other Versions of Sampling Theorem

- We will also discuss “flat top” sampling using rectangular pulses.
  - Becomes identical to instantaneous sampling approach as pulses become short
- Random signals are band-limited if the power spectral density satisfies: $P_X(f) = 0, \text{ for } |f| > B$
  - May also be represented by samples taken at rate 2*B*
- Bandpass signals with bandwidth *B* may be represented by *complex-valued* samples taken at rate B or by *real-valued* samples taken at rate *2B*

15

RM Buehrer
Virginia Tech
Fall 2007

Practical Limitations

- Band-limited Signals
  - No signal has a spectrum that goes to identically zero at some finite frequency
  - We treat some level (say 30dB below the strongest frequency component as *essentially zero.*
  - Signals are typically filtered with a low pass filter with very steep roll-off in order to ensure the signal falls below a certain value
- Ideal reconstruction
  - Using sinc functions for interpolation (i.e., using a perfect “brick-wall” frequency filter) results in *ideal recontruction*
  - Non-ideal reconstruction can be accomplished using other forms of interpolation
    - Result in non-ideal frequency filters

Example 5.1

- Consider the following time domain signal:

$$w(t) = \text{sinc}(100000t)$$

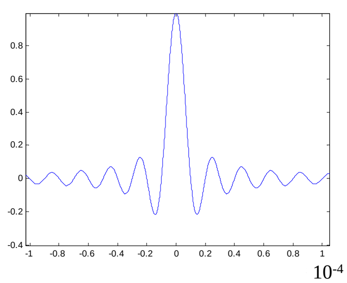
- The Fourier Transform is :

$$W(f) = \frac{1}{100000}\Pi\left(\frac{f}{100000}\right)$$

- The bandwidth of this signal is:
  - B=50 kHz
- Therefore samples must be taken at least at rate:
  - R > 2 B = 100,000 samples/second

RM Buehrer
Virginia Tech
Fall 2007

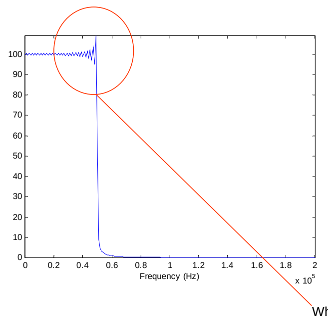
Example 5.1 (cont.)



$$w(t) = \left[\text{Sa}(100000\pi t)\right]$$

RM Buehrer
Virginia Tech
Fall 2007

Example 5.1 (cont.)

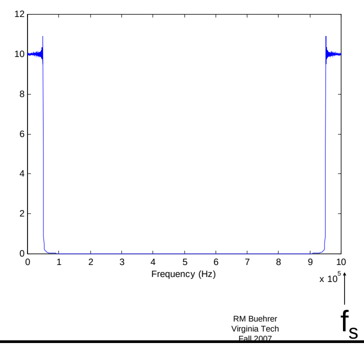


$$W(f) = \frac{1}{100000}\Pi\left(\frac{f}{100000}\right)$$

Why isn't this a perfect square pulse?

RM Buehrer
Virginia Tech
Fall 2007

Example 5.1 (cont.)

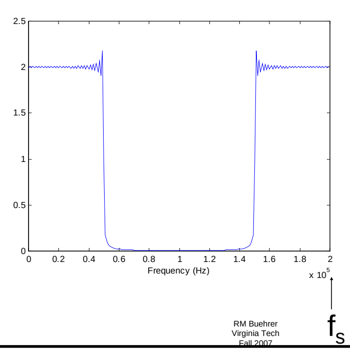


$f_s >> 2B$

B = 50kHz
f_s = 1MHz

RM Buehrer
Virginia Tech
Fall 2007

Example 5.1 (cont.)

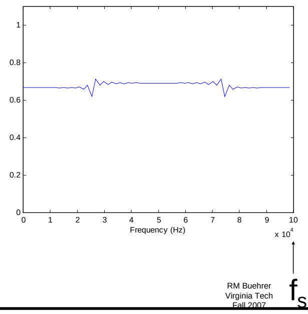


$f_s > 2B$

B = 50kHz
f_s = 200kHz

RM Buehrer
Virginia Tech
Fall 2007

Example 5.1 (cont.) : Aliasing

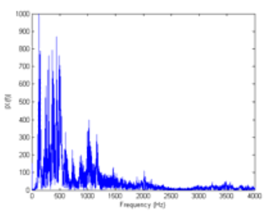


$f_s < 2B$

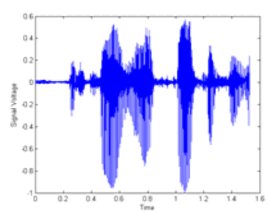
B = 50kHz
f_s = 75kHz

Example 5.2

Original Spectrum



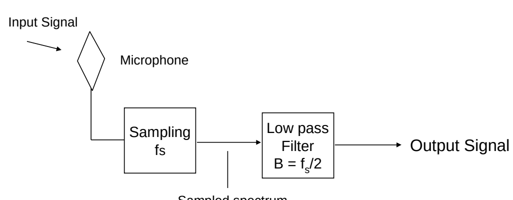
Time Signal



SamplingOriginal.wav

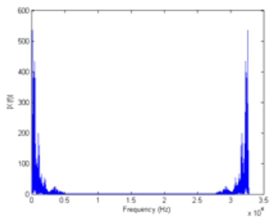
Example 5.2: System

- Simple sampling and reconstruction

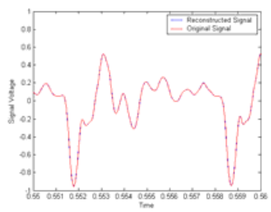


Example 5.2: f_s = 32kHz

No Aliasing



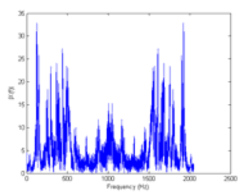
Perfect Reconstruction



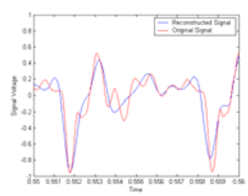
Sound Quality

Example 5.2: f_s = 2kHz

Substantial Aliasing

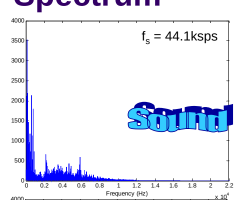


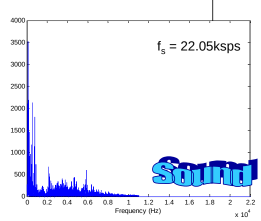
Imperfect Reconstruction

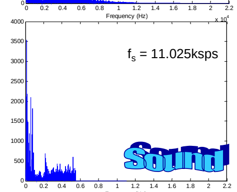


Sound Quality

Example 5.3 – Reconstructed Spectrum







Two aspects of aliasing can be discerned:

1. Loss of high frequency content
2. Distortion of lower frequency content

Practical Sampling Rates

- Speech:
  - Telephone quality speech has a bandwidth of 4 kHz
  - Most digital telephone systems sample at 8000 samples/sec
- Audio:
  - The highest frequency the human ear can hear is approximately 15 kHz
  - CDs sample at rate 44,100 samples/sec
- Video:
  - The human eye requires samples at a rate of at least 20 frames/sec to achieve smooth motion

Summary

- Today we have examined a key aspect of digital communications: Sampling
- Nyquist's Sampling Theorem tells us that sampling introduces no distortion provided that we sample at a rate equal to or greater than twice the highest frequency
- In practical scenarios we typically filter the signal before sampling in order to prevent aliasing

ECE4634
Digital Communications
Fall 2007

Appendix
Matlab Representation of Signals

Representation of Signals in Matlab

- Matlab uses vectors which are inherently sampled signals with limited duration (0 to *T*)
- Recall the Fourier Transform $X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}$
- The sampled signal can be written as

$$x(\Delta t)\approx\sum_{n=0}^{N-1}x(n\Delta t)\delta(t-n\Delta t)$$

- Which has a Fourier Transform $X(f)=\sum_{n=0}^{N-1}x(n\Delta t)e^{-j2\pi fn\Delta t}$
- We are interested in discrete points thus we limit *f* to values {*0, 1/T, 2/T, ….(N-1)/T*} $f=\frac{k}{T}=\frac{k}{N\Delta t}$
- Thus we have:

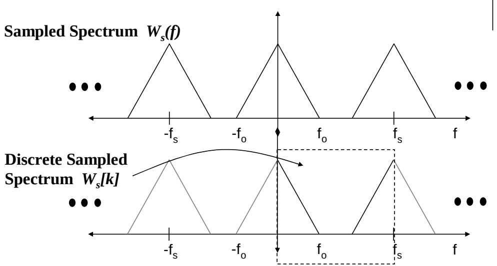
$$X[k]=\sum_{n=0}^{N-1}x[n]e^{j2\pi n\frac{k}{N}}\quad k=0,1\ldots N-1$$

This is termed the Discrete Fourier Transform

Representation of Signals in Matlab

- Sampling rate must be high enough to avoid aliasing (f_s > 2B)
- Use `fft` command to obtain spectrum
- Frequency domain resolution can be increased by "zero padding" - adding zeros at end of signal
- Typically we desire a plot of the *magnitude* of the spectral density of the signal [`plot(abs(fft(x)))`]
- Matlab requires that you assign the units of the scale [`f=0:fs/points:fs-fs/points`]
  - The `fft` function only returns values between 0 and f_s.

Matlab Plots



- Since the DFT only provides frequency domain points from f = 0 to f_s/2, the plot only shows a fraction of the actual sampled spectrum

ECE4634
Digital Communications
Fall 2007

Appendix
Time domain view of the Sampling Theorem

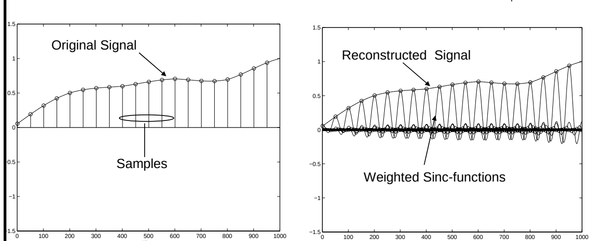
34

Another View of the Sampling Theorem

$$W(f) = W_s(f)\Pi\left(\frac{f}{2B}\right)$$

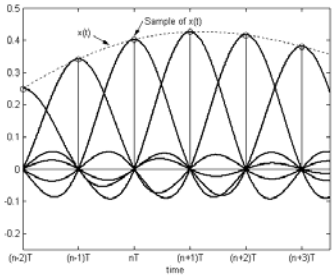
$$w(t) = w_s(t) * \Im^{-1}\left\{\Pi\left(\frac{f}{2B}\right)\right\}$$
$$= w_s(t) * \mathrm{sinc}\left(2\mathrm{Bt}\right)$$
$$= \left(\sum_{n=-\infty}^{\infty} w\left(nT_s\right)\delta\left(t - nT_s\right)\right) * \mathrm{sinc}\left(2\mathrm{Bt}\right)$$
$$= \sum_{n=-\infty}^{\infty} w\left(nT_s\right)\mathrm{sinc}\left(2Bt - n2BT_s\right)$$
$$= \sum_{n=-\infty}^{\infty} w\left(nT_s\right)\mathrm{sinc}\left(\frac{t}{T_s} - n\right)$$

Time-Domain View of the Sampling Theorem



sin(x)/x is also referred to as the *Sampling Function*

Ideal Reconstruction



- Sinc functions provide ideal reconstruction of values between samples
- Compared to linear interpolation or some other form of interpolation, using sinc functions provides *ideal interpolation*

37

RM Buehrer
Virginia Tech
Fall 2007