ECE 5654 - Digital Communications Spring 2005

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Lecture #10 – Noncoherent Demodulation

Coherent Receivers

- PSK $\cos(2\pi f_c t + \theta_i)$
  - 2 basis functions: $f_1(t) = \cos(2\pi f_c t)$
    $f_2(t) = \sin(2\pi f_c t)$
- QAM $A_i \cos(2\pi f_c t + \theta_i)$
  - 2 basis functions: $f_1(t) = \cos(2\pi f_c t)$
    $f_2(t) = \sin(2\pi f_c t)$
- FSK $\cos(2\pi f_i t)$
  - *M* basis functions:

$$f_1(t) = \cos(2\pi f_1 t)$$

$$\vdots$$

$$f_M(t) = \cos(2\pi f_M t)$$

Each set of basis functions assumes that there is no unknown phase term in the received signal except the data. This assumes perfect phase synchronization.

Coherent Receivers

Received Signal before phase synchronization

- PSK $$\cos(2\pi f_c t + \theta(t) + \varphi_o)$$
- QAM $$A(t)\cos(2\pi f_c t + \theta(t) + \varphi_o)$$
- FSK $$\cos(2\pi f_i t + \varphi_o)$$

In reality there is some unknown phase term ϕ_o (relative to the local carrier frequency) which is due to propagation. Phase synchronization is required to eliminate this phase difference. Phase-lock loops or pilots can be used to adjust for this phase difference.

Noncoherent Receivers

- Up to this point, we have assumed that all receivers are coherent (i.e. they are able to detect and track the phase of the signal).
  - For telephone lines, fixed microwave links, and some fixed satellite links, coherent reception is frequently possible.
- While there are practical circuits (phase lock loops) which accomplish this, in many cases strict phase synchronization is not possible.
  - For many mobile and wireless systems, multipath components and movement of the receiver make phase synchronization difficult.

Types of Non-coherent Reception

- PSK and QAM signals represent information with the phase of the signal.
- In many cases, even though it is not possible to detect the absolute phase of a signal, it is possible to detect the difference in phase from one symbol to the next. Thus PSK can be implemented via *Differential Encoding/Detection*
- For FSK with sufficiently spaced tones, demodulation can be accomplished with no phase information using a non-coherent receiver.

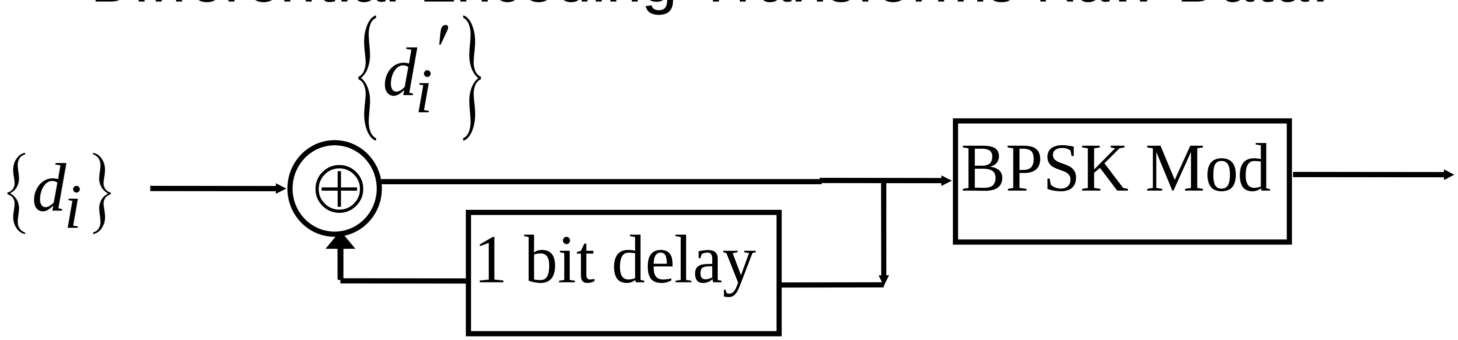
Differential Encoding of Data

- Consider BPSK modulation:

$$0 \Rightarrow s(t) = \sqrt{2P}\cos(2\pi f_c t)\Big|_0^T$$

$$1 \Rightarrow s(t) = \sqrt{2P}\cos(2\pi f_c t + \pi)\Big|_0^T = -\sqrt{2P}\cos(2\pi f_c t)\Big|_0^T$$

- Differential Encoding Transforms Raw Data:



- Resulting modulation rule:
  - Change the phase if input data is a 1
  - Keep phase same if input data is 0

Differential Encoding

- Differential Encoding can be used with either *coherent* detection or *differential* detection
- Coherent detection still requires differential *decoding*
    - eliminates phase ambiguities due to phase tracking loops (phase lock loops can eliminate phase difference but leave a phase ambiguity)
- Non-coherent detection requires *differential detection* which also inherently involves differential decoding
- If data is differentially encoded, two levels of receivers can be employed
    - Non-coherent, cheap, worse performance
    - Coherent, more expensive, better performance

Example of Differential Encoding

d_i :		0	1	0	1	1	1	0
d_{i-1}' :		0	0	1	1	0	1	0
d_i' :		0	1	1	0	1	0	0
Phase:	0	0	π	π	0	π	0	0

Initial condition

Does initial condition matter?

d_i :		0	1	0	1	1	1	0
d_{i-1}' :		1	1	0	0	1	0	1
d_i' :		1	0	0	1	0	1	1
Phase:	π	π	0	0	π	0	π	π

Initial condition

Absolute values change, but we still follow rule that 1→ change phase, 0→same p

Differential Decoding Data (DPSK)

- If we *coherently* detect the signal we must follow it with differential decoding
- Differential decoding is accomplished by multiplying the current bit estimate by the previous:

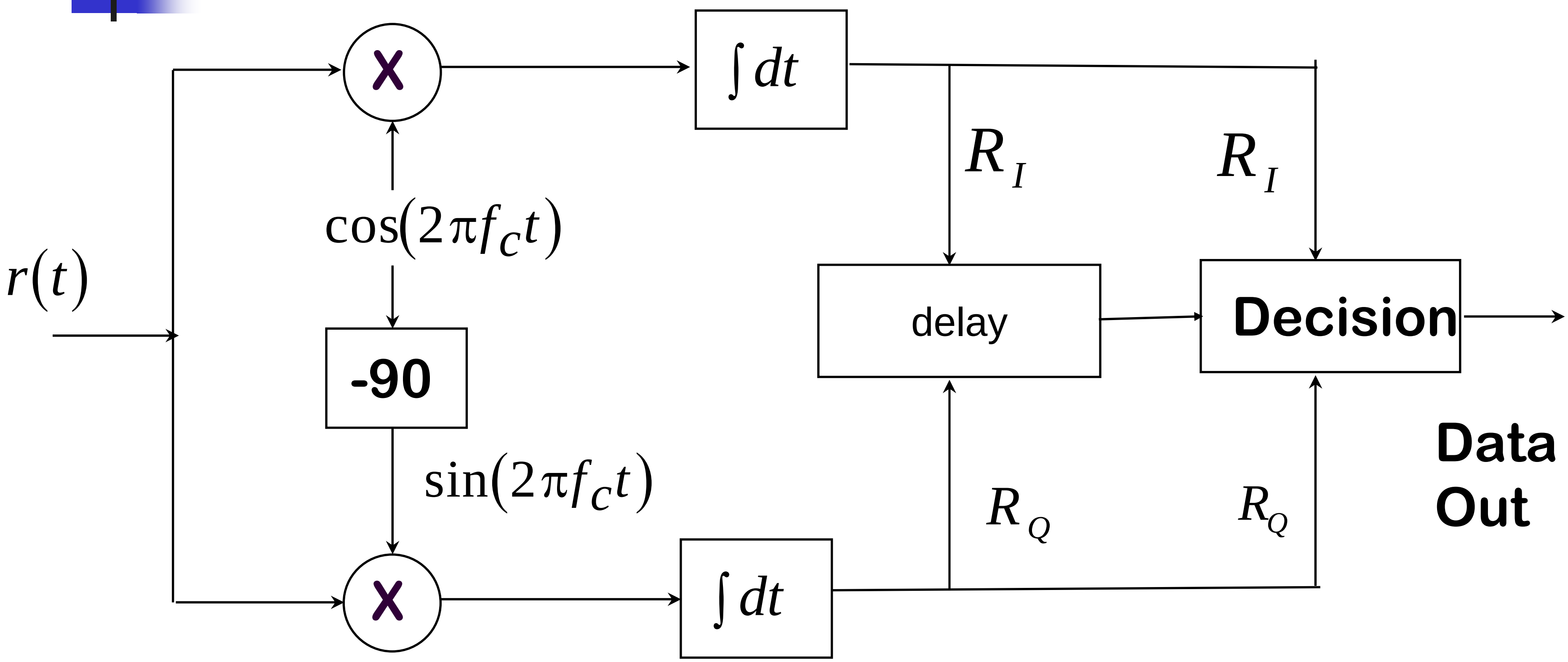
modulo 2 addition for bits

Rx Phase	0	0	π	π	0	π	0	0
$\hat{d}_i'$:	0	0	1	1	0	1	0	0
$\hat{d}_{i-1}'$:		0	0	1	1	0	1	0
$\hat{d}_i$:		0	1	0	1	1	1	0

Initial condition

Original Bits

Differential Detection (DBPSK)



Note that due to the lack of phase coherence, we require *two* correlators for BPSK instead of one

Differential Detection of D-MPSK

Unknown phase offset

$$R_{I,k} = \sqrt{E_s}\cos\left[\frac{2\pi}{M}m_k(t)+\theta_o\right]$$

$$R_{Q,k} = \sqrt{E_s}\sin\left[\frac{2\pi}{M}m_k(t)+\theta_o\right]$$

$$\underbrace{\begin{aligned} Z_{I,k} &= R_{I,k}R_{I,k-1} + R_{Q,k}R_{Q,k-1} \\ Z_{Q,k} &= R_{Q,k}R_{I,k-1} - R_{I,k}R_{Q,k-1} \end{aligned}}_{k\text{th decision metric}}$$

$$Z_{I,k} = \cos\left(\frac{2\pi}{M}\left[m_k(t) - m_{k-1}(t)\right]\right)$$

$$Z_{Q,k} = \sin\left(\frac{2\pi}{M}\left[m_k(t) - m_{k-1}(t)\right]\right)$$

m_k(t) = message symbol {0,1,…M-1}

- Phase offset eliminated!
- Difference between consecutive phases tells us the change in the phase which is the info

Differential Detection

Alternatively, we may view this in terms of complex variab

$$r_k = R_{I,k} + jR_{Q,k}$$
$$= \sqrt{E_s}\, e^{j\left(\frac{2\pi}{M} m_k(t) + \theta_o\right)}$$

$$Z_k = r_k (r_{k-1})^*$$
$$= \sqrt{E_s}\, e^{j\left(\frac{2\pi}{M} m_i(t) + \theta_o\right)} \sqrt{E_s}\, e^{-j\left(\frac{2\pi}{M} m_{i-1}(t) + \theta_o\right)}$$
$$= E_s e^{j\left(\frac{2\pi}{M}\left[m_i(t) - m_{i-1}(t)\right]\right)}$$

- Phase offset eliminated!
- Difference between consecutive phases tells us the change in the phase which is the info

Example of Differential Decoding

$\theta(i)$:	θ_o	$\theta_o+\pi$	$\theta_o+\pi$	θ_o	$\theta_o+\pi$	θ_o	θ_o
$\theta(i-1)$:	θ_o	θ_o	$\theta_o+\pi$	$\theta_o+\pi$	θ_o	$\theta_o+\pi$	θ_o
$\Delta\theta(i)$:	0	π	0	π	π	π	0
$\hat{d}_i$:	0	1	0	1	1	1	0

Initial condition

Original Bits

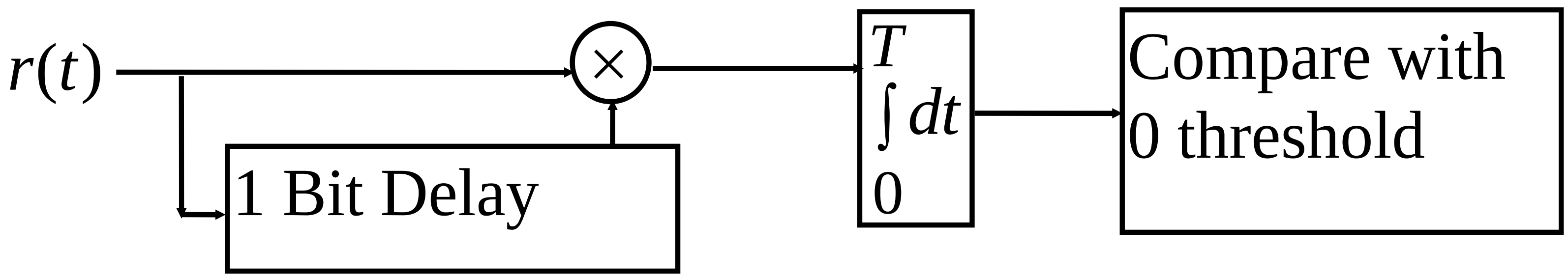
Example of Differential Decoding - With Error

$\theta(i)$:	θ_o	$\theta_o+\pi+N$	$\theta_o+\pi$	θ_o	$\theta_o+\pi$	θ_o	θ_o
$\theta(i-1)$:	θ_o	θ_o	$\theta_o+\pi+N$	$\theta_o+\pi$	θ_o	$\theta_o+\pi$	θ_o
$\Delta\theta(i)$:	0	$\pi+N$	N	π	π	π	0
$\hat{d}_i$:	0	?	?	1	1	1	0

Noise in one phase value causes 2 consecutive bit errors

Differential Reception

- We can think of differential reception as using the noisy version of the received signal as its phase reference for the correlation operation:



- In a coherent receiver we would mix the signal with a phase coherent sinusoid instead of the a delayed version of the signal itself

Probability of Error for Differential Reception

- If $\mathrm{Re}\left[r_i \cdot r_{i-1}*\right] \geq 0$, decide 0

  If $\mathrm{Re}\left[r_i \cdot r_{i-1}*\right] < 0$, decide 1
- Note that for *b*=0:

$$\mathrm{Re}\left\{r_i \cdot r_{i-1}*\right\} = E_b + \sqrt{E_b}\left(n_i + n_{i-1}*\right) + n_i \cdot n_{i-1}*$$

- For moderate E_b/N_0 , $n_i \cdot n_{i-1}*$ becomes small
- One approximation is that (since noise is doubled), DPSK is 3dB worse than BPSK.
- However, a more elaborate analysis shows that the performance is actually not that bad:

$$P_b = \frac{1}{2} e^{-E_b/N_0}$$

Probability of Error

- The received signal vectors for consecutive symbols:

$$r_k = R_{I,k} + jR_{I,k} + n_i$$
$$= \sqrt{E_s} e^{j(\theta_i + \theta_o)} + n_i$$
$$r_{k-1} = \sqrt{E_s} e^{j(\theta_{i-1} + \theta_o)} + n_{i-1}$$

- An error occurs if $Z < 0 | b = 0$
  - where

$$Z = \mathrm{Re}\left\{ r_k \left(r_{k-1} \right)^* \right\}$$
$$= \frac{1}{2}\left(r_k \left(r_{k-1} \right)^* + \left(r_k \right)^* r_{k-1} \right)$$

- Thus

$$P_s = \Pr\left\{ Z < 0 | \theta_k = \theta_{k-1} \right\}$$

Probability of Error (cont.)

- This is a special case of the general quadratic combination of Gaussian random variables
- If we define

$$D = A|X|^2 + B|Y|^2 + CX^*Y + CXY^*$$

  - where *A*,*B*, and *C* are arbitrary constants and *X* and *Y* are Gaussian random variables
- Then it can be shown that

$$P_s = Q_1(a,b) - \frac{v_2/v_1}{1+v_2/v_1} I_o(ab)\exp\left(-\frac{1}{2}\left(a^2+b^2\right)\right)$$

  - where $Q_1(a,b)$ is the generalized Marcum *Q*-function, $I_o(x)$ is the modified zeroth order Bessel of the first kind, and *a*, *b*, v_1 and v_2 are constants defined by the moments of *X* and *Y*

Probability of Error (cont.)

- Now

$$a = \sqrt{\frac{2v_1^2 v_2 \left(\alpha_1 v_1 - \alpha_2\right)}{\left(v_1 + v_2\right)^2}} \quad b = \sqrt{\frac{2v_1 v_2^2 \left(\alpha_1 v_1 + \alpha_2\right)}{\left(v_1 + v_2\right)^2}}$$

$$v_1 = \sqrt{w^2 + \frac{1}{4\left(\mu_{XX}\mu_{YY} - \left|\mu_{XY}\right|^2\right)\left(\left|C\right|^2 - AB\right)}} - w$$

$$v_2 = \sqrt{w^2 + \frac{1}{4\left(\mu_{XX}\mu_{YY} - \left|\mu_{XY}\right|^2\right)\left(\left|C\right|^2 - AB\right)}} + w$$

$$w = \frac{A\mu_{XX} + B\mu_{YY} + C\mu_{XY} + C\mu_{XY}}{4\left(\mu_{XX}\mu_{YY} - \left|\mu_{XY}\right|^2\right)\left(\left|C\right|^2 - AB\right)}$$

$$\alpha_1 = 2\left(\left|C\right|^2 - AB\right)\left(\left|\overline{X}\right|^2 \mu_{YY} + \left|\overline{Y}\right|^2 \mu_{XX} - \overline{X}^* \overline{Y} \mu_{XY} - \overline{X}\,\overline{Y}^* \mu_{XY}\right)$$

$$\alpha_2 = A\left|\overline{X}\right|^2 + B\left|\overline{Y}\right|^2 + C\overline{X}^* \overline{Y} + C\overline{X}\,\overline{Y}^*$$

Probability of Error (cont.)

- For our case things simplify greatly:

$$w = 0$$

$$v_1 = v_2 = \frac{1}{\sigma^2}$$

$$\sigma^2 = \mu_{XX} = \mu_{YY}$$

$$\alpha_1 = E_s \sigma^2$$

$$\alpha_2 = E_s$$

- and

$$a = 0 \quad b = \sqrt{\frac{E_s}{\sigma^2}}$$

Probability of Error (cont.)

- Thus, we have

$$P_s = Q_1\left(0, \sqrt{\frac{E_s}{\sigma^2}}\right) - \frac{1}{2} I_o(0) \exp\left(-\frac{1}{2}\frac{E_s}{\sigma^2}\right)$$

- Now,

$$I_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$Q_1(a,b) = \exp\left(-\frac{1}{2}\left(a^2 + b^2\right)\right) \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab)$$

- Thus, simplifying

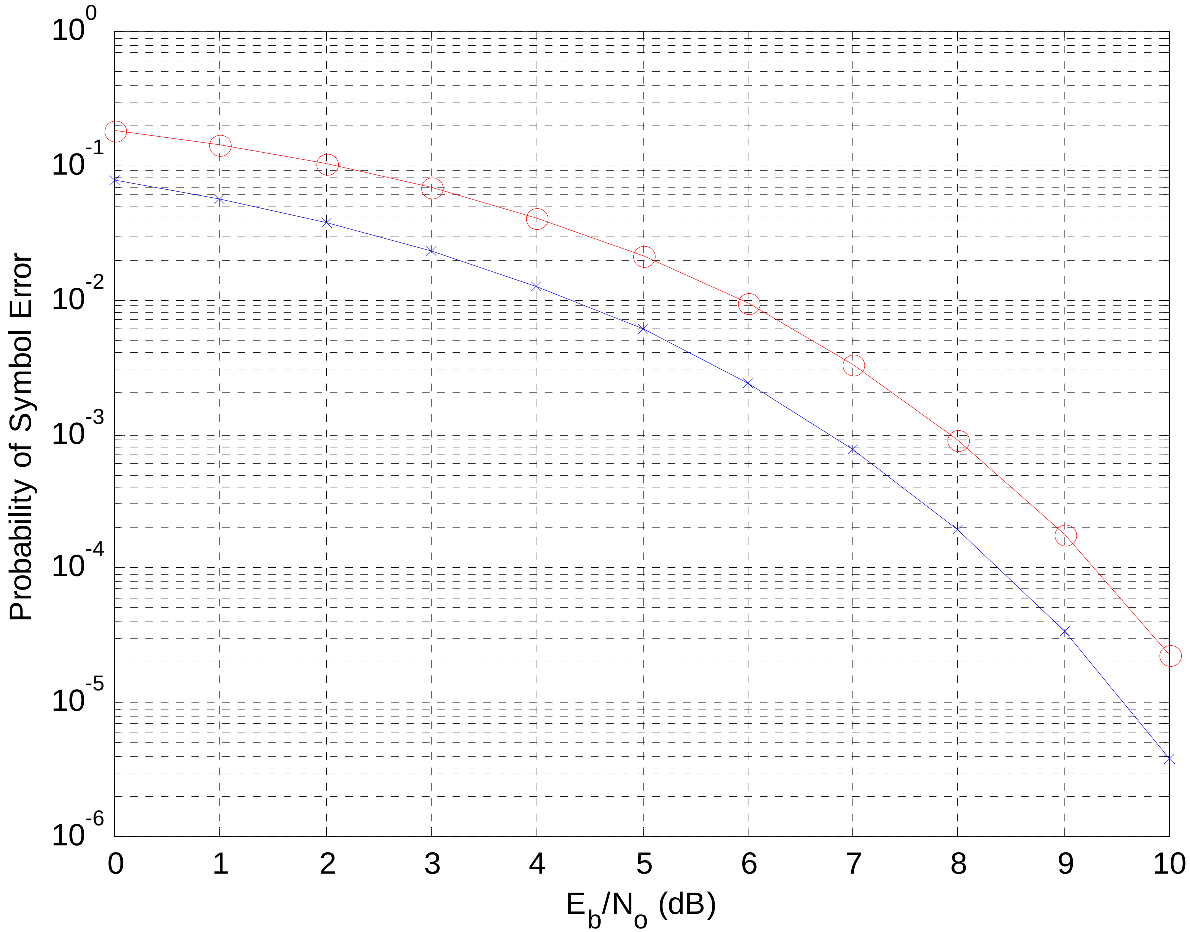
$$P_s = \exp\left(-\frac{1}{2}\frac{E_s}{\sigma^2}\right) - \frac{1}{2} * 1 * \exp\left(-\frac{1}{2}\frac{E_s}{\sigma^2}\right)$$

$$= \frac{1}{2}\exp\left(-\frac{E_b}{N_o}\right)$$

Note: $E_s = E_b$

$$\sigma^2 = \frac{N_o}{2}$$

Performance Comparison of BPSK and DPSK



- Performance is not quite as bad as the 3 dB estimate would indicate. That estimate is more accurate for D-QPSK

Higher Order Differential Reception

- We have focused on the binary case of DPSK, but we can use differential encoding and reception with any type of phase modulation.
- With DQPSK, we have four possible phase rotations: 0, 90, 180, 270.
- π /4 DQPSK modulation inserts an extra phase rotation of π /4 radians (=45^o) with each symbol to reduce possibility of timing errors (due to many consecutive 0 phase shifts)
  - π /4 DQPSK is used in the US Digital Cellular standard IS-54/IS-136

Non-coherent Reception: FSK

- For FSK signals which are sufficiently spaced in the frequency domain, it is possible to demodulate with no phase information at all.
  - Coherent modulation/demod allows frequency spacing of 1/2T
  - Non-coherent modulation/demod requires frequency spacing of 1/T
- The form of the optimum non-coherent demodulator for FSK can be interpreted as an energy detector.
- FSK modulation with non-coherent reception is cheap and robust for wireless applications.

Model for Non-coherent Modulation

- Consider binary FSK with two signals:

$$s_1(t) = \sqrt{2P}\cos(2\pi f_1 t + \phi_1)\Big|_0^T$$

$$= \sqrt{2P}\cos(2\pi f_1 t)\cos(\phi_1) - \sqrt{2P}\sin(2\pi f_1 t)\sin(\phi_1)\Big|_0^T$$

$$s_2(t) = \sqrt{2P}\cos(2\pi f_2 t + \phi_2)\Big|_0^T$$

$$= \sqrt{2P}\cos(2\pi f_2 t)\cos(\phi_2) - \sqrt{2P}\sin(2\pi f_2 t)\sin(\phi_2)\Big|_0^T$$

- where ϕ_1 and ϕ_2 are unknown random phases with uniform distribution:

$$p_{\phi_1}(x) = p_{\phi_2}(x) = \begin{cases} 1/2\pi & x \in [0, 2\pi) \\ 0, & else \end{cases}$$

Basis Functions for FSK With Unknown Phase

- No matter what the two phases are, the signals can be expressed as a linear combination of the four basis functions:

$$f_{1c}(t) = \sqrt{2/T}\cos(2\pi f_1 t)\Big|_0^T,\ f_{1s}(t) = -\sqrt{2/T}\sin(2\pi f_1 t)\Big|_0^T,$$

$$f_{2c}(t) = \sqrt{2/T}\cos(2\pi f_2 t)\Big|_0^T,\ f_{2s}(t) = -\sqrt{2/T}\sin(2\pi f_2 t)\Big|_0^T$$

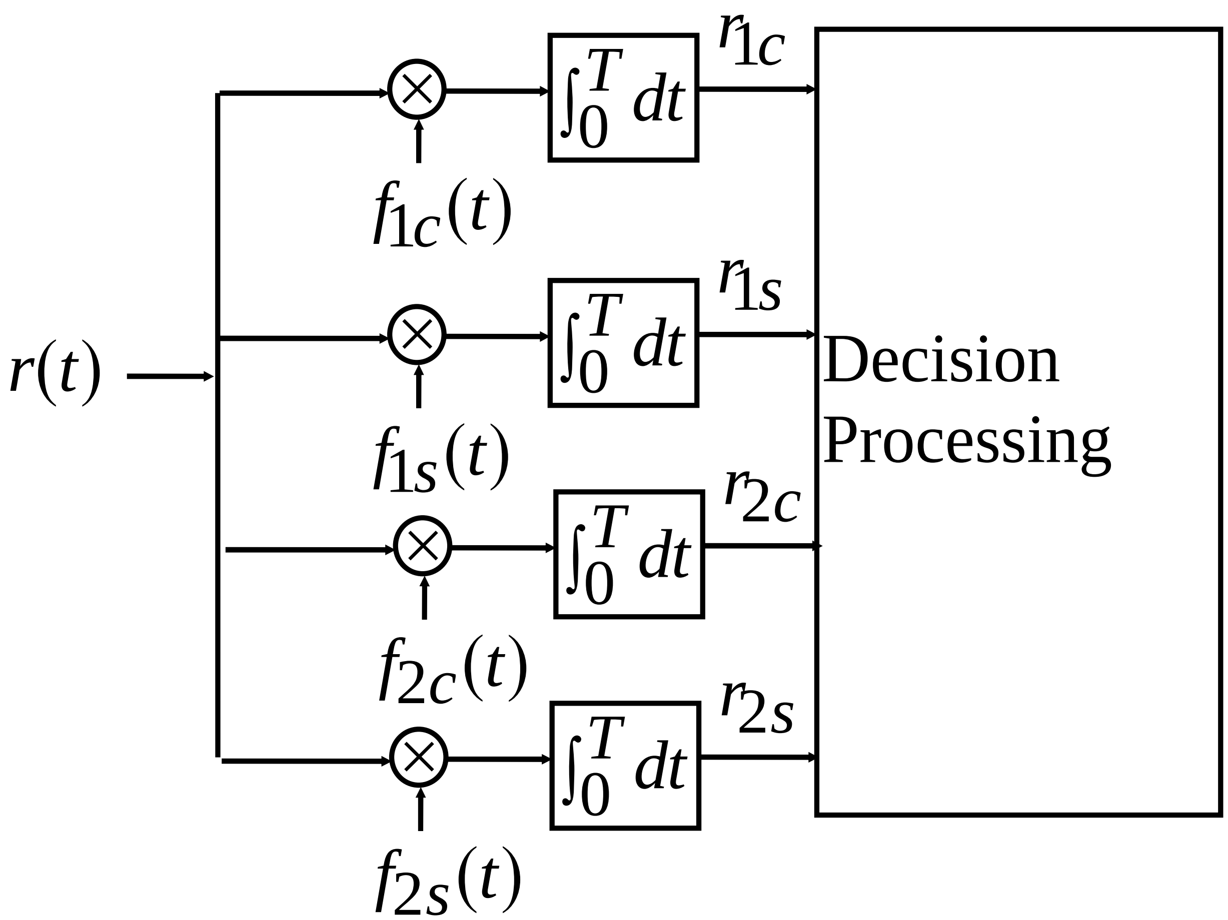
- The signals have the representation:

$$s_1(t) = \sqrt{E_b}\cos(\phi_1) f_{1c}(t) + \sqrt{E_b}\sin(\phi_1) f_{1s}(t)$$

$$s_2(t) = \sqrt{E_b}\cos(\phi_2) f_{2c}(t) + \sqrt{E_b}\sin(\phi_2) f_{2s}(t)$$

- Correlation with these basis function produces a set of *sufficient statistics*

Noncoherent Receiver Structure



Processing for Noncoherent Receiver

- Decision Rule: Choose signal 1 if

$$\Pr[\mathbf{s}_1]\, p(\mathbf{r}|\mathbf{s}_1) \geq \Pr[\mathbf{s}_2]\, p(\mathbf{r}|\mathbf{s}_2) \qquad \text{(MAP Rule)}$$

$$p(\mathbf{r}|\mathbf{s}_1) \geq p(\mathbf{r}|\mathbf{s}_2) \qquad \text{(ML Rule)}$$

- Conditional pdf:

$$p(\mathbf{r}|\mathbf{s}_1,\phi_1) = \frac{1}{\pi N_0} \exp\left[-\frac{\left(r_{1c}-\sqrt{E}\cos\phi_1\right)^2 + \left(r_{1s}-\sqrt{E}\sin\phi_1\right)^2}{N_0}\right]$$

$$\times \frac{1}{\pi N_0} \exp\left[-\frac{r_{2c}{}^2 + r_{2s}{}^2}{N_0}\right]$$

Assuming AWGN!

Processing for Noncoherent Receiver (continued)

- Similarly:

$$p(\mathbf{r}|\mathbf{s}_2,\phi_2)=\frac{1}{\pi N_0}\exp\left[-\frac{\left(r_{2c}-\sqrt{E}\cos\phi_2\right)^2+\left(r_{2s}-\sqrt{E}\sin\phi_2\right)^2}{N_0}\right]$$

$$\times\frac{1}{\pi N_0}\exp\left[-\frac{r_{1c}{}^2+r_{1s}{}^2}{N_0}\right]$$

- For the ML decision we need to evaluate:

$$p(\mathbf{r}|\mathbf{s}_1)\geq p(\mathbf{r}|\mathbf{s}_2)$$

$$\Leftrightarrow\frac{1}{2\pi}\int_0^{2\pi}p(\mathbf{r}|\mathbf{s}_1,\phi_1)d\phi_1\geq\frac{1}{2\pi}\int_0^{2\pi}p(\mathbf{r}|\mathbf{s}_2,\phi_2)d\phi_2$$

Processing for Noncoherent Receiver (continued)

- Before we evaluate this, we remove the constant term:

$$\left(\frac{1}{\pi N_0}\right)^2 \exp\left[-\frac{r_{1c}^2 + r_{1s}^2 + r_{2c}^2 + r_{2s}^2 + E}{N_0}\right]$$

- We have the inequality:

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\left[\frac{2\sqrt{E}r_{1c}\cos\phi_1 + 2\sqrt{E}r_{1s}\sin\phi_1}{N_0}\right]d\phi_1$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \exp\left[\frac{2\sqrt{E}r_{2c}\cos\phi_2 + 2\sqrt{E}r_{2s}\sin\phi_2}{N_0}\right]d\phi_2$$

Bessel Functions

- By definition:

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos\phi_1+2\sqrt{E}r_{1c}\sin\phi_1}{N_0}\right]d\phi_1$$

$$=I_0\left(\frac{2\sqrt{E\left(r_{1c}{}^2+r_{1s}{}^2\right)}}{N_0}\right)$$

- where $I_0(\cdot)$ is a modified Bessel function of the zeroth order.

Decision Rule for Noncoherent FSK

- The decision rule becomes: choose signal 1 if:

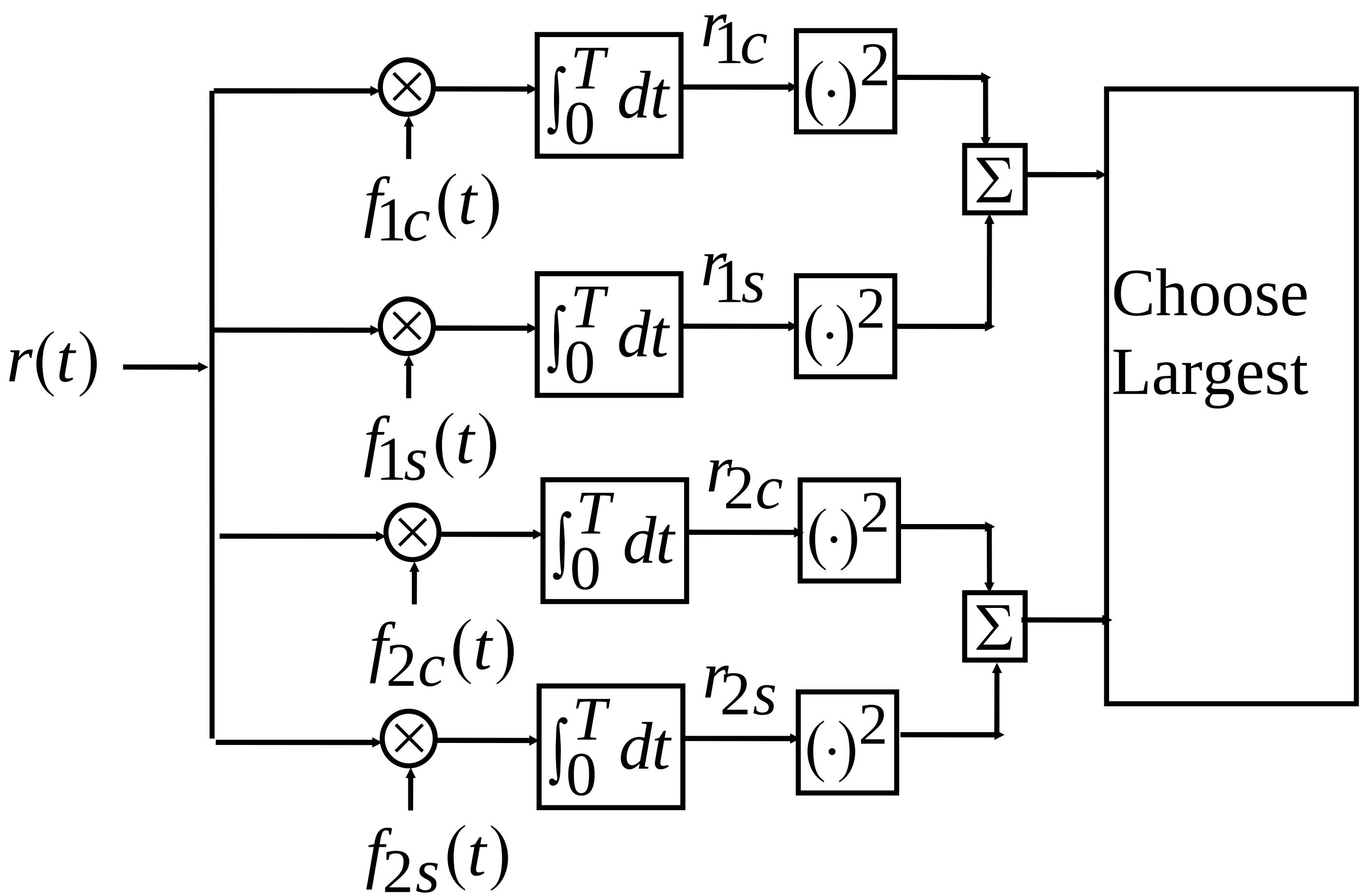
$$I_0\left(\frac{2\sqrt{E\left(r_{1c}^2+r_{1s}^2\right)}}{N_0}\right) \geq I_0\left(\frac{2\sqrt{E\left(r_{2c}^2+r_{2s}^2\right)}}{N_0}\right)$$

- The only thing we need to know about the Bessel function is that it is monotonically increasing.
- Therefore, choose signal 1 if:

$$\sqrt{r_{1c}^2+r_{1s}^2} \geq \sqrt{r_{2c}^2+r_{2s}^2}$$

- This has a convenient interpretation: Compare the energy in the two frequencies and pick the larger.

Structure of Optimum Noncoherent Receiver for Binary FSK



Error Probability for Noncoherent Binary FSK

- If r_{1c} and r_{1s} have Gaussian distributions, then $\sqrt{r_{1c}^2 + r_{1s}^2}$ has a Rayleigh distribution.
- Integrating to find the probability that

$$\sqrt{r_{1c}^2 + r_{1s}^2} \geq \sqrt{r_{2c}^2 + r_{2s}^2}$$

given signal 2 was sent yields:

$$P_b(e) = \frac{1}{2}\exp\left[-E_b/2N_0\right]$$

Note that this is 3dB worse than DPSK!

Probability of Error for Non-coherent *M* -ary FSK

- By exact computation, it can be shown that:

$$P_s(e) = \sum_{n=1}^{M-1} (-1)^{n+1} \cdot \binom{M-1}{n} \frac{1}{n+1} \exp\left[-\frac{nE_b \log_2 M}{(n+1)N_0}\right]$$

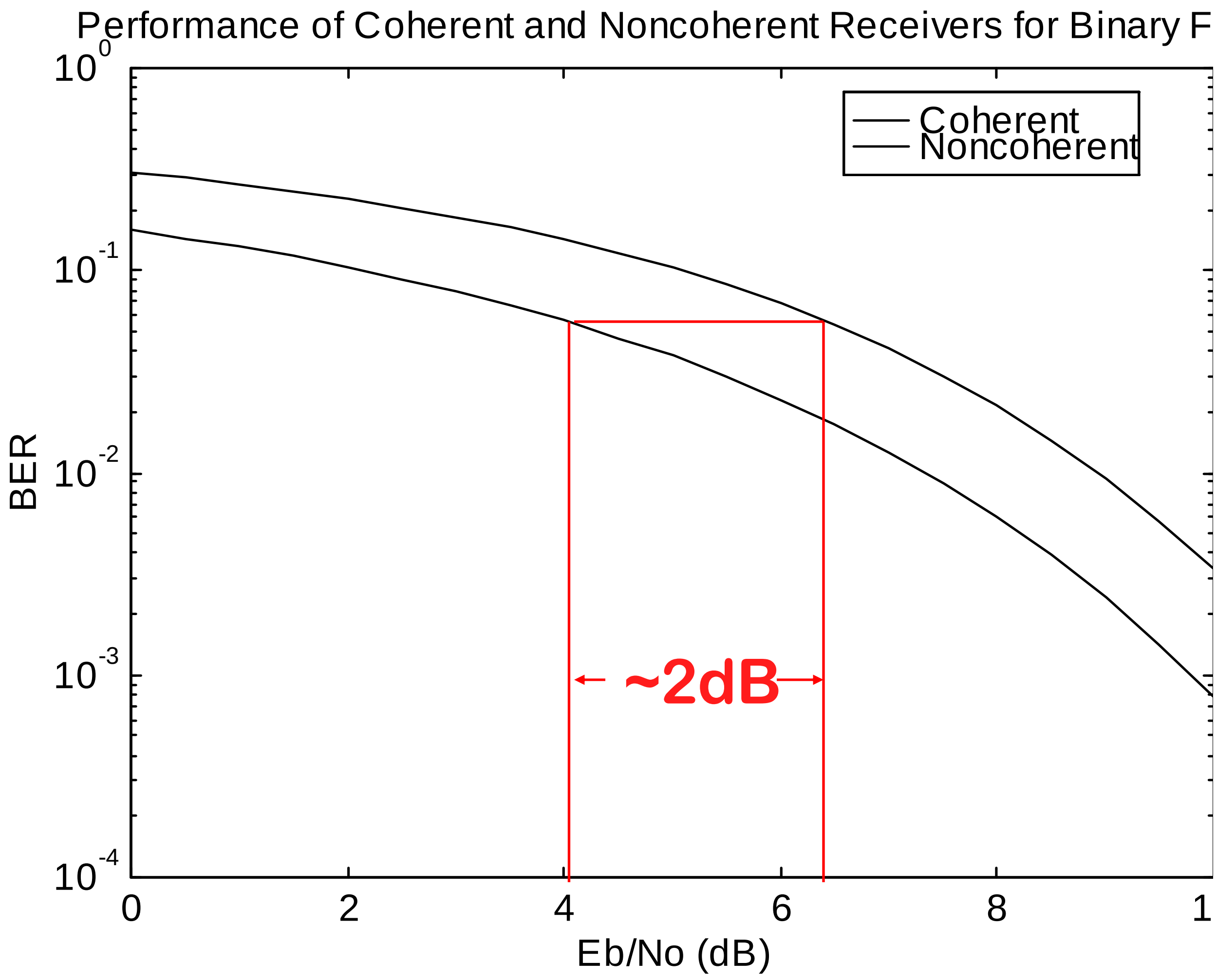
- By the union bound we have:

$$P_s(e) \leq \frac{M-1}{2} \exp\left[-\frac{E_b \log_2 M}{2N_0}\right]$$

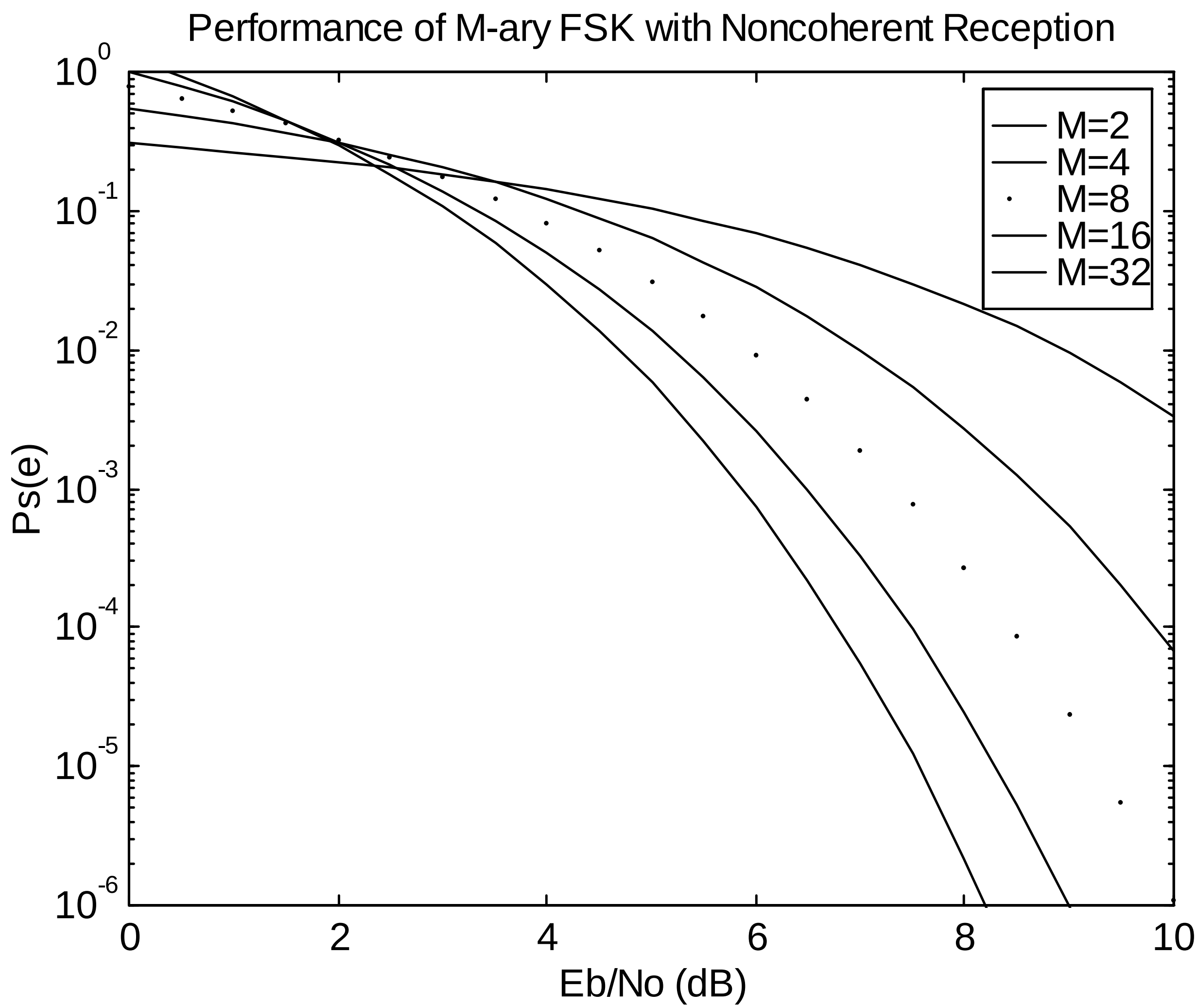
- Bit error rate is related to symbol error rate by:

$$P_b(e) \leq \frac{2^{(\log_2 M - 1)}}{(2^{\log_2 M}) - 1} P_s(e)$$

Comparison of Coherent and Noncoherent Binary FSK



Performance of *M* -ary FSK Using Union Bound



Energy efficiency (in terms of E_b/N_o) improves with *M*

Summary of Noncoherent Receivers

- Differential detection is possible for phase modulation.
- Noncoherent detection is possible for frequency modulation.
- Once again, the signal space representation leads to the optimal receiver and a method for evaluating performance.
- Noncoherent FSK can be very robust.
- High order FSK can be very energy efficient but not very bandwidth efficient.