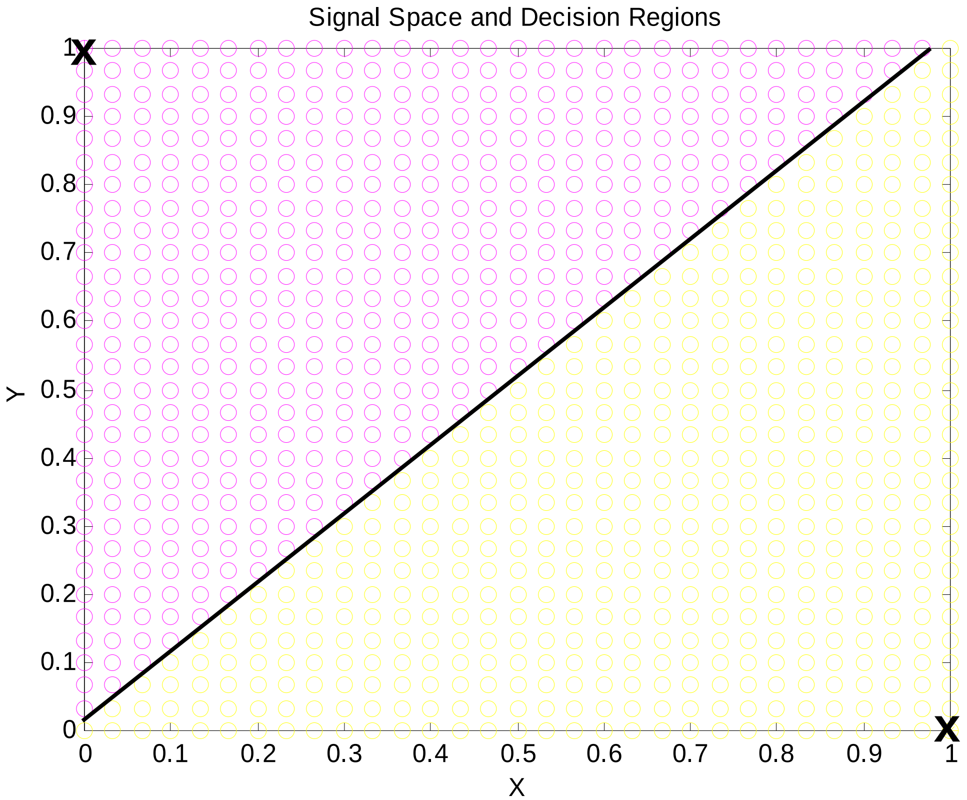
EE 5654 - Digital Communications - Spring 2005
Homework 2
Solution
Due Thursday 2/10/05

1. How do the decision regions for coherent BFSK change for unequal probabilities (0.9, 0.1) and an E_b/N_o = 5dB? Use sigspace.m.

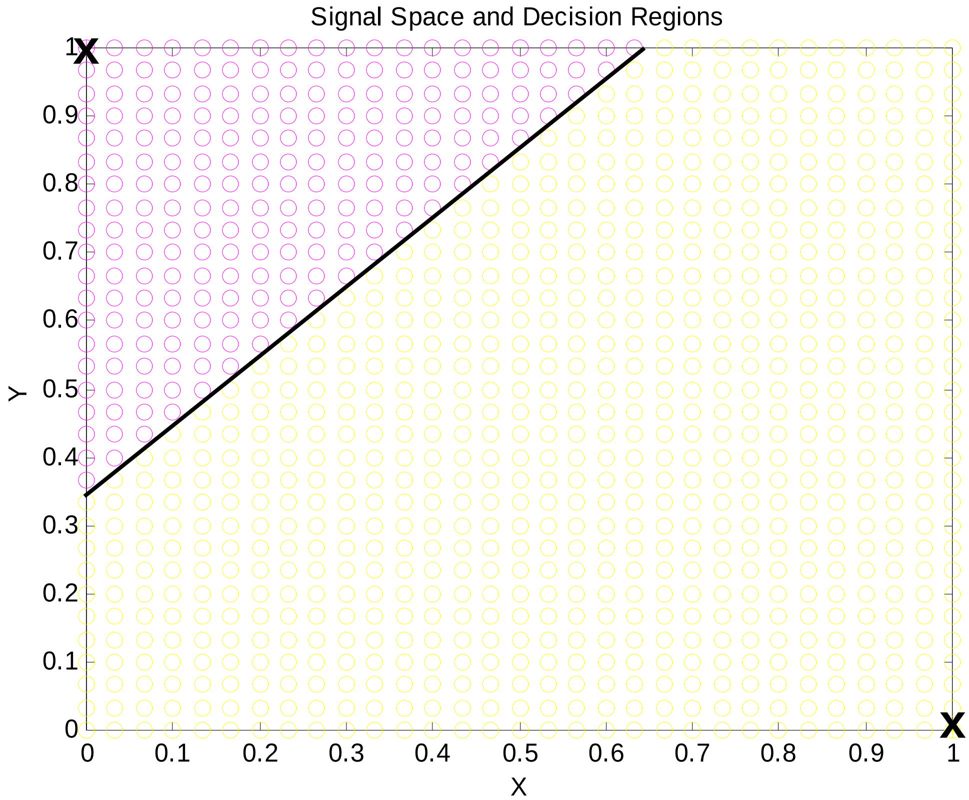
The signal space or vector representation of BFSK is

$$\mathbf{s}_1 = \sqrt{E_b}\,\begin{bmatrix}1 & 0\end{bmatrix}$$
$$\mathbf{s}_2 = \sqrt{E_b}\,\begin{bmatrix}0 & 1\end{bmatrix}$$

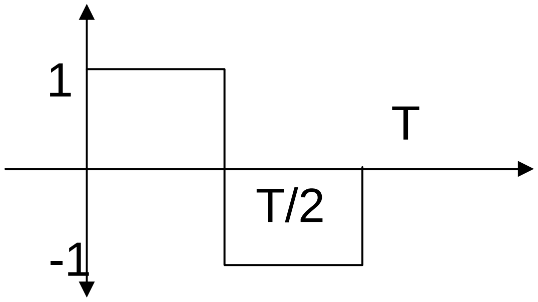
The decision regions are thus determined by the perpendicular bisector of the line connecting these two points as shown below [`sigspace([1 0; 0 1], 5)`]:

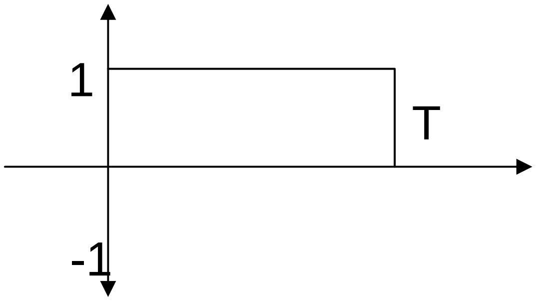


Changing the a priori probabilities has the effect of moving the decision boundary towards s_2 since s_1 is more probable and has a correspondingly larger decision region. Note that the boundary is still perpendicular to the line connecting the two points: below [`sigspace([1 0 0.9; 0 1 0.1], 5)`]:


Signal Space and Decision Regions
Y
X
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X
X

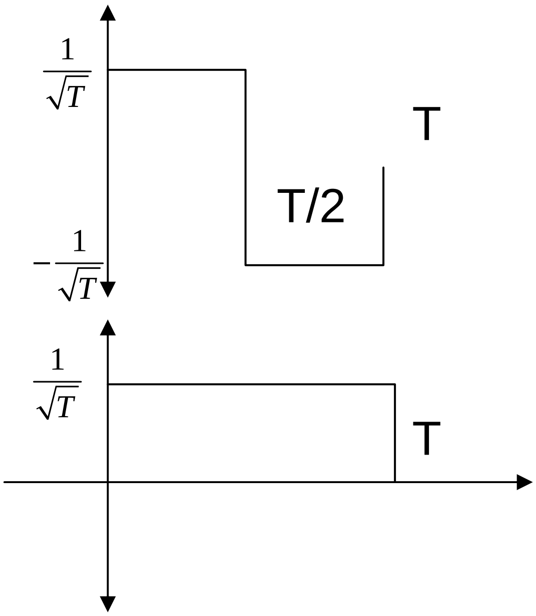
2. Assume that binary data is transmitted via two waveforms:





- (a) Determine basis functions for this signal set.
- (b) What is the dimensionality of the signal set?
- (c) Draw the reduced complexity optimal receiver.
- (d) Draw the resulting constellation diagram.
- (e) Draw the decision boundaries assuming that $p_1 = p_2$.
- (f) Determine the probability of error.

(a) We can solve this part in one of two ways. Either we can use Gramm-Schmidt or we can use inspection. Using Gramm-Schmidt we find that the two signals are orthogonal and thus the basis functions are:



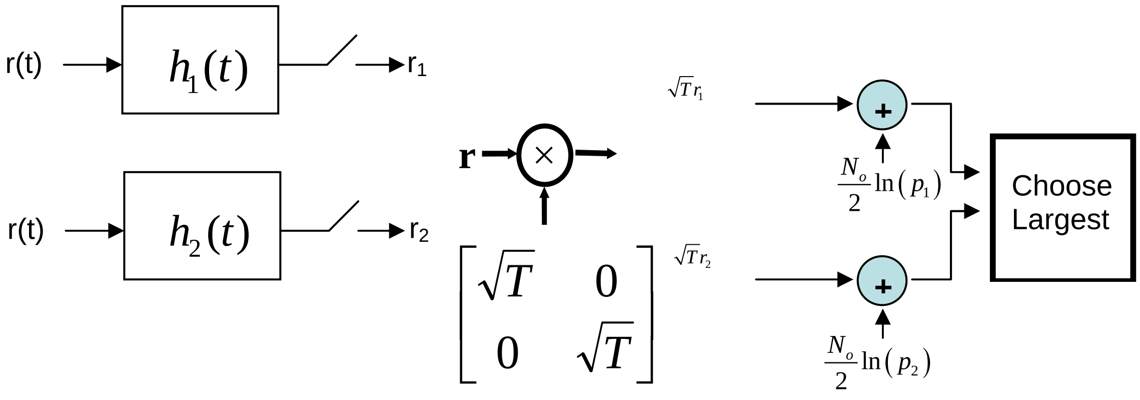
(b) The dimensionality of the signal set is 2 and the vectors are

$$\mathbf{s}_1 = \sqrt{T}\begin{bmatrix}1 & 0\end{bmatrix}$$
$$\mathbf{s}_2 = \sqrt{T}\begin{bmatrix}0 & 1\end{bmatrix}$$

In terms of energy per symbol E_s we have

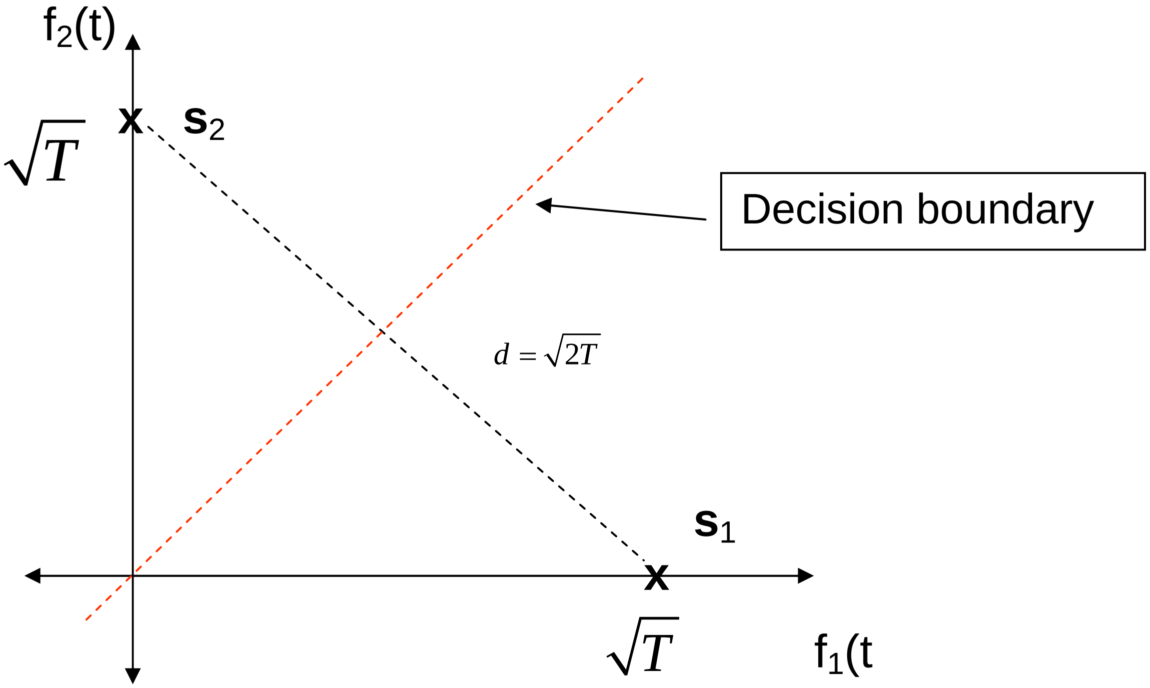
$$\mathbf{s}_1 = \sqrt{E_s}\begin{bmatrix}1 & 0\end{bmatrix}$$
$$\mathbf{s}_2 = \sqrt{E_s}\begin{bmatrix}0 & 1\end{bmatrix}$$

(c) The optimal receiver can be implemented as:



Where $h_1(t)$ and $h_2(t)$ are the impulse responses of two filters and are equal to time reversal of the basis functions. Note that since we have two dimensions and two symbols we could also simply use a correlator receiver with no additional complexity.

(d) From (b) we can draw the constellation diagram as



(e) The decision boundary is the perpendicular bisector of the line connecting the two points since it represents the line of points equidistant from $\mathbf{s}_1$ and $\mathbf{s}_2$.

(f) The probability of error is quite simply found as

$$P_s(e) = Q\left(\frac{d}{\sqrt{2N_o}}\right)$$
$$= Q\left(\frac{\sqrt{2T}}{\sqrt{2N_o}}\right)$$

Now, we can find the energy per bit as $E_s = 1*T$ which is the same as the energy per bit E_b since there is one symbol per bit. Thus,

$$\begin{aligned} P_s(e) &= Q\left(\sqrt{\frac{T}{N_o}}\right) \\ &= Q\left(\sqrt{\frac{E_b}{N_o}}\right) \end{aligned}$$

3. Show that the ML receiver is equivalent to a receiver which maps received signals to the closest signal point.

The MAP receiver uses the decision rule

$$\begin{aligned}\hat{\mathbf{s}} &= \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\left\{p(\mathbf{s}_m)\,p(\mathbf{r}|\mathbf{s}_m)\right\}\\ &= \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\left\{p(\mathbf{s}_m)(\pi N_0)^{-K/2}\exp\left(-\sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2 \Big/ N_0\right)\right\}\\ &= \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2\end{aligned}$$

The Maximum Likelihood (ML) receiver assumes that all symbols are equally likely *a priori* and thus the decision rule is:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \quad \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2$$

This can be re-written as the minimization of:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\min} \quad -2\sum_{k=1}^{K} r_k s_{m,k} + \sum_{k=1}^{K} s_{m,k}{}^2$$

Now, we can add the received energy since it will not affect the decision:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\min} \quad \sum_{k=1}^{K} r_k^2 - 2\sum_{k=1}^{K} r_k s_{m,k} + \sum_{k=1}^{K} s_{m,k}{}^2$$

Simplifying gives us

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\min} \quad \sum_{k=1}^{K}\left(r_k^2 - 2r_k s_{m,k} + s_{m,k}{}^2\right)$$

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\min} \quad \sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

Thus, we are choosing the symbol **s** which is closest (i.e., minimum Euclidean distance) to the received signal vector.

QED

~~otherwise it decides in favor of s_1. The decision rule may be expressed as:~~

$$\frac{\text{PM}(\mathbf{r},\mathbf{s}_0)}{\text{PM}(\mathbf{r},\mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2-r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

~~or equivalently :~~

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

~~The optimum threshold is $\frac{1}{2}A\sqrt{T}$.~~

~~**(b)** The average probability of error is:~~

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\text{SNR}}\right]
\end{aligned}$$

~~where~~

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

~~Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.~~

Problem 5.5 :

Since $\{f_n(t)\}$ constitute an orthonormal basis for the signal space : $r(t) = \sum_{n=1}^{N} r_n f_n(t),\ s_m(t) =$

$\sum_{n=1}^{N} s_{mn} f_n(t)$. Hence, for any m :

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= 2\int_0^T r(t)s_m(t)dt - \int_0^T s_m^2(t)dt \\ &= 2\int_0^T \sum_{n=1}^{N} r_n f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t)dt - \int_0^T \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t)dt \\ &= 2\sum_{n=1}^{N} r_n \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t)dt - \sum_{n=1}^{N} s_{mn} \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t)dt \\ &= 2\sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2 \end{aligned}$$

where we have exploited the orthonormality of $\{f_n(t)\}$: $\int_0^T f_n(t)f_l(t)dt = \delta_{nl}$. The last form is indeed the original form of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$.

Problem 5.6 :

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E\left[|n(T)|^2\right]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{aligned} E\left[|n(T)|^2\right] &= \int_0^T \int_0^T E\left[z(a)z^*(b)\right] h_l(T-a)h_l^*(T-b)dadb \\ &= 2N_0 \int_0^T |h_l(T-t)|^2 dt \end{aligned}$$

so we want to maximize :

$$SNR = \frac{\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2}{2N_0 \int_0^T |h_l(T-t)|^2 dt}$$

From Schwartz inequality :

$$\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt$$

Hence :

$$SNR \leq \frac{1}{2N_0} \int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$