1. In this problem you will determine the impact of a constant phase offset to BPSK modulation. Specifically, derive the probability of error for the optimal receiver with equally likely symbols when there is a constant phase offset θ_o. The calculation should assume that the standard decision rule is applied to the received signal (i.e., the receiver does not take into account the constant phase offset.)

For BPSK there are two possible received symbols (including the phase offset):

$$s_1(t) = \sqrt{2P}\cos\left(2\pi f_c t + \theta_o\right)\Big|_0^T$$
$$s_2(t) = -\sqrt{2P}\cos\left(2\pi f_c t + \theta_o\right)\Big|_0^T$$

However, the receiver is not aware of the constant offset and thus assumes a single basis function:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos\left(2\pi f_c t\right)\Big|_0^T$$

The optimal receiver correlates the received signal with the single basis function and compares the result with the threshold zero. The received signal conditioned on the two possible symbols (assuming additive white Gaussian noise) is:

$$\mathbf{r}|\mathbf{s}_1 = \sqrt{E_b}\cos\left(\theta_o\right) + \mathbf{n}$$
$$\mathbf{r}|\mathbf{s}_2 = -\sqrt{E_b}\cos\left(\theta_o\right) + \mathbf{n}$$

where $\mathbf{n}$ *is a Gaussian random variable with zero mean and variance* $\sigma^2 = N_o/2$. *If we compare the* $\mathbf{r}$ *with zero, we have a probability of error (assuming equally likely symbols) as*

$$P_s(e) = \frac{1}{2}\Pr\left[\hat{\mathbf{s}} = \mathbf{s}_1 \middle| \mathbf{s} = \mathbf{s}_2\right] + \frac{1}{2}\Pr\left[\hat{\mathbf{s}} = \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_1\right]$$

$$\Pr\left[\hat{\mathbf{s}} = \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_1\right] = \Pr\left[\mathbf{r} < 0 \middle| \mathbf{s} = \mathbf{s}_1\right]$$

If $|\theta_o| < \pi/2$ *we have*

$$\Pr\left[\mathbf{r} < 0 \middle| \mathbf{s} = \mathbf{s}_1\right] = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(r - \sqrt{E_b}\cos\theta_o\right)^2}{N_0}\right) dr$$
$$= Q\left(\sqrt{\frac{2E_b}{N_o}}\cos\theta_o\right)$$

If $|\theta_o| < \pi/2$ we have

$$\Pr\left[\mathbf{r} < 0 \middle| \mathbf{s} = \mathbf{s}_1\right] = 1 - Q\left(\sqrt{\frac{2E_b}{N_o}}\left|\cos\theta_o\right|\right)$$

By symmetry, the same holds for symbol 2.

2. Repeat the problem for QPSK modulation.

For QPSK, we have four symbols:

$$s_1(t) = \sqrt{2P}\cos\left(2\pi f_c t + \frac{\pi}{4} + \theta_o\right)\Big|_0^T$$

$$s_2(t) = \sqrt{2P}\cos\left(2\pi f_c t + \frac{3\pi}{4} + \theta_o\right)\Big|_0^T$$

$$s_3(t) = \sqrt{2P}\cos\left(2\pi f_c t - \frac{\pi}{4} + \theta_o\right)\Big|_0^T$$

$$s_4(t) = \sqrt{2P}\cos\left(2\pi f_c t - \frac{3\pi}{4} + \theta_o\right)\Big|_0^T$$

For QPSK, since the symbols are symmetric, the average probability of symbol error is equal to the probability of error given that symbol 1 was sent. Assuming that symbol 1 was sent we have using the improved Union Bound:

$$\begin{aligned} P_s(e) &\approx \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_1\right] + \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_4 \middle| \mathbf{s} = \mathbf{s}_1\right] \\ &\approx Q\left(\frac{2(d_1/2)}{\sqrt{2N_o}}\right) + Q\left(\frac{2(d_2/2)}{\sqrt{2N_o}}\right) \end{aligned}$$

where d_1 and d_2 are the distances to the respective decision boundaries as shown in Figure 1. Solving for the distances we have:

$$\begin{aligned} d_1/2 &= \sqrt{E_s}\cos\left(\frac{\pi}{4} + \theta_o\right) \\ &= \sqrt{E_s}\left\{\cos\left(\frac{\pi}{4}\right)\cos(\theta_o) - \sin\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\} \\ d_2/2 &= \sqrt{E_s}\sin\left(\frac{\pi}{4} + \theta_o\right) \\ &= \sqrt{E_s}\left\{\sin\left(\frac{\pi}{4}\right)\cos(\theta_o) + \cos\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\} \end{aligned}$$

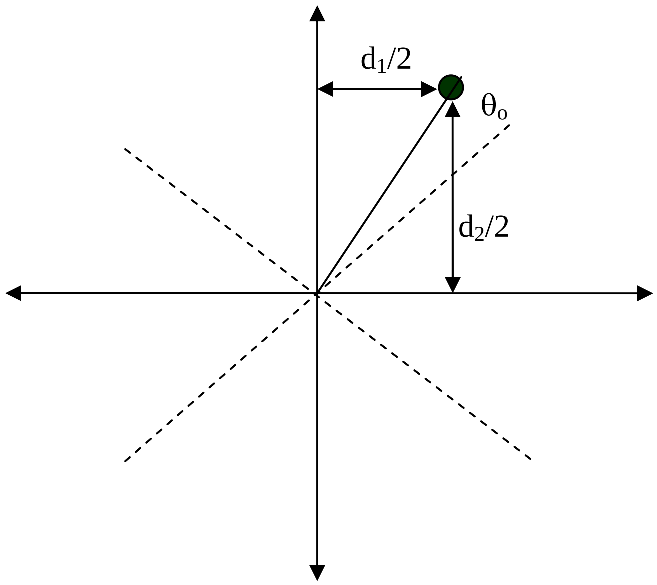

Figure 1: Location of Symbol 1 after Phase rotation due to θ_o

Plugging the values for d_1 and d_2 into the equation we can obtain the probability of error:

$$P_s(e) \approx Q\left(\frac{2(d_1/2)}{\sqrt{2N_o}}\right) + Q\left(\frac{2(d_2/2)}{\sqrt{2N_o}}\right)$$

$$= Q\left(\frac{2\left(\left(\sqrt{E_s}\left\{\cos\left(\frac{\pi}{4}\right)\cos(\theta_o) - \sin\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\}\right)\right)}{\sqrt{2N_o}}\right) + \ldots$$

$$Q\left(\frac{2\left(\left(\sqrt{E_s}\left\{\sin\left(\frac{\pi}{4}\right)\cos(\theta_o) + \cos\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\}\right)\right)}{\sqrt{2N_o}}\right)$$

$$= Q\left(\sqrt{\frac{4E_b\left\{\cos\left(\frac{\pi}{4}\right)\cos(\theta_o) - \sin\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\}^2}{N_o}}\right) + \ldots$$

$$Q\left(\sqrt{\frac{4E_b\left\{\sin\left(\frac{\pi}{4}\right)\cos(\theta_o) + \cos\left(\frac{\pi}{4}\right)\sin(\theta_o)\right\}^2}{N_o}}\right)$$

Note that if $\theta_o = 0$, the probability of error collapses to

$$P_s(e) = 2Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

as expected. Note also that we have assumed that $\theta_o < \pi/4$. Otherwise the probability of error is nearly 50%.

3. Repeat problem #1 for M-PSK

We can use the same procedure for M-PSK:

$$P_s(e) \approx \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_1\right] + \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_M \middle| \mathbf{s} = \mathbf{s}_1\right]$$
$$\approx Q\left(\frac{2(d_1/2)}{\sqrt{2N_o}}\right) + Q\left(\frac{2(d_2/2)}{\sqrt{2N_o}}\right)$$

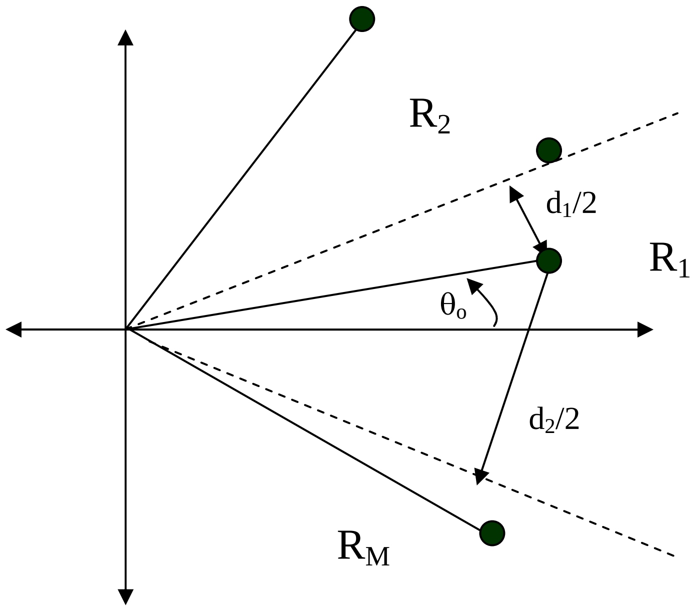

Figure 2: *M-PSK*

From the Figure 2, we can calculate $d_1/2$ and $d_2/2$ as

$$d_1/2 = \sqrt{E_s}\sin\left(\frac{\pi}{M} - \theta_o\right)$$
$$d_2/2 = \sqrt{E_s}\sin\left(\frac{\pi}{M} + \theta_o\right)$$

Thus, we have

$$
\begin{aligned}
P_s(e) &\approx \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_1\right] + \Pr\left[\hat{\mathbf{s}} = \mathbf{s}_M \middle| \mathbf{s} = \mathbf{s}_1\right] \\
&= Q\left(\frac{2\left(\sqrt{E_s}\sin\left(\frac{\pi}{M} - \theta_o\right)\right)}{\sqrt{2N_o}}\right) + Q\left(\frac{2\left(\sqrt{E_s}\sin\left(\frac{\pi}{M} - \theta_o\right)\right)}{\sqrt{2N_o}}\right) \\
&= Q\left(\sqrt{\frac{4E_s\sin^2\left(\frac{\pi}{M} - \theta_o\right)}{2N_o}}\right) + Q\left(\sqrt{\frac{4E_s\sin^2\left(\frac{\pi}{M} + \theta_o\right)}{2N_o}}\right) \\
&= Q\left(\sqrt{\frac{2E_b\log_2(M)\sin^2\left(\frac{\pi}{M} - \theta_o\right)}{N_o}}\right) + Q\left(\sqrt{\frac{2E_b\log_2(M)\sin^2\left(\frac{\pi}{M} + \theta_o\right)}{N_o}}\right)
\end{aligned}
$$

Note that we assume that $\theta_o < \pi/M$. Otherwise the bit error rate is approximately 50%.

4. How would you adapt the receiver in problem #1 if the receiver knew the phase offset?

To adapt the receiver we simply need to (a) rotate the incoming signal by $-\theta_o$ radians or we must adjust the symbol boundaries to account for the phase offset.

5. Derive the probability of error of the optimal coherent BPSK receiver in terms of Eb/No if white Laplacian noise is added to the signal. The Laplacian pdf is

$$f\left(x\right)=\frac{1}{\sigma}e^{-(2/\sigma)|x|}$$

where σ is the standard deviation of the noise and assume that $\sigma^2 = N_o/2$ at the output of the matched filter. Plot your result along with the probability of error for AWGN.

The received signal assuming signal 1 was sent is

$$r=\sqrt{E_s}+n$$

Since the distribution is symmetric, we can determine the probability of error for symbol 1. The probability of error for symbol one is:

$$\begin{aligned}\Pr\left[\hat{\mathbf{s}}=\mathbf{s}_2\middle|\mathbf{s}=\mathbf{s}_1\right]&=\Pr\left[\mathbf{r}<0\middle|\mathbf{s}=\mathbf{s}_1\right]\\&=\Pr\left[n<-\sqrt{E_s}\right]\\&=\Pr\left[n>\sqrt{E_s}\right]\\&=\int_{\sqrt{E_s}}^{\infty}\frac{1}{\sigma}e^{-(2/\sigma)x}dx\\&=\frac{1}{2}\int_{\sqrt{4E_s/\sigma}}^{\infty}e^{-y}dy\\&=\frac{1}{2}e^{-\sqrt{8E_b/N_o}}\end{aligned}$$

This is plotted in Figure 3 along with Gaussian noise.

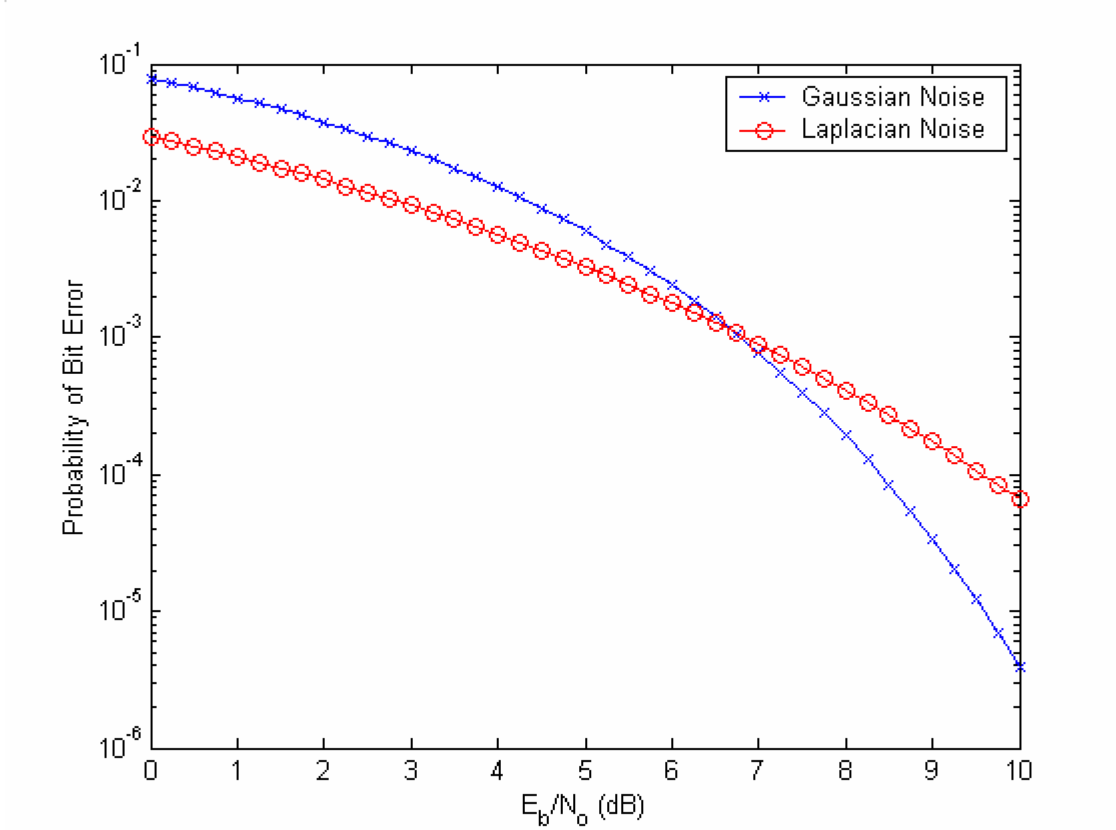

Figure 3: Probability of Error for Laplacian and Gaussian Noise