Digital Communications
Midterm Exam I
September 22, 2006

SOLUTION

I pledge that I have neither given nor received any assistance on this exam.

____
(signed)

______________________________ Name (print)

______________________________ Student Number

1. (20 points) Short Answer/Multiple Choice

(a) [5 points] The Fourier Transform of a periodic time-domain signal is

(a) ~~periodic in the frequency domain~~
(b) discrete in the frequency domain
(c) ~~continuous in the frequency~~ domain
(d) None of the above

(b) If a time-domain signal is integrated, then in the frequency domain

(a) high-frequency terms are emphasized
(b) ~~the signal is differentiated~~
(c) low-frequency terms are emphasized
(d) None of the above

(c) [5 points] Explain why companding improves the SNR of the PCM quantization process in the presence of a uniform quantizer.

A uniform quantizer is the optimal scalar quantizer for a uniformly distributed source. However, it does not necessarily provide the best SNR performance for sources with other distributions, e.g., voice signals. A compander distorts the distribution of a voice signal so that the uniform quantizer provides better SNR by spreading the distribution (i.e., making it closer to uniform).

(d) [5 points] Explain why restricting the bandwidth of a square pulse causes intersymbol interference.

A square pulse has an extremely sharp edge (i.e., the derivative is undefined at the end of the pulse). This requires a lot of high frequency content. By restricting the bandwidth we do not allow the signal to change as fast, thus causing pulses to smear into the neighboring pulses. Another way of looking at it is that bandwidth limits always cause a signal to lengthen in time, thus causing ISI.

2. (25 points) The Fourier Transform

$$x(t) = rect\left(\frac{t}{10}\right)\cos(200\pi t)$$

(a) [5 points] Is this an energy signal or a power signal?

The signal is limited in time and is clearly an energy signal.

(b) [15 points] Determine the autocorrelation function of *x(t)* (clearly state any assumptions made):

The easiest way to obtain the autocorrelation function is to use the frequency domain. This way, we can use tables and transforms. Specifically we use the relationship

$$R_x(\tau) \rightleftharpoons \psi_x(f)$$

First, we know that

$$F\left\{rect\left(\frac{t}{10}\right)\right\} = T\text{sinc}(10f)$$

Secondly, the modulation property states that

$$F\left\{x(t)\cos(2\pi f_o t)\right\} = \frac{1}{2}X(f - f_o) + \frac{1}{2}X(f + f_o)$$

Thus,

$$X(f) = \frac{10}{2}\text{sinc}\big(10(f-100)\big) + \frac{10}{2}\text{sinc}\big(10(f+100)\big)$$

Now the ESD is defined as

$$\psi_x(f) = \left|X(f)\right|^2$$

Thus, we have

$$\begin{aligned}\psi_x(f) &= \left|\frac{10}{2}\operatorname{sinc}\left(10(f-100)\right)+\frac{10}{2}\operatorname{sinc}\left(10(f+100)\right)\right|^2 \\ &= \left(\frac{10}{2}\operatorname{sinc}\left(10(f-100)\right)+\frac{10}{2}\operatorname{sinc}\left(10(f+100)\right)\right)\left(\frac{10}{2}\operatorname{sinc}\left(10(f-100)\right)+\frac{10}{2}\operatorname{sinc}\left(10(f+100)\right)\right)^* \\ &= \left(\frac{10}{2}\operatorname{sinc}\left(10(f-100)\right)+\frac{10}{2}\operatorname{sinc}\left(10(f+100)\right)\right)^2 \\ &= 25\operatorname{sinc}^2\left(10(f-100)\right)+50\operatorname{sinc}\left(10(f-100)\right)\operatorname{sinc}\left(10(f+100)\right)+25\operatorname{sinc}^2\left(10(f+100)\right)\end{aligned}$$

Now, assuming that $50\operatorname{sinc}\left(10(f-100)\right)\operatorname{sinc}\left(10(f+100)\right)\approx 0$ *we have*

$$\begin{aligned}\psi_x(f) &\approx 25\operatorname{sinc}^2\left(10(f-100)\right)+25\operatorname{sinc}^2\left(10(f+100)\right) \\ &= 50\operatorname{sinc}^2(10f)\otimes\left\{\frac{1}{2}\delta(f-100)+\frac{1}{2}\delta(f+100)\right\}\end{aligned}$$

Using the transform pair

$$\mathrm{T}\operatorname{sinc}^2(Tf) \rightleftharpoons tri\left(\frac{t}{T}\right) = \begin{cases} 1-\left|\dfrac{t}{T}\right| & |t|\le T \\ 0 & else\end{cases}$$

along with the modulation property we can write:

$$R_x(\tau)\approx 5tri\left(\frac{t}{10}\right)\cos(200\pi t)$$

(c) [5 points] Determine the null-to-null bandwidth of the signal.

The spectrum is defined by the sinc function centered at fc = 100Hz. The first nulls on either side of the center frequency occur when the argument of the sinc function equal +/- 1:

$$10\left(f_H - 100\right) = 1$$
$$f_H = 100.1Hz$$

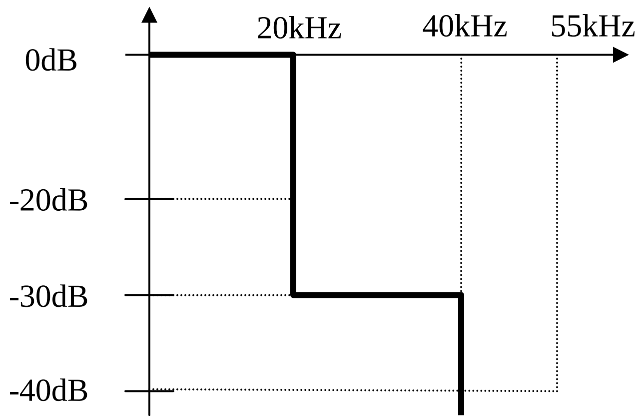
$$10\left(f_L - 100\right) = -1$$
$$f_L = 99.9Hz$$

Thus,

$$B = f_L - f_L = 0.2Hz$$

3. (15 points) Sampling

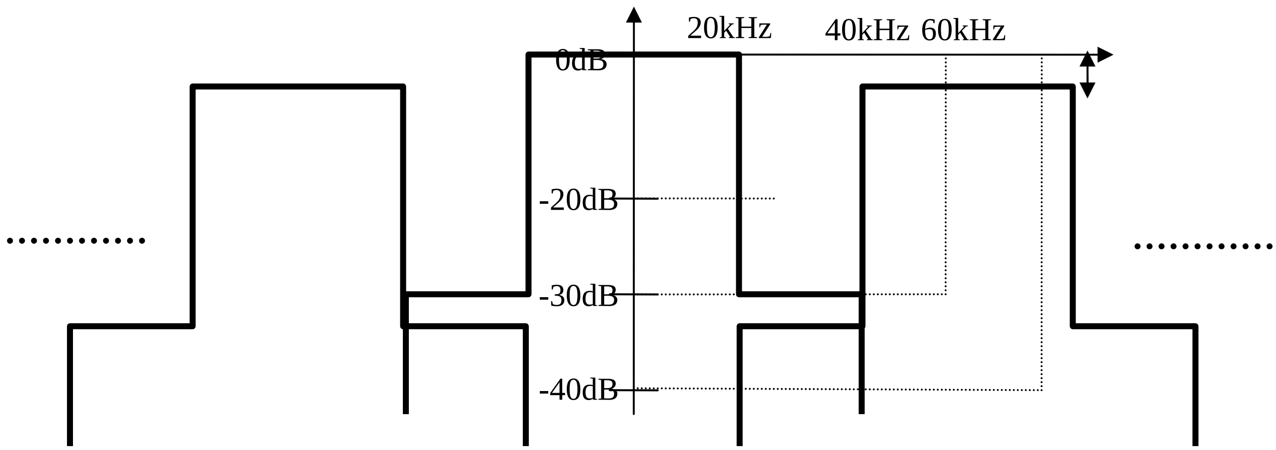
Consider a signal with the following power spectral density:



(a) (5 points) Assuming the use of PAM based on natural sampling (gating), what is the minimum pulse rate that will guarantee perfect reconstruction at the receiver? (Assume the use of an ideal reconstruction filter.)

The minimum sampling rate is the same as in ideal sampling, $f_s = 2B = 80kHz$.

(b) (10 points) Sketch the spectrum of the PAM signal if f_s = 60kHz and the pulse width is 1μs?



The spectrum is plotted above. The original spectrum is repeated at integer multiples of 60kHz with each replica reduced by the factor

$$\mathrm{sinc}\left(\frac{n\tau}{T_s}\right) = \mathrm{sinc}\left(\frac{n*10^{-6}}{1/6*10^{-4}}\right) = \mathrm{sinc}\left(n*6*10^{-2}\right)$$

For the first term, the factor is very close to one (0.994), thus the spectra are brought down by a fraction of a dB. Note that the plot is slightly exaggerated to show that the spectra are down from the original. Also, note that in gating there is no spectral distortion.

4. (40 points) PCM

(a) [10 points] A signal is to be sampled, quantized and sent across a channel with a bandwidth of 100kHz. The signal to be quantized has an analog bandwidth of 10kHz. Assuming that the pulse shape below is used and that the bandwidth of the pulse is measured as the 40dB bandwidth, determine the maximum SNR at the output of a uniform quantizer assuming that the source is uniformly distributed. Assume binary signaling.

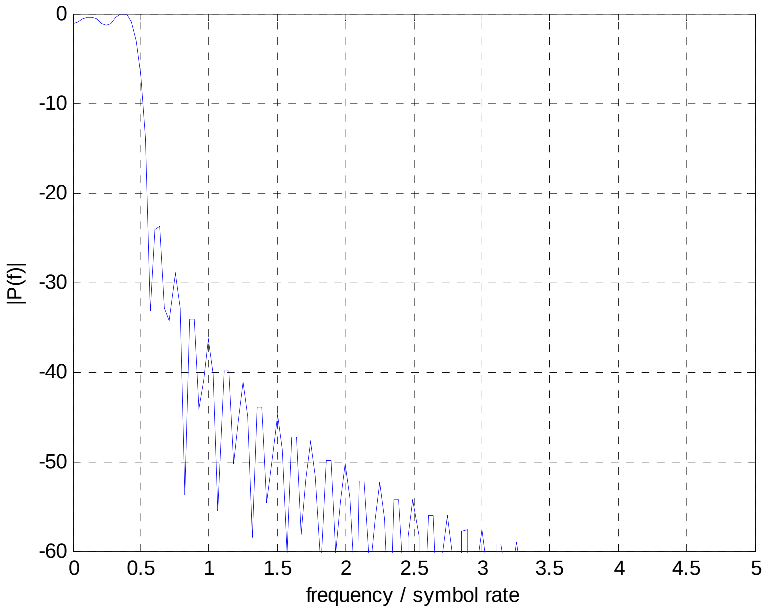


Figure 1: Energy Spectral Density of the Pulse Shape Used in Problem 4

For a uniform distribution and a uniform quantizer, we have

$$SNR(dB) = 6.02n$$

where n is the number of bits per sample.

From the plot, the 40dB bandwidth is approximately 1.2R_s. Since the toal available bandwidth is 100kHz, the allowable symbol rate is

$$1.2R_s \leq 100kHz$$
$$R_s \leq 83.3kHz$$

Since we have binary signaling, R_b = R_s and thus using Nyquist sampling, we have

$$R_s = nf_s \le 83.3kHz$$
$$n \le \frac{83.3kHz}{20kHz} = 4.2$$
$$n = 4$$

Thus, the SNR is

$$SNR(dB) = 6.02n$$
$$= 24.1dB$$

(b) [10 points] What is the SNR if the signal isn't uniformly distributed, but instead is a voice signal with a peak-to-average ratio of 15dB?

If the signal is a voice signal we have the formula for a uniform quantizer

$$SNR(dB) = 6.02n + 4.77 - 20\log\left(\frac{V_{peak}}{V_{rms}}\right)$$
$$= 24.1dB + 4.77 - 15$$
$$= 13.9dB$$

(c) [10 points] Could we do better with a delta modulation as opposed to PCM for the signal described in part (b)? Assume that Δ is chosen to avoid slope overload.

For a Delta Modulated system we have with Δ chosen to avoid slope overload

$$SNR = \frac{3f_s^3}{(1600\pi)^2 B}\left(\frac{V_{rms}}{V_{peak}}\right)^2$$

Now since we use one bit per sample we have

$$R_s = nf_s \le 83.3kHz$$
$$f_s \le 83.3kHz$$

Further, using the bandwidth of 10kHz and converting the peak-to-average ratio to linear: 10log$_{10}$*(15) = 31.6*

$$
\begin{aligned}
SNR &= \frac{3(83300)^3}{(1600\pi)^2 10000}\frac{1}{31.6} \\
&= 217.2 \\
&= 23.4dB
\end{aligned}
$$

(d) [10 points] Compare the first-null BW of a PCM system using square pulses and a unipolar return-to-zero line code with a delta modulation system with the same line code if the required SNR for both systems is 60dB. (Assume the signal defined in part (b) and assume that the step size is chosen to avoid overload noise).

For unipolar RZ signaling, the first null is 2Rs. Thus for PCM we have

$$
\begin{aligned}
BW &= 2R_s = 2nf_s \\
&= 2n20kHz
\end{aligned}
$$

The SNR can be written as

$$
\begin{aligned}
SNR(dB) &= 6.02n + 4.77 - 20\log\left(\frac{V_{peak}}{V_{rms}}\right) \\
&= 6.02n - 10.23
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
60dB &\le 6.02n - 10.23 \\
n &\ge 11.7 \\
n &= 12
\end{aligned}
$$

The bandwidth is then

$$
\begin{aligned}
BW &= 2n20kHz \\
&= 480kHz
\end{aligned}
$$

For Delta Modulation

$$SNR = \frac{3 f_s^3}{(1600\pi)^2 B}\left(\frac{V_{rms}}{V_{peak}}\right)^2$$
$$10^6 \le \frac{3 f_s^3}{(1600\pi)^2 10000}\frac{1}{31.6}$$
$$f_s \ge \sqrt[3]{(1600\pi)^2 10000 * 31.6/3}$$
$$f_s \ge 1.4 Msps$$

*With B = 2*f_s *we have*

B = 2.8MHz