

ECE4634 Digital Communications Fall 2007

Instructor: R. Michael Buehrer
Lecture #7: Digital Pulse
Modulation : PCM



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Overview



- We are currently studying *baseband communication systems*
- Previously we looked at
 - Sampling continuous time waveforms
 - Nyquist sampling theorem
 - Representing time samples using *discrete baseband* communication systems where samples are used to modulate a pulse train
- Digital systems are discrete in time and have a finite number of messages
- Pulse Code Modulation is one form of *digital* baseband communications
 - Often called *analog-to-digital conversion* since it is so common
 - However, this is only *one form* of analog-to-digital conversion
- What to read – 5.4-5.6

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Lecture Objectives



- In this lecture we will
 - Introduce the digital communication concept of Pulse Code Modulation
 - Examine (uniform scalar) quantization in detail
 - Derive the SNR of quantization
 - Demonstrate the trade-off that exists in quantization between fidelity and bit rate (i.e., bandwidth)

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What is Modulation?



- Modulation is changing a parameter of a carrier in order to convey information
- Two common carriers are
 - Sinusoid (bandpass systems)
 - Pulse stream (baseband systems)
- In PCM (Pulse Code Modulation) we modulate a stream of pulses (typically we assume square pulses for convenience)

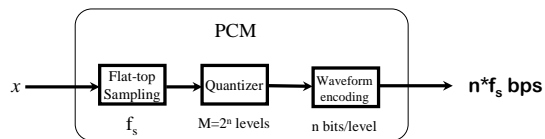
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Pulse Code Modulation (PCM)



- Pulse Code Modulation refers to a system that creates a baseband signal that is generated directly from the binary quantizer output



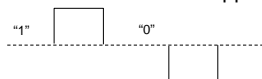
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Waveform Encoding



- Once the information is converted to bits, it must be mapped onto waveforms (i.e., modulation)
- If each bit is mapped to one of two different waveforms, we term this *binary encoding*
- If m bits are mapped to $M = 2^m$ waveforms, we term this *M-ary encoding*
- We will discuss specific waveforms next class
- For now assume that we use the mapping



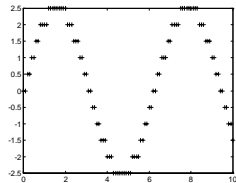
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Digital Representation of Analog Signals



- Quantization of sampled analog signals makes the samples discrete in amplitude:



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Quantization



- Continuous time signals are sampled at discrete time intervals
- Sampling may be performed without distortion provided signal is sampled at Nyquist rate
- Continuous-valued samples of data require an infinite # of bits to represent with perfect precision.
- Quantization is the process of approximating continuous-valued samples with a finite # of bits.
- Quantization always introduces some distortion.

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Notation Associated with Quantization



- Let X be a random variable representing a sample of data.
- Then $\tilde{X} = f_Q(X)$ is the quantized value of X .
- A quantizer has M quantization levels:

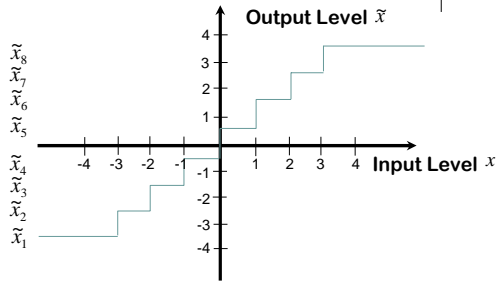
$$\tilde{X} \in \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_M\}$$
- The M levels correspond to M quantization regions.
- The endpoints of the quantization regions are specified by $M + 1$ values: $\{x_0, x_1, \dots, x_M\}$, where $x_0 = -\infty, x_M = \infty$
- Then:

$$x_{k-1} \leq x < x_k \Rightarrow \tilde{x} = f_Q(x) = \tilde{x}_k$$

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Graphical Description of Quantization



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Table Representation of Quantizer



k	x_{k-1}	x_k	\tilde{x}_k	Output Bits
1	$-\infty$	-3	-3.5	000
2	-3	-2	-2.5	001
3	-2	-1	-1.5	010
4	-1	0	-0.5	011
5	0	1	0.5	100
6	1	2	1.5	101
7	2	3	2.5	110
8	3	∞	3.5	111

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Concise Representation of Quantizer



- Usually, it is sufficient just to list the quantization levels.
- Example: $\{-3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5\}$
- Why?
 - We assume that all points are quantized to the nearest quantization level
 - This determines where the borders of the quantization regions are
 - Any other borders would increase the error introduced by the quantizer

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Practical Methods for Implementing Analog to Digital Converters



Lower Complexity

- Counting or Ramp ADC
 - test value is incremented in equal steps until it is greater than input sample
- Serial or Successive Approximation ADC
 - uses binary search to narrow range of input sample until desired accuracy is reached
- Parallel or Flash ADC
 - input sample is compared with all possible quantization levels at once

Faster

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Distortion



- Quantization introduces distortion into a signal.
- We want to minimize average distortion D , where

$$\begin{aligned}
 D &= E[(X - \tilde{X})^2] \\
 &= \int_{-\infty}^{\infty} (x - \tilde{x})^2 f(x) dx \\
 &= \sum_{k=1}^M \int_{x_{k-1}}^{x_k} (x - \tilde{x}_k)^2 f(x) dx
 \end{aligned}$$

- This measure of distortion is sometimes also called mean square error (MSE)
- MSE penalizes large errors more than small errors

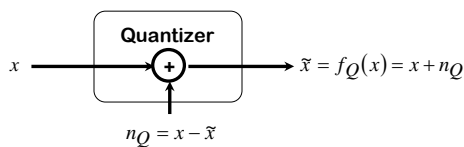
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Another Way of Viewing Quantization



- Quantization adds a random "noise" to the true value of the sample point
- Then $\text{MSE} = E[n_Q^2]$ may be thought of as noise power
- We can define a signal-to-noise ratio (SNR) to measure performance



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Signal to Noise Ratio Calculations for Quantizers



- Average SNR

$$\left(\frac{S}{N}\right)_{\text{avg}} = \frac{E[X^2]}{E[n_Q^2]} = \frac{E[X^2]}{D} = \frac{\int_{-\infty}^{\infty} x^2 f_X(x) dx}{\int_{-\infty}^{\infty} (x - \tilde{x})^2 f_X(x) dx}$$

- Sometimes we also define "peak" SNR

- This is less important.

$$\left(\frac{S}{N}\right)_{\text{peak}} = \frac{X_{\text{peak}}^2}{E[n_Q^2]} = \frac{X_{\text{peak}}^2}{D}$$

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Example 7.1: SNR Calculation



- Let: $\{\tilde{x}_1 = -3.5, \tilde{x}_2 = -2.5, \tilde{x}_3 = -1.5, \tilde{x}_4 = -0.5, \tilde{x}_5 = 0.5, \tilde{x}_6 = 1.5, \tilde{x}_7 = 2.5, \tilde{x}_8 = 3.5\}$

- Let: $f(x) = \begin{cases} 1/8, & -4 \leq x \leq 4 \\ 0, & \text{else} \end{cases}$

$$E[X^2] = \int_{-4}^4 x^2 \cdot \frac{1}{8} dx = \frac{x^3}{24} \Big|_{-4}^4 = \frac{128}{24} = \frac{16}{3}$$

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Example 7.1 (continued)



- Distortion:

$$D = MSE = \sum_{k=1}^M \int_{\tilde{x}_{k-1}}^{\tilde{x}_k} (x - \tilde{x}_k)^2 f_X(x) dx$$

$$= \sum_{k=1}^8 \int_{-5+k}^{-4+k} (x - (-4.5+k))^2 \cdot \frac{1}{8} dx$$

- Examine the $k=5$ term:

$$\int_0^1 (x - 0.5)^2 \cdot \frac{1}{8} dx = \frac{1}{8} \int_0^1 x^2 - x + 0.25 dx = \frac{1}{8} \left[\frac{x^3}{3} - \frac{x^2}{2} + 0.25x \right]_0^1$$

$$= \frac{1}{8} \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right] = \frac{1}{8} \cdot \frac{1}{12}$$

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Example 7.1 (continued)



- All 8 terms are identical. $\therefore \text{MSE} = 1/12$

$$\left(\frac{S}{N}\right)_{\text{avg}} = 10 \log_{10} \frac{E[X^2]}{D} = 10 \log_{10} \left(\frac{16/3}{1/12}\right) = 18.1 \text{dB}$$

- A standard rule of thumb is that each additional bit adds 6 dB to the SNR of uniform quantizers operating on a uniform pdf.

SNR for Uniform Quantization



- General result: $\left(\frac{S}{N}\right)_{\text{avg}} = M^2$
 - Assumes uniform quantizer with M levels
 - Assumes that input samples have uniform distribution with identical range as quantizer
 - Though some texts do not make it clear, this result applies only when these special set of conditions hold. Otherwise, we have to use integral formula
- However, a useful Rule of Thumb:
 - Each additional bit (doubling M) increases SNR by 6 dB

$$\left(\frac{S}{N}\right)_{\text{avg}} = 6.02n + \alpha$$

- Where α depends on the distribution of the signal

Example 7.2: PCM Calculation



- **Problem:**
 - Suppose that an analog music signal is found to have a bandwidth of 15 kHz and that samples of the signal may be modeled as having a uniform distribution.
 - Find the minimum first-null bandwidth (assuming the use of square pulses) at which it would be possible to transmit a PCM signal while maintaining an average SNR of at least 58 dB.

Example 7.2 (continued)



- **Solution:**

$$f_s \geq 2B = 30,000 \text{ samples / sec}$$

$$10 \log_{10} M^2 \geq 58 \text{ dB} \Rightarrow M \geq 794 \Rightarrow M \geq 1024$$

$$\Rightarrow n \geq 10 \text{ bits/sample}$$

- We will assume n must be an integer, thus $n = 10$
- Minimum data rate: $R = f_s n \geq 300 \text{ kbps}$
- First null BW: $B = f_s n = 300 \text{ kHz}$
- Assumes rectangular pulses are used as the waveforms

Note: Bandwidth comes into play twice here. Once for the original analog signal and once for the resulting digital signal.

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Non-Uniform Quantization



- A quantizer for which

$$\bar{x}_{k-1} - \bar{x}_k = \Delta, \forall k \in \{1, \dots, L-1\}$$

is called a uniform quantizer.

- It is sometimes better to use non-uniform spacing
- Examine the distortion measure: $D = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx$
- We wish to make $(x - \bar{x})^2$ small when $f(x)$ is large.
- We can accept larger $(x - \bar{x})^2$ when $f(x)$ is small.
- **Basic principle:** concentrate quantization levels in area of largest pdf.

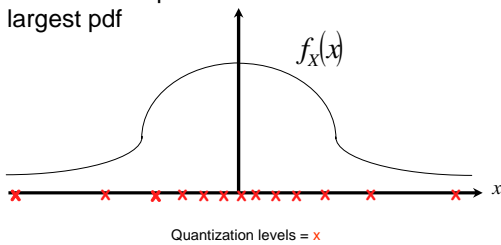
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Non-uniform Quantization



- Concentrate quantization levels in areas of largest pdf



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Difference between Uniform and Non-uniform Quantization



- Let: $p(x) = e^{-\frac{x^2}{2}} / \sqrt{2\pi}$
- Let: $\{x_1 = -1.494, x_2 = -0.498, x_3 = 0.498, x_4 = 1.494\}$
- Numerical integration shows that:
$$D = 0.1188, E[X^2] = 1, \left(\frac{S}{N}\right)_{avg} = 10 \log_{10} \left(\frac{1}{0.1188}\right) = 9.25dB$$
- Non-uniform quantization will yield better results
 - The "best" possible quantizer has

$$\left(\frac{S}{N}\right)_{avg} = 12dB$$

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Drill Problem



- If time....

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Summary



- Today we have examined Pulse Code Modulation which converts an analog signal to a digital signal.
- Comprised of
 - Sampling
 - Quantization
 - Pulse modulation
- Next class we will examine additional techniques for converting analog signals to baseband digital signals

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