

# ECE4634

## Digital Communications

### Fall 2007

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Instructor: R. Michael Buehrer  
Lecture #7: Digital Pulse  
Modulation : PCM



Analog and Digital Communications



# Overview

- We are currently studying *baseband communication systems*
- Previously we looked at
  - Sampling continuous time waveforms
    - Nyquist sampling theorem
  - Representing time samples using *discrete baseband* communication systems where samples are used to modulate a pulse train
- Digital systems are discrete in time and have a finite number of messages
- Pulse Code Modulation is one form of *digital* baseband communications
  - Often called *analog-to-digital conversion* since it is so common
  - However, this is only *one form* of analog-to-digital conversion
- What to read – 5.4-5.6



# Lecture Objectives

- In this lecture we will
  - Introduce the digital communication concept of Pulse Code Modulation
  - Examine (uniform scalar) quantization in detail
  - Derive the SNR of quantization
  - Demonstrate the trade-off that exists in quantization between fidelity and bit rate (i.e., bandwidth)



# What is Modulation?

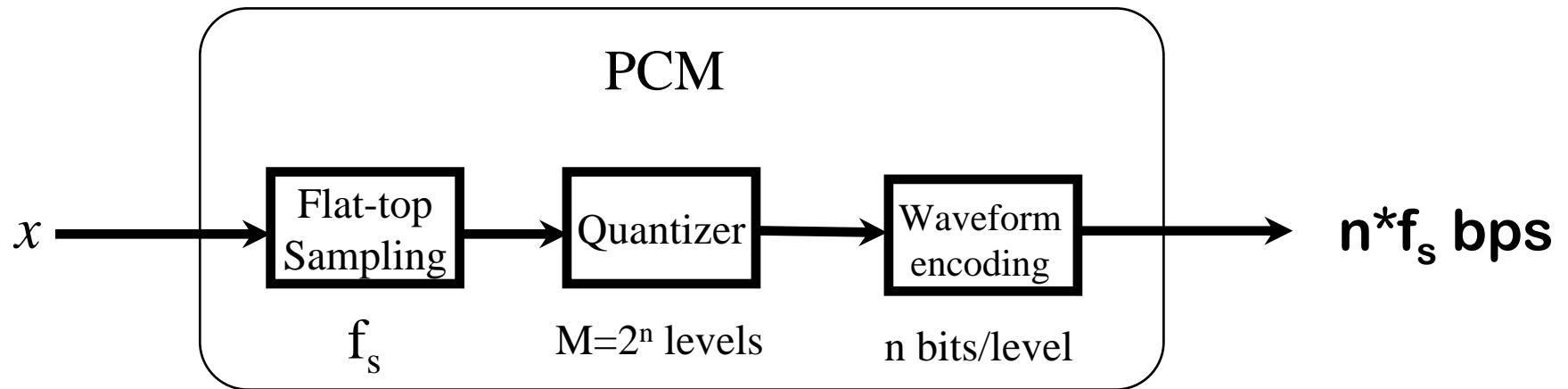
- Modulation is changing a parameter of a carrier in order to convey information
- Two common carriers are
  - Sinusoid (bandpass systems)
  - Pulse stream (baseband systems)
- In PCM (Pulse Code Modulation) we modulate a stream of pulses (typically we assume square pulses for convenience)

# Pulse Code Modulation (PCM)



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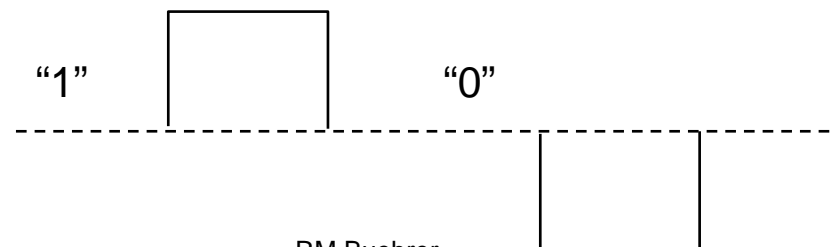
- Pulse Code Modulation refers to a system that creates a baseband signal that is generated directly from the binary quantizer output





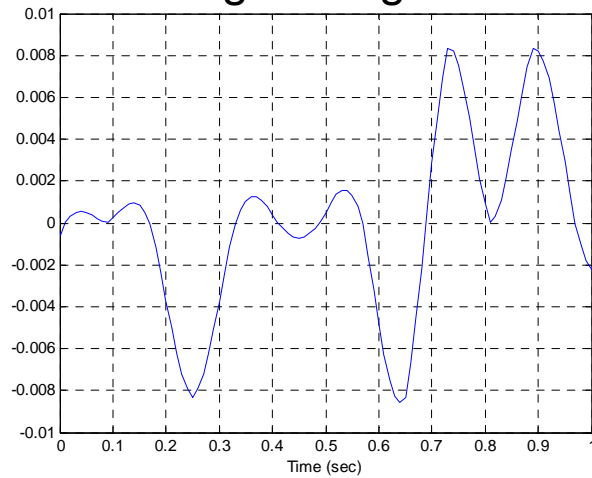
# Waveform Encoding

- Once the information is converted to bits, it must be mapped onto waveforms (i.e., modulation)
- If each bit is mapped to one of two different waveforms, we term this *binary encoding*
- If  $m$  bits are mapped to  $M = 2^m$  waveforms, we term this  $M$ -ary encoding
- We will discuss specific waveforms next class
- For now assume that we use the mapping

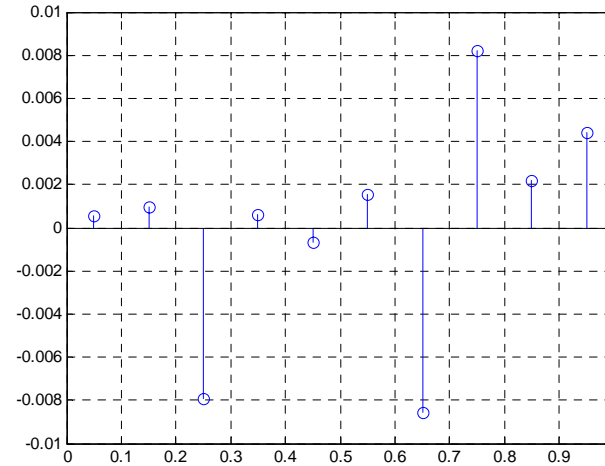


# PCM – Example

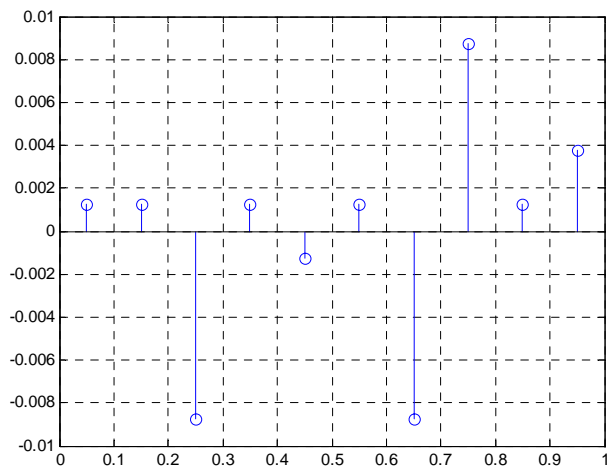
Original Signal



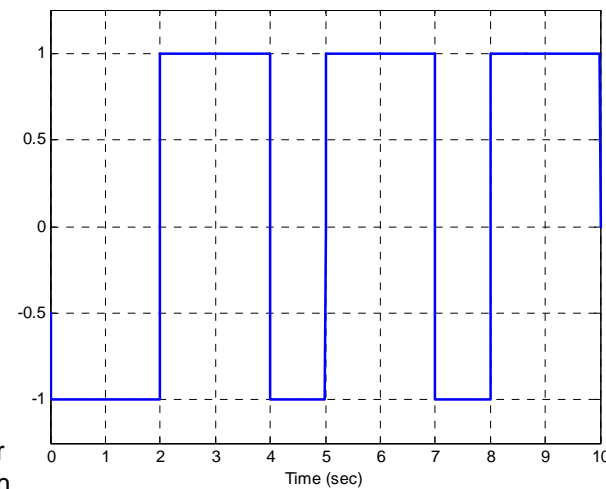
Sampled Signal



Quantized Signal



Resulting Digital Signal



# Bandwidth of PCM Signals



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- Sample rate:  $f_s$  samples/second.
- Bit rate out of the quantizer:

$$f_s \log_2 M = f_s \cdot n \text{ bits / second}$$

- Bandwidth of the resulting digital signal depends on the pulses used
- Minimum theoretical bandwidth (with optimal pulse shaping which we will discuss later):  $f_s \cdot n/2$  Hz
- First null bandwidth (with rectangular pulse waveforms):  
$$f_s \cdot n \text{ Hz}$$
- We'll talk more about the effects of pulse shaping in a few days

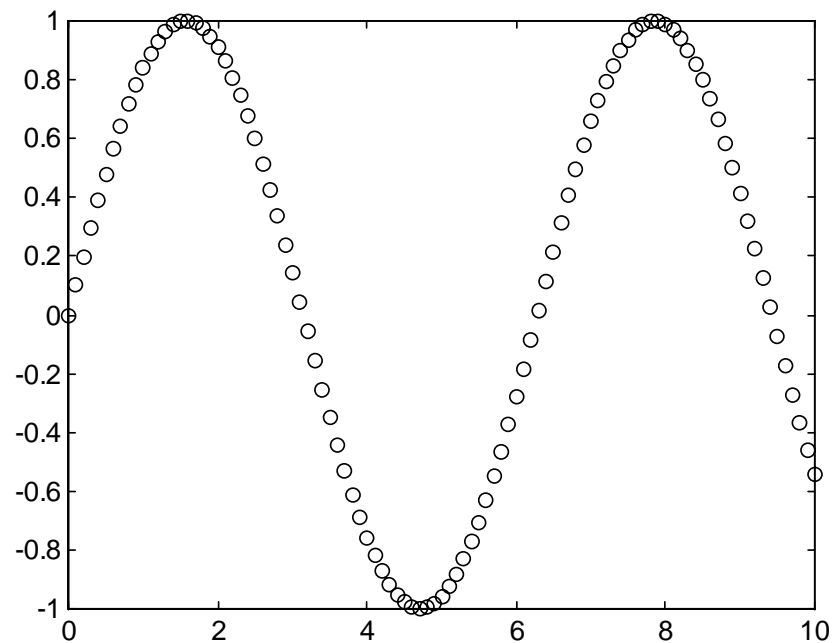
Note that the resulting bandwidth depends on the digital waveform (i.e., pulse shape) that is used.

# Digital Representation of Analog Signals



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- Sampling analog signals makes them discrete in time:

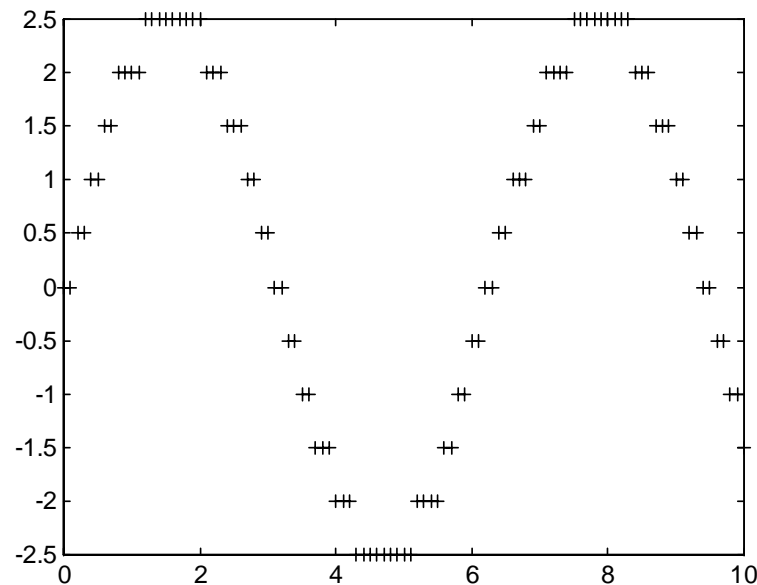


# Digital Representation of Analog Signals



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- Quantization of sampled analog signals makes the samples discrete in amplitude:



# Quantization



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- Continuous time signals are sampled at discrete time intervals
- Sampling may be performed without distortion provided signal is sampled at Nyquist rate
- Continuous-valued samples of data require an infinite # of bits to represent with perfect precision.
- Quantization is the process of approximating continuous-valued samples with a finite # of bits.
- Quantization always introduces some distortion.

# Notation Associated with Quantization



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- Let  $X$  be a random variable representing a sample of data.
- Then  $\tilde{X} = f_Q(X)$  is the quantized value of  $X$ .
- A quantizer has  $M$  quantization levels:

$$\tilde{X} \in \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_M\}$$

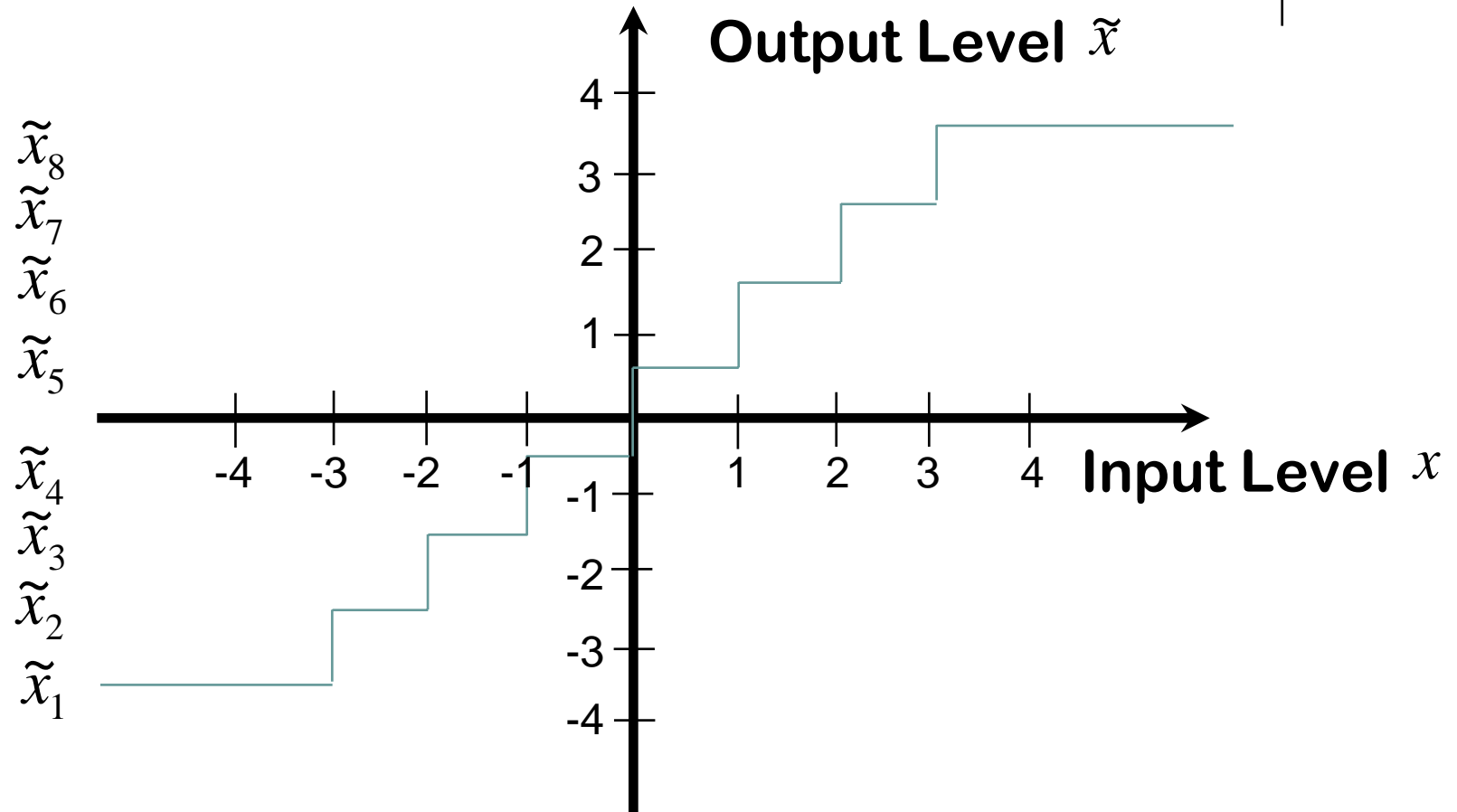
- The  $M$  levels correspond to  $M$  quantization regions.
- The endpoints of the quantization regions are specified by  $M + 1$  values:  $\{x_0, x_1, \dots, x_M\}$ , where  $x_0 = -\infty, x_M = \infty$
- Then:

$$x_{k-1} \leq x < x_k \Rightarrow \tilde{x} = f_Q(x) = \tilde{x}_k$$

# Graphical Description of Quantization



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# Table Representation of Quantizer



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$k$	$x_{k-1}$	$x_k$	$\tilde{x}_k$	Output Bits
1	$-\infty$	-3	-3.5	000
2	-3	-2	-2.5	001
3	-2	-1	-1.5	010
4	-1	0	-0.5	011
5	0	1	0.5	100
6	1	2	1.5	101
7	2	3	2.5	110
8	3	$\infty$	3.5	111

# Concise Representation of Quantizer



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- Usually, it is sufficient just to list the quantization levels.
- Example:  $\{-3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5\}$
- Why?
  - We assume that all points are quantized to the nearest quantization level
  - This determines where the borders of the quantization regions are
  - Any other borders would increase the error introduced by the quantizer

# Practical Methods for Implementing Analog to Digital Converters



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Lower Complexity

- Counting or Ramp ADC
  - test value is incremented in equal steps until it is greater than input sample
- Serial or Successive Approximation ADC
  - uses binary search to narrow range of input sample until desired accuracy is reached
- Parallel or Flash ADC
  - input sample is compared with all possible quantization levels at once

Faster

# Distortion



- Quantization introduces distortion into a signal.
- We want to minimize average distortion  $D$  , where

$$\begin{aligned} D &= E \left[ \left( X - \tilde{X} \right)^2 \right] \\ &= \int_{-\infty}^{\infty} (x - \tilde{x})^2 f(x) dx \\ &= \sum_{k=1}^M \int_{x_{k-1}}^{x_k} (x - \tilde{x}_k)^2 f(x) dx \end{aligned}$$

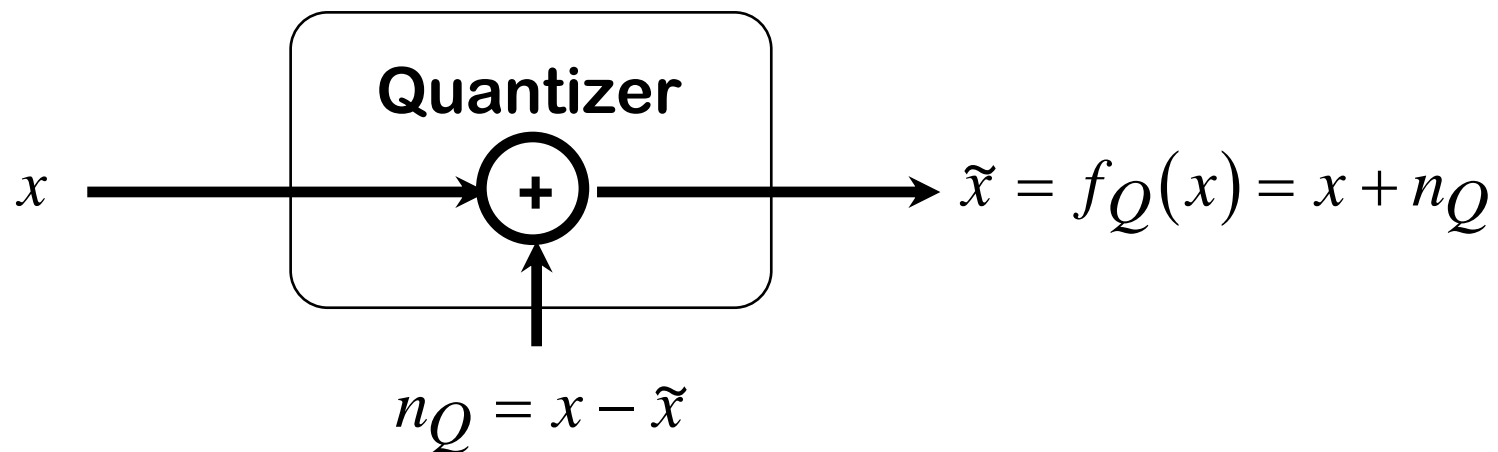
- This measure of distortion is sometimes also called mean square error (MSE)
- MSE penalizes large errors more than small errors

# Another Way of Viewing Quantization



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- Quantization adds a random “noise” to the true value of the sample point
- Then  $\text{MSE} = E[n_Q^2]$  may be thought of as noise power
- We can define a signal-to-noise ratio (SNR) to measure performance



# Signal to Noise Ratio Calculations for Quantizers



- Average SNR

$$\left(\frac{S}{N}\right)_{\text{avg}} = \frac{E[X^2]}{E[n_Q^2]} = \frac{E[X^2]}{D} = \frac{\int_{-\infty}^{\infty} x^2 f_X(x) dx}{\int_{-\infty}^{\infty} (x - \tilde{x})^2 f_X(x) dx}$$

- Sometimes we also define “peak” SNR
  - This is less important.

$$\left(\frac{S}{N}\right)_{\text{peak}} = \frac{X_{\text{peak}}^2}{E[n_Q^2]} = \frac{X_{\text{peak}}^2}{D}$$

# Example 7.1: SNR Calculation



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- Let:  $\{\tilde{x}_1 = -3.5, \tilde{x}_2 = -2.5, \tilde{x}_3 = -1.5, \tilde{x}_4 = -0.5, \tilde{x}_5 = 0.5,$   
 $\tilde{x}_6 = 1.5, \tilde{x}_7 = 2.5, \tilde{x}_8 = 3.5\}$

- Let:  $f(x) = \begin{cases} 1/8, & -4 \leq x \leq 4 \\ 0, & \text{else} \end{cases}$

$$E[X^2] = \int_{-4}^4 x^2 \cdot \frac{1}{8} dx = \frac{x^3}{24} \Big|_{-4}^4 = \frac{128}{24} = \frac{16}{3}$$

# Example 7.1 (continued)



- Distortion:

$$\begin{aligned} D = MSE &= \sum_{k=1}^M \int_{x_{k-1}}^{x_k} (x - \tilde{x}_k)^2 f_X(x) dx \\ &= \sum_{k=1}^8 \int_{-5+k}^{-4+k} (x - (-4.5 + k))^2 \cdot \frac{1}{8} dx \end{aligned}$$

- Examine the  $k = 5$  term:

$$\begin{aligned} \int_0^1 (x - 0.5)^2 \cdot \frac{1}{8} dx &= \frac{1}{8} \cdot \int_0^1 x^2 - x + 0.25 dx = \frac{1}{8} \cdot \left[ \frac{x^3}{3} - \frac{x^2}{2} + 0.25x \right]_0^1 \\ &= \frac{1}{8} \cdot \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right] = \frac{1}{8} \cdot \frac{1}{12} \end{aligned}$$

# Example 7.1 (continued)



- All 8 terms are identical.  $\therefore \text{MSE} = 1/12$

$$\left(\frac{S}{N}\right)_{\text{avg}} = 10 \log_{10} \frac{E[X^2]}{D} = 10 \log_{10} \left(\frac{16/3}{1/12}\right) = 18.1 \text{dB}$$

- A standard rule of thumb is that each additional bit adds 6 dB to the SNR of uniform quantizers operating on a uniform pdf.

# SNR for Uniform Quantization



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- General result: 
$$\left(\frac{S}{N}\right)_{avg} = M^2$$
  - Assumes uniform quantizer with  $M$  levels
  - Assumes that input samples have uniform distribution with identical range as quantizer
  - Though some texts do not make it clear, this result applies only when these special set of conditions hold. Otherwise, we have to use integral formula
- However, a useful Rule of Thumb:
  - Each additional bit (doubling  $M$ ) increases SNR by 6 dB

$$\left(\frac{S}{N}\right)_{avg} = 6.02n + \alpha$$

- Where  $\alpha$  depends on the distribution of the signal

# Example 7.2: PCM Calculation



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- **Problem:**

- Suppose that an analog music signal is found to have a bandwidth of 15 kHz and that samples of the signal may be modeled as having a uniform distribution.
- Find the minimum first-null bandwidth (assuming the use of square pulses) at which it would be possible to transmit a PCM signal while maintaining an average SNR of at least 58 dB.

# Example 7.2 (continued)



- **Solution:**

$$f_s \geq 2B = 30,000 \text{ samples / sec}$$

$$10 \log_{10} M^2 \geq 58 \text{dB} \Rightarrow M \geq 794 \Rightarrow M \geq 1024$$

$$\Rightarrow n \geq 10 \text{ bits/sample}$$

- We will assume  $n$  must be an integer, thus  $n = 10$
- Minimum data rate:  $R = f_s n \geq 300 \text{kbps}$
- First null BW:  $B = f_s n = 300 \text{kHz}$
- Assumes rectangular pulses are used as the waveforms

Note: Bandwidth comes into play twice here. Once for the original analog signal and once for the resulting digital signal.

# Non-Uniform Quantization



- A quantizer for which

$$\tilde{x}_{k-1} - \tilde{x}_k = \Delta, \forall k \in \{1, \dots, L-1\}$$

is called a uniform quantizer.

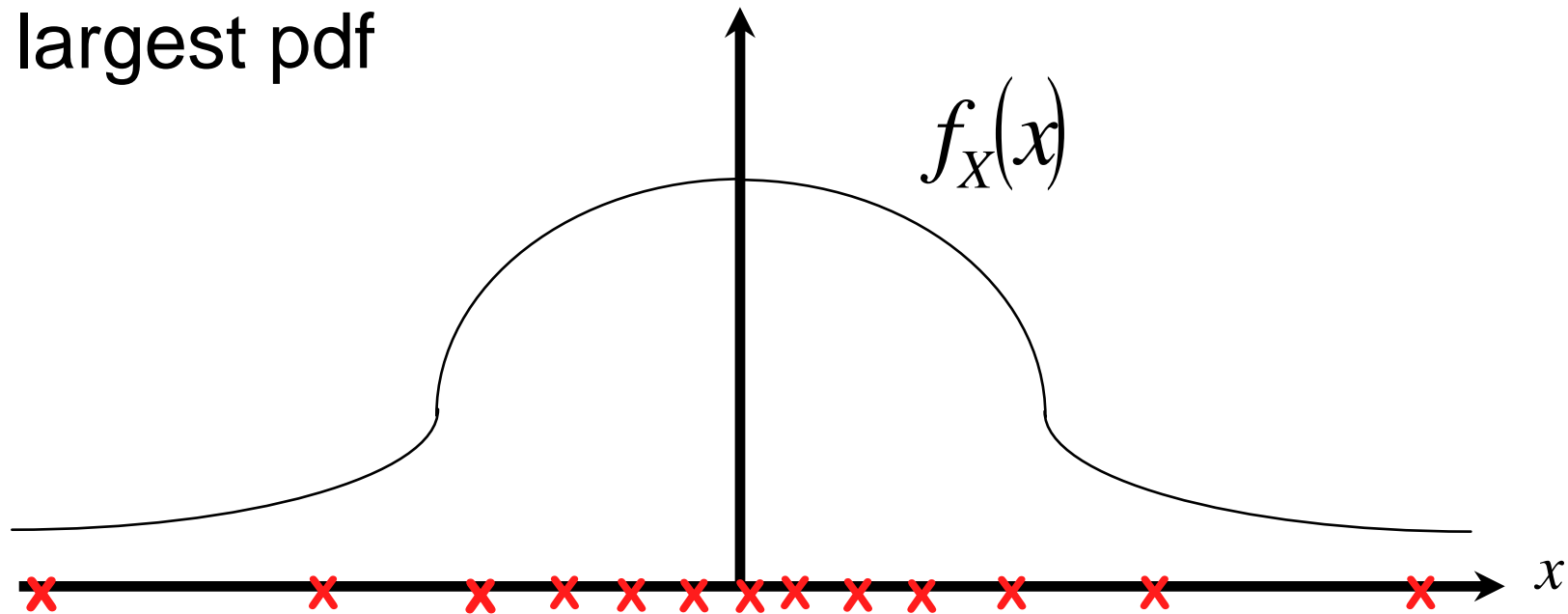
- It is sometimes better to use non-uniform spacing
- Examine the distortion measure:  $D = \int_{-\infty}^{\infty} (x - \tilde{x})^2 f(x) dx$
- We wish to make  $(x - \tilde{x})^2$  small when  $f(x)$  is large.
- We can accept larger  $(x - \tilde{x})^2$  when  $f(x)$  is small.
- **Basic principle:** concentrate quantization levels in area of largest pdf.

# Non-uniform Quantization



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- Concentrate quantization levels in areas of largest pdf



Quantization levels =  $x$

# Difference between Uniform and Non-uniform Quantization



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- Let:  $p(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$
- Let:  $\{\tilde{x}_1 = -1.494, \tilde{x}_2 = -0.498, \tilde{x}_3 = 0.498, \tilde{x}_4 = 1.494\}$
- Numerical integration shows that:

$$D = 0.1188, E[X^2] = 1, \left(\frac{S}{N}\right)_{avg} = 10 \log_{10} \left( \frac{1}{0.1188} \right) = 9.25dB$$

- Non-uniform quantization will yield better results
  - The “best” possible quantizer has

$$\left(\frac{S}{N}\right)_{avg} = 12dB$$

# Drill Problem

- If time....



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# Summary

- Today we have examined Pulse Code Modulation which converts an analog signal to a digital signal.
- Comprised of
  - Sampling
  - Quantization
  - Pulse modulation
- Next class we will examine additional techniques for converting analog signals to baseband digital signals