EE 5654 - Digital Communications - Spring 2004
Homework 1
Due Tuesday 1/27/04
Solution

Problem 2-7
From the definition of the characteristic function we know that

$$\varphi(jv_1, jv_2 jv_3 jv_4) = E\left[e^{j(v_1x_1+v_2x_2+v_3x_3+v_4x_4)}\right]$$

Further, from (2.1-83) we make the extension:

$$E\left[X_1X_2X_3X_4\right] = (-j)^4 \left.\frac{\partial^4\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1\partial v_2\partial v_3\partial v_4}\right|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given random variables we know that the characteristic function is

$$\varphi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v'Mv}}$$

where $\mathbf{v} = [v_1, v_2, v_3, v_4]^T$ and $\mathbf{M}_{ij} = \mu_{ij}$. We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that:

$$\frac{d\varphi(j\mathbf{v})}{dv_i} = -\boldsymbol{\mu}^T{}_i\mathbf{v}e^{-\frac{1}{2}\mathbf{v'Mv}}$$

where $\boldsymbol{\mu}_i = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]^T$. Also note that

$$\frac{d\boldsymbol{\mu}_j^T\mathbf{v}}{dv_i} = \mu_{ij} = \mu_{ji}$$

Taking the first partial derivative:

$$\frac{\partial\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1} = \frac{\partial}{\partial v_1}\left\{e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

$$= -\boldsymbol{\mu}_1^T\mathbf{v}e^{-\frac{1}{2}\mathbf{v'Mv}}$$

Proceeding to the second partial derivative:

$$\frac{\partial\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1 \partial v_2} = \frac{\partial}{\partial v_2}\left\{-\boldsymbol{\mu}_1^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

$$= -\mu_{12} e^{-\frac{1}{2}\mathbf{v'Mv}} - \boldsymbol{\mu}_1^T \mathbf{v} \frac{\partial}{\partial v_2}\left\{e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

$$= -\mu_{12} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_1^T \mathbf{v} \boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}$$

The next step is to take the third partial derivative:

$$\frac{\partial\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1 \partial v_2 \partial v_3} = \frac{\partial}{\partial v_3}\left\{-\mu_{12} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_1^T \mathbf{v} \boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

$$= \mu_{12}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \mu_{13}\boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_1^T \mathbf{v} \frac{\partial}{\partial v_3}\left\{\boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

$$= \mu_{12}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \mu_{13}\boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_1^T \mathbf{v} \left\{\mu_{23} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_2^T \mathbf{v}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

Finally we take the last partial derivative:

$$\frac{\partial\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} = \frac{\partial}{\partial v_4}\left\{\mu_{12}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \mu_{13}\boldsymbol{\mu}_2^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_1^T \mathbf{v} \left\{\mu_{23} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_2^T \mathbf{v}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}\right\}$$

$$= \mu_{12}\left(\mu_{34} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_3^T \mathbf{v}\boldsymbol{\mu}_4^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right) + \mu_{13}\left(\mu_{24} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_2^T \mathbf{v}\boldsymbol{\mu}_4^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right)\ldots$$

$$+\mu_{14}\left\{\mu_{23} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_2^T \mathbf{v}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\} + \boldsymbol{\mu}_1^T \mathbf{v} \frac{\partial}{\partial v_4}\left\{\mu_{23} e^{-\frac{1}{2}\mathbf{v'Mv}} + \boldsymbol{\mu}_2^T \mathbf{v}\boldsymbol{\mu}_3^T \mathbf{v} e^{-\frac{1}{2}\mathbf{v'Mv}}\right\}$$

Where we didn't bother to finish taking the derivative of the last term since it will be multiplied by zero. Hence,

$$\left.\frac{\partial^4\varphi(jv_1, jv_2 jv_3 jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}\right|_{v_1=v_2=v_3=v_4=0} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Problem 2-14

We wish to find $R_{YY}(\tau)$ for stationary real normal process Y(t) in terms of $R_{XX}(\tau)$ where $Y(t) = X^2(t)$.

From 2-7 we know that

$$E\left[X_1X_2X_3X_4\right] = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Thus, defining $X_1 = X_2 = X(t)$, $X_3 = X_4 = X(t+\tau)$,

$$\begin{aligned}
E\left[Y(t)Y(t+\tau)\right] &= E\left[X(t)X(t)X(t+\tau)X(t+\tau)\right] \\
&= E\left[X_1X_2X_3X_4\right] \\
&= \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13} \\
&= R_{XX}(0)R_{XX}(0) + R_{XX}(\tau)R_{XX}(\tau) + R_{XX}(\tau)R_{XX}(\tau) \\
&= R_{XX}^2(0) + 2R_{XX}^2(\tau)
\end{aligned}$$

Problem 2-20

A discrete random process has an autocorrelation function

$$\phi(k) = \left(\frac{1}{2}\right)^{|k|}$$

The power spectral density is found as

$$\begin{aligned}
\Phi(f) &= \sum_{k=-\infty}^{\infty} \phi(k) e^{-j2\pi fk} \\
&= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} e^{-j2\pi fk} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{-j2\pi fk} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{j2\pi fk} - 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{-j2\pi fk} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j2\pi f}\right)^{k} + \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{-j2\pi f}\right)^{k} - 1 \\
&= \frac{1}{1 - \frac{1}{2} e^{j2\pi f}} + \frac{1}{1 - \frac{1}{2} e^{-j2\pi f}} - 1 \\
&= \frac{1 - \frac{1}{2} e^{-j2\pi f} + 1 - \frac{1}{2} e^{j2\pi f}}{\left(1 - \frac{1}{2} e^{-j2\pi f}\right)\left(1 - \frac{1}{2} e^{j2\pi f}\right)} - 1 \\
&= \frac{2 - \cos(2\pi f)}{5/4 - \cos(2\pi f)} - 1 \\
&= \frac{3}{5 - 4\cos(2\pi f)}
\end{aligned}$$

Problem 4

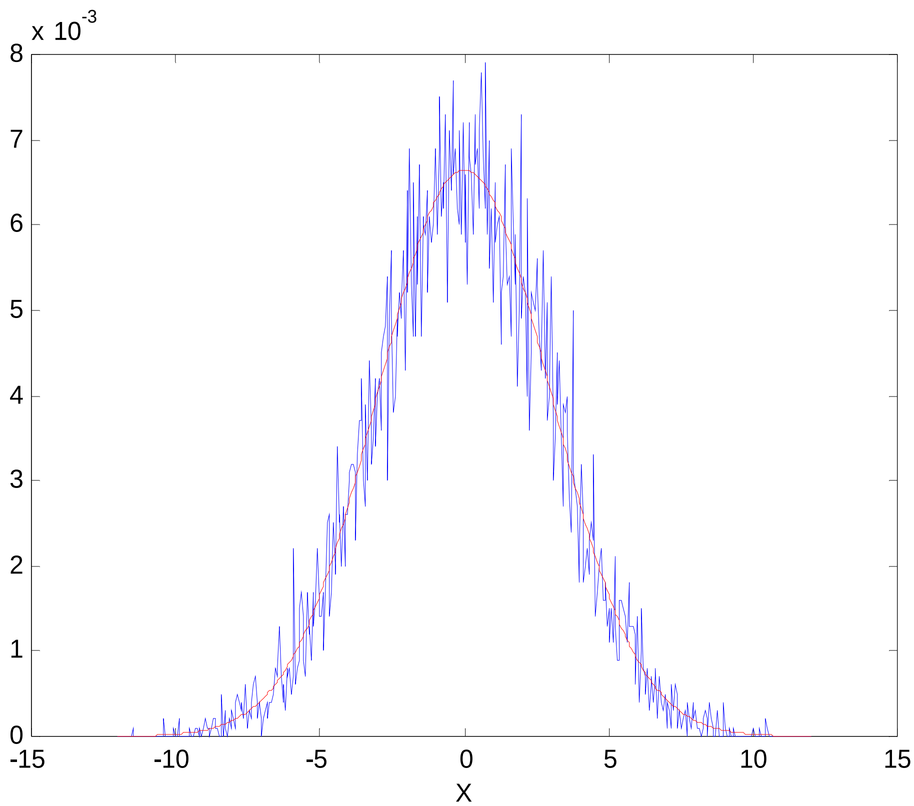
To generate 10,000 instances of a zero mean Gaussian random variable with variance of 9 in Matlab, we use the command

```
x = 3*randn(1,10000);
```

since `randn` returns zero mean GRV's with a variance of 1 and multiplying a zero mean GRV by c creates a GRV with variance c^2.

Below is a normalized histogram of the data along with the theoretical pdf of $N(0,3)$. The commands used were

```
range = -15:0.01:15;
h = hist(x, range);
h = h/sum(h);
plot(range, h);
thy = 1/sqrt(2*pi*9)*exp(-1/(2*9)*range.^2);
hold on;
plot(range, thy*0.01, 'r');
```



Problem 5

Since $\Sigma_{XY} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ the GRV's are independent and have variance of 9. We use the same procedure to generate 2 GRV's from above

```
x = 3*randn(2,10000);
```

We create Z by

```
z = sqrt(x(1,:).^2 + x(2,:).^2);
```

The resulting random variable z is a Rayleigh random variable. We plot the histogram in the same manner as above: