Digital Communications
Final Exam
May 10, 2004

I pledge that I have neither given nor received any assistance on this exam.

____
(signed)

Name (print)

Student Number

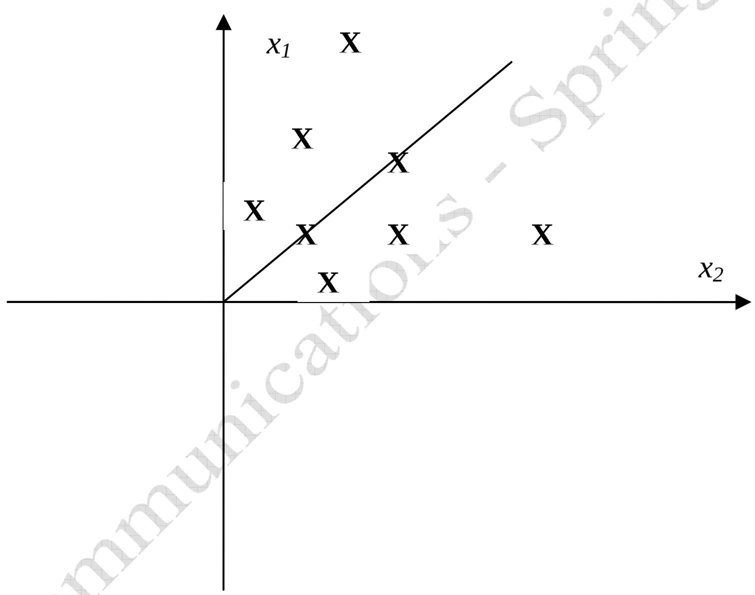
1. (20 points) Short answer. Please answer the following questions.

a. (5 points) Vector quantization: Consecutive samples of a memoryless source are to be quantized with a vector (of length 2) quanitzer with 1.5 bits per sample. The two-dimensional pdf of the source outputs is

$$p_{X_1X_2}\left(x_1, x_2\right) = e^{-\left(x_1+x_2\right)}$$

On the graph below, plot the *approximate* quantization vectors.

The vectors should be concentrated in areas of high pdf (i.e., near zero). Also, the vectors should be in the region $x_1 > 0$ $x_2 > 0$. Since the pdf is symmetric, the vectors should be symmetrically distributed about $x_1 = x_2$



b. (5 points) Ricean fading: What type of fading is more severe Ricean fading with $K = 0$, Ricean fading with K = 10, or Rayleigh fading?

Ricean K=0 is the same as Rayleigh fading and both are more severe than Ricean with K=10.

c. (5 points) Capacity: You see a presentation at a conference where a new coding/modulation scheme achieves an error rate of 10^{-8} at a spectral efficiency of 2bits/s/Hz with an SNR of 3dB. Does this seem reasonable? Why or why not?

The capacity in terms of spectral efficiency is

$$\frac{C}{W} = \log_2\left(1 + \frac{S}{N}\right)$$

for an SNR of 3dB we have

$$\frac{C}{W} = \log_2\left(1+10^{0.3}\right)$$
$$= 1.56 \text{ b/s/Hz}$$

Thus, the spectral efficiency quoted is greater than capacity for an SNR of 3dB. Thus, we should be very skeptical that such a low BER can be achieved.

d. (5 points) MSK: Which is more energy efficient MSK, QPSK, or BFSK?

MSK is equally energy efficient as QPSK provided that it is demodulated two symbols at a time and coherently. Otherwise, MSK has performance equal to BFSK which is 3dB less efficient than QPSK.

2. (20 points) Block coding: As a communications engineer for SpaceCom, Inc. you are designing a modulation and coding scheme for a satellite communications link. The system requires a 200kbps data rate in a 200kHz channel. The RF engineers inform you that based on filtering limitations, the system will use a root raised cosine pulse shaping scheme (i.e., to achieve an overall pulse shape which is raised cosine) with alpha = 0.5. The carrier frequency is 1.8GHz. A matched filter receiver is to be used. The following modulation and coding schemes are to be considered: QPSK, 8-PSK, (n=63, k=36, t=5) BCH code, (n=63, k=24, t=7) BCH code, (n=63, k=16, t=11) BCH code and no coding. The received power is -42dBm and the noise power spectral density is -105dBm/Hz. The desired bit error rate is 10^{-5}. What modulation/coding scheme gives the best performance while meeting the specifications?

First we must check the spectral efficiency of the various combinations. These can be easily shown to be

	No Coding	*(63,36)*	*(63,36)*	*(63,36)*
QPSK	*2*	*1*	*0.75*	*0.5*
8-PSK	*3*	*1.5*	*1.125*	*0.75*

The required spectral efficiency can be easily determined as

$$\begin{aligned}\zeta &= \frac{200kHz}{R_s} \\ &= \frac{200kbps}{200kHz/(1+\alpha)} \\ &= 1.5b/symbol\end{aligned}$$

Thus, only the uncoded cases and 8-PSK with the ½ rate code could be used. The performance will depend on the E_b/N_o received and the scheme we choose. The E_b/N_o can be calculated as

$$\begin{aligned}\left(\frac{E_b}{N_o}\right)_{\text{info}} &= -42dBm - 10\log\{200000Hz\} - (-105dBm/Hz) \\ &= 10dB\end{aligned}$$

The performance of the three can be shown to be

	Performance
QPSK	$Q\left(\sqrt{\frac{2E_b}{N_o}}\right)=Q\left(\sqrt{20}\right)=4*10^{-6}$
8-PSK	$\frac{2}{3}Q\left(\sqrt{\frac{2E_b}{N_o}k\sin^2\left(\frac{\pi}{M}\right)}\right)=Q\left(\sqrt{20*3*\sin^2\left(\frac{\pi}{8}\right)}\right)=1*10^{-3}$
8-PSK with ½ rate coding	$P_b=\frac{1}{2}\left\{\binom{n}{t+1}p^{t+1}\left(1-p\right)^{n-t-1}+\binom{n}{t+2}p^{t+2}\left(1-p\right)^{n-t-2}\right\}=\binom{63}{6}p^6\left(1-p\right)^{57}=1*10^{-5}$ $p=\frac{2}{3}Q\left(\sqrt{\frac{2E_b}{N_o}k*r*\sin^2\left(\frac{\pi}{M}\right)}\right)=Q\left(\sqrt{20*3*\frac{36}{63}\sin^2\left(\frac{\pi}{8}\right)}\right)=8.8*10^{-3}$

Thus, both coded 8-PSK and uncoded QPSK meet the requirement. However, uncoded QPSK gives slightly better performance and is much less complex and thus should be preferred..

3. (25 points) Rayleigh Fading: A BPSK system is being deployed in a mobile environment. It is determined that the mobile has sufficient power to ensure that an E_b/N_o value of 20dB is achieved over 90% of the coverage area.

(a - 5 points) What is the performance of the system assuming perfect channel estimation?

The probability of error can be calculated as

$$P_e = \frac{1}{4 E_b/N_o} = 2.5*10^{-3}$$

(b – 10 points) It is determined that a BER of 10^{-4} is required for this application. A clever VT RF engineer has found a way to install additional antennas in the mobile unit to provide diversity against fading. How many antennas would be required to achieve the desired performance assuming maximal ratio combining?

The probability of error for L path diversity can be calculated as

$$P_e = \left(\frac{1}{4 E_b/N_o}\right)^L \binom{2L-1}{L}$$

Trying L=2 provides

$$P_e = \left(\frac{1}{4*100}\right)^2 \binom{3}{2} = 1.8*10^{-5}$$

Thus, we find that two path diversity can achieve the desired performance.

(c – 10 points) A Turbo coding chip is found that will provide a coding gain of 7dB at a bit error rate of 10^{-4}. The cost per handset is determined to be $10 while the cost of RF electronics is $5 per antenna. Is this a superior solution to (b) ?

To determine this we must first calculate the Eb/No required for BPSK to achieve a 10^{-4} error rate.

$$P_e = \frac{1}{4 E_b / N_o} = 1*10^{-4}$$

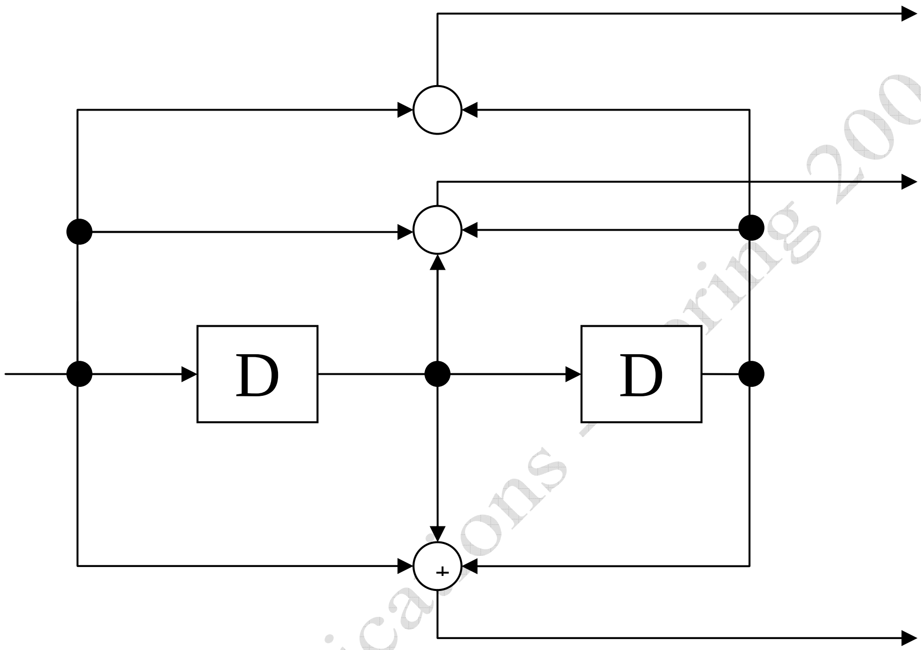
$$E_b / N_o = 2500$$

$$= 34dB$$

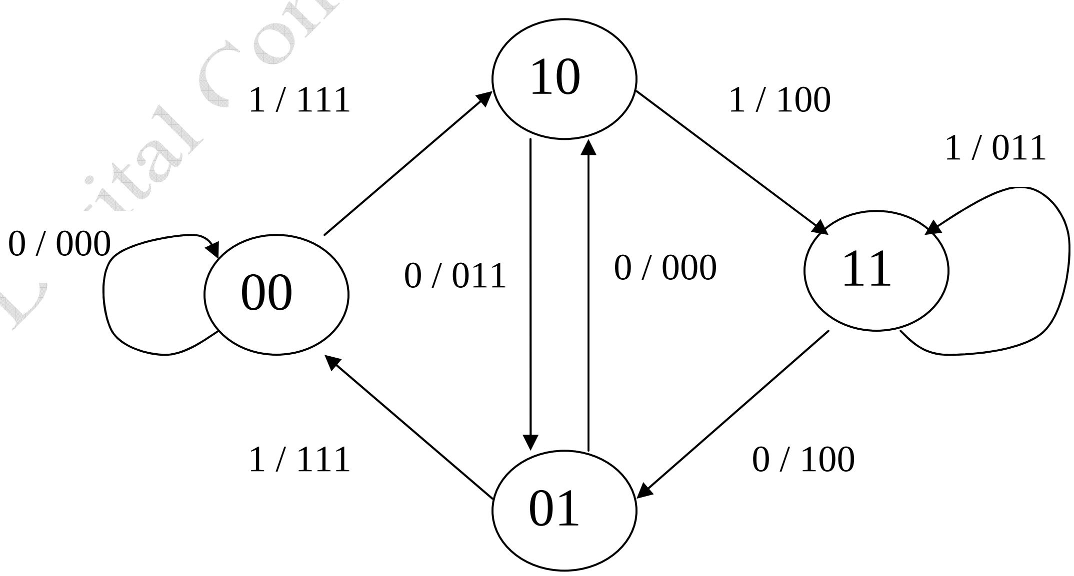
Since the Turbo Code provides a 7dB gain, we need 34dB – 7dB = 27dB. However, two antenna diversity achieves a better BER using 7dB lower Eb/No (20dB). The cost of the two schemes is the same, but the better performance leads us to opt for the diversity system.

(4) (35 points) Convolutional Codes: A rate 1/3 K=3 convolutional code (g = 577) is to be used in a BPSK modulated system.

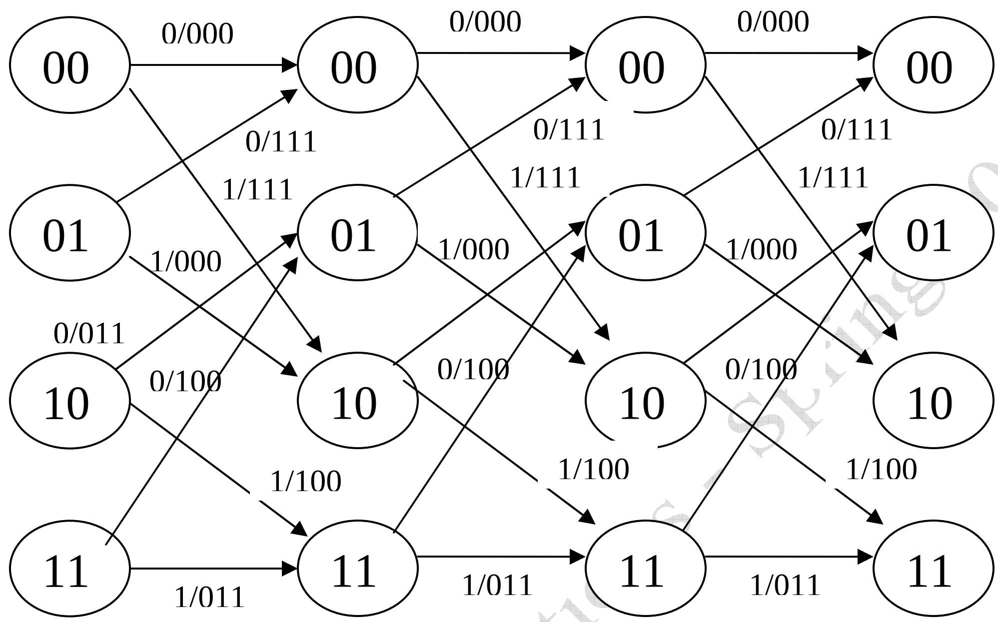
(a – 5 points) Draw the encoder.



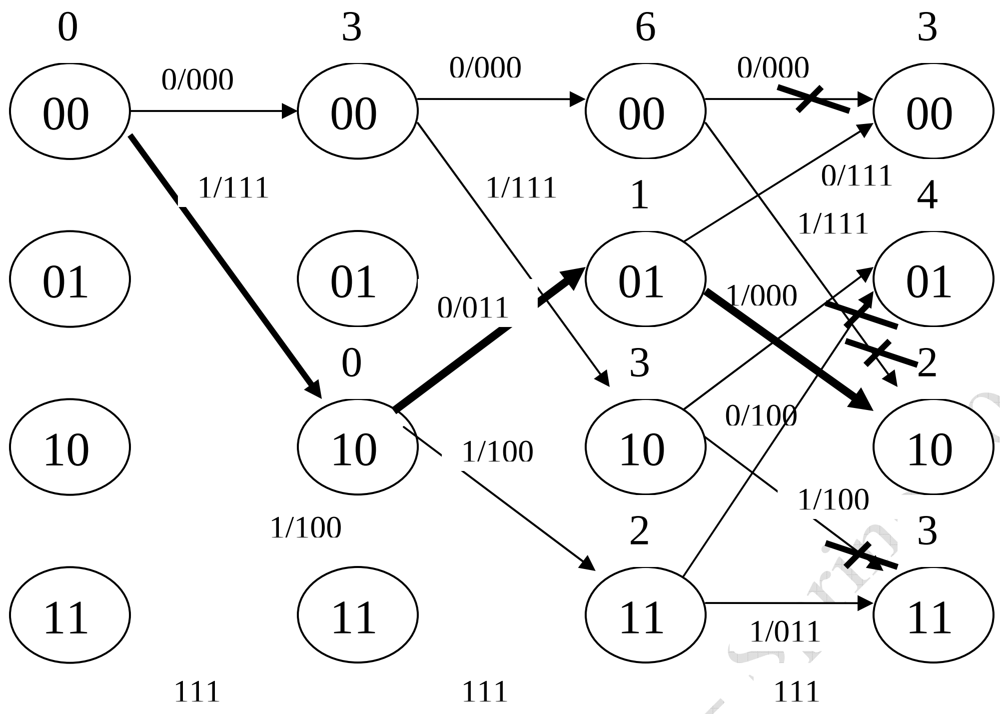
(b – 5 points) Draw the state diagram.



(c – 5 points) Draw the trellis for 3 time epochs.

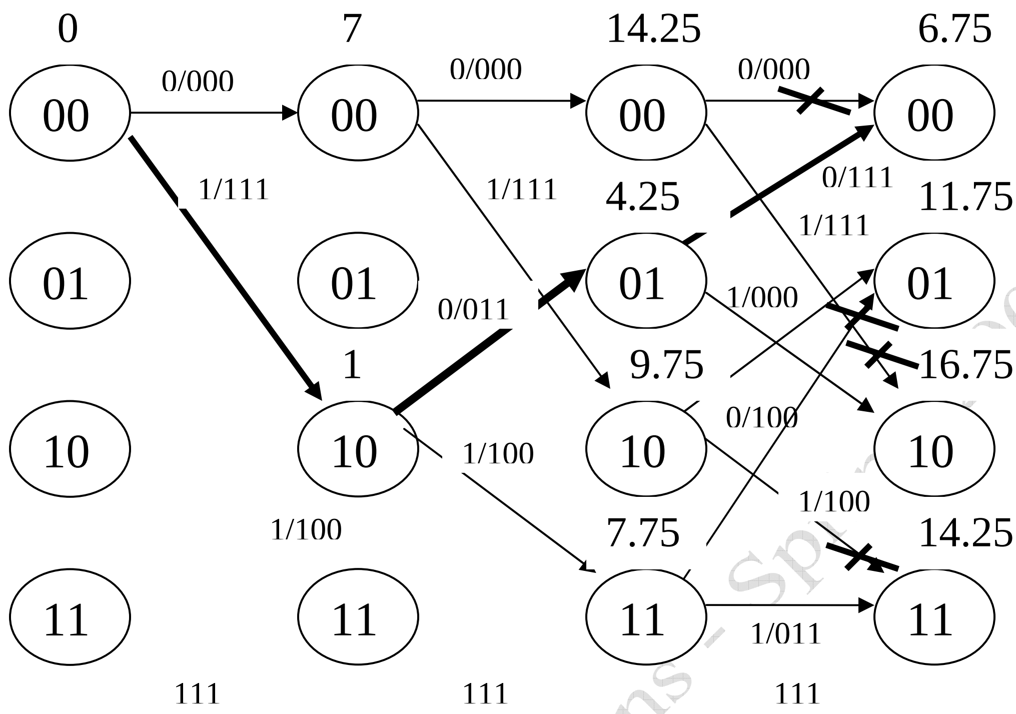


(d – 10 points) The sequence 2 1 1 0.25 2 2 -0.25 1 -0.25 is seen at the receiver. Decode the sequence using hard decision decoding (simply choose the lowest state metric for termination)



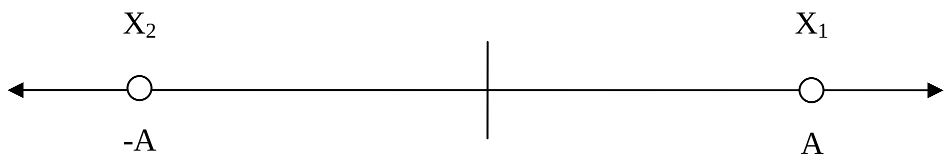
Observing the final set of states, we find that state 10 has the lowest state metric. Tracing back provides a decoded sequence of 101.

(e – 10 points) Repeat (d) for soft decision decoding.



We see that with soft decision decoding we get a different answer. The new answer is 100.

5. (25 points) MAP and Maximum likihood detection: You are using the modulation scheme shown below in a non-AWGN channel. The channel noise is additive and uniformly distributed on [-B, B]. Assume that *B* > *A*.



(a – 10 points) Determine the MAP decision rule for this modulation and noise.

The MAP decision rule can be stated as follows:

$$P\left(X_1|Z\right) \underset{X_1}{\overset{X_2}{\gtrless}} P\left(X_2|Z\right)$$

where Z is the received signal equal to

$$Z = X + N$$

Since the a posteriori probabilities are difficult to obtain we use Bayes rule:

$$P\left(X_1|Z\right) \underset{X_1}{\overset{X_2}{\gtrless}} P\left(X_2|Z\right)$$

$$\frac{P\left(Z|X_1\right)P\left(X_1\right)}{P\left(Z\right)} \underset{X_1}{\overset{X_2}{\gtrless}} \frac{P\left(Z|X_2\right)P\left(X_2\right)}{P\left(Z\right)}$$

$$P\left(Z|X_1\right)P\left(X_1\right) \underset{X_1}{\overset{X_2}{\gtrless}} P\left(Z|X_2\right)P\left(X_2\right)$$

$$\frac{P\left(Z|X_1\right)P\left(X_1\right)}{P\left(Z|X_2\right)P\left(X_2\right)} \underset{X_1}{\overset{X_2}{\gtrless}} 1$$

The conditional probability of Z is found as

$$P\left(Z\middle|X_1\right)=\begin{cases}\dfrac{1}{2B} & A-B\le Z\le A+B\\ 0 & else\end{cases}$$

$$P\left(Z\middle|X_2\right)=\begin{cases}\dfrac{1}{2B} & -A-B\le Z\le -A+B\\ 0 & else\end{cases}$$

Using the above values we have

$$\frac{P\left(Z\middle|X_1\right)P\left(X_1\right)}{P\left(Z\middle|X_2\right)P\left(X_2\right)}=\begin{cases}0 & -A-B\le Z\le A-B\\ \dfrac{P\left(X_1\right)}{P\left(X_2\right)} & A-B\le Z\le -A+B\\ \infty & -A+B\le Z\le A+B\\ undefined & else\end{cases}$$

Thus, our decision rule is

$$\begin{array}{ll}-A-B\le Z\le A-B & \widehat{X}=X_2\\ A-B\le Z\le -A+B & \widehat{X}=\arg\max_X\left(P\left(X\right)\right)\\ -A+B\le Z\le A+B & \widehat{X}=X_1\end{array}$$

(b – 5 points) Determine the ML decision rule.

The ML rule assumes that the apriori probabilities are equal. Thus,

$$\begin{array}{ll}-A-B\le Z\le A-B & \widehat{X}=X_2\\ A-B\le Z\le -A+B & \widehat{X}=random\\ -A+B\le Z\le A+B & \widehat{X}=X_1\end{array}$$

(c – 10 points) Determine the probability of error for an SNR (A^2/σ_n^2) = 2 the *apriori probabilities* are equal. (Note that $\sigma_n^2 = \frac{(b-a)^2}{12}$ for a uniform distribution.) Is this better or worse than Gaussian Noise?

Since the noise distribution is symmetric, we can find the probability of error for X_1 and use symmetry. If $Z > -A+B$, we will decode the bit correctly. If $A-B<Z<-A+B$, we must flip a coin. Thus, the probability of error is 50%. The probability of error is then

$$P_e = \frac{1}{2}\Pr\left(Z < -A + B \middle| X = A\right)$$

For the given problem

$$\frac{A^2}{\sigma_n^2} = 2$$

$$\sigma_n^2 = \frac{A^2}{2}$$

Additionally, we have

$$\sigma_n^2 = \frac{(b-a)^2}{12} = \frac{(2B)^2}{12} = \frac{B^2}{3}$$

Thus, $B = \sqrt{\frac{3}{2}}A$. *The probability of error is*

$$
\begin{aligned}
P_e &= \frac{1}{2}\Pr\left(Z < -A + B \middle| X = A\right) \\
&= \frac{1}{2}\int_{A-B}^{-A+B} \frac{1}{2B}\,dx \\
&= \frac{1}{4}\sqrt{\frac{2}{3}}\frac{1}{A}\int_{A-\sqrt{\frac{3}{2}}A}^{-A+\sqrt{\frac{3}{2}}A} dx \\
&= \frac{1}{4}\sqrt{\frac{2}{3}}\frac{1}{A}\left(-A+\sqrt{\frac{3}{2}}A - A + \sqrt{\frac{3}{2}}A\right) \\
&= \frac{1}{4}\sqrt{\frac{2}{3}}\frac{1}{A}A\left(\sqrt{6}-2\right) \\
&= 0.0918
\end{aligned}
$$

The probability of error for AWGN is

$$
\begin{aligned}
P_b &= Q\left(\sqrt{\frac{A^2}{\sigma_n^2}}\right) \\
&= 0.079
\end{aligned}
$$

Thus, we can achieve better performance with AWGN. However, note that if B < A, the probability of error is zero. This corresponds to an SNR > 3. This is, of course, not true with AWGN.