EE 5654 - Digital Communications Spring 2005

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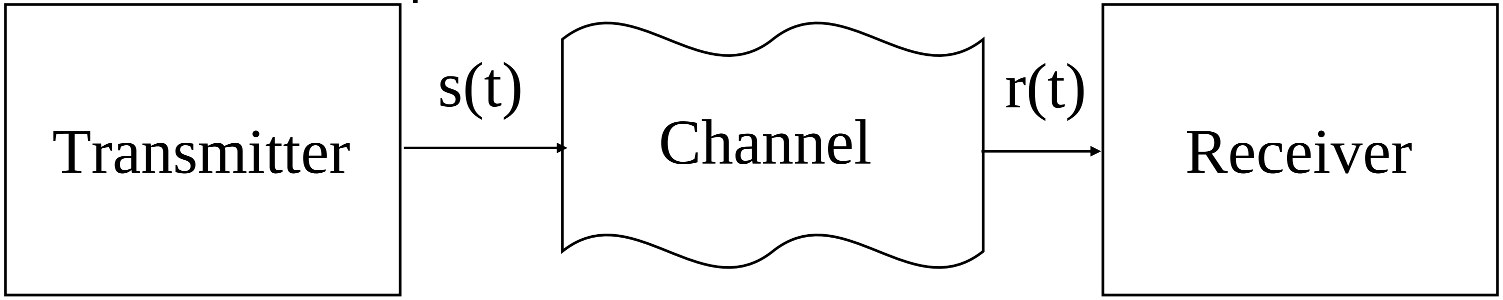
Lecture #12 – Fading Channels and Diversity

Summary

- Previously we examined the performance of modulation in an AWGN channel
- Fading was not considered, which is justifiable in satellite or fixed channels
- However, in mobile radio environments, fading is an important consideration
  - Today we will discuss fading channels and examine the performance of modulation in fading channels
- Types of fading
  - Rayleigh
  - Ricean
- Techniques to combat fading
  - spatial diversity
  - frequency diversity
  - coding

The Channel

- The channel is the medium through which the system communicates information
- In general $r(t) = s(t) \otimes h(t,\tau) + n(t)$ where *h(t,τ)* is baseband equivalent of the channel impulse response, *ν(t)* is thermal noise and $\otimes$ is the convolution operation.



Transmitter

s(t)

Channel

r(t)

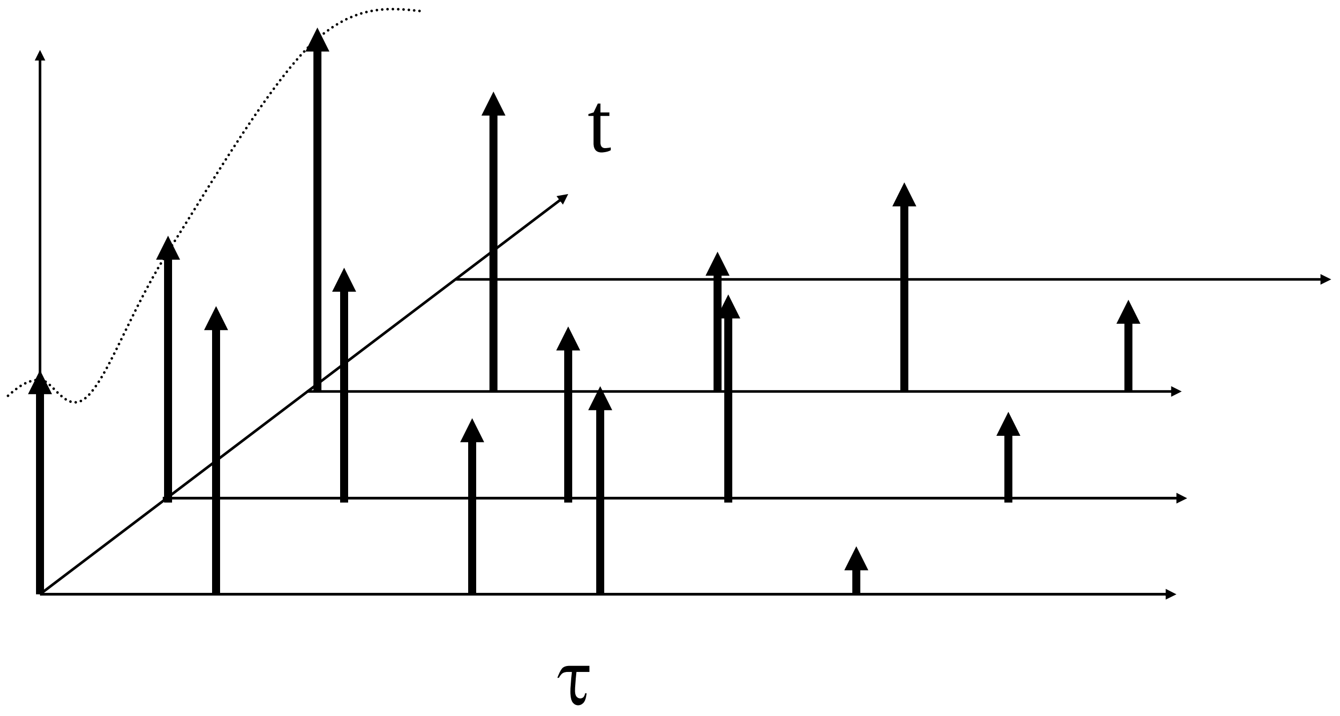
Receiver

- Typically we model the channel impulse response as a sum of discrete mulitpath components

$$h(t,\tau) = \sum_{i=1}^{N} \alpha_i(t,\tau) e^{j\theta_i(t,\tau)} \delta\left(\tau - \tau_i(t)\right)$$

Channel Impulse Response

- Time Varying Impulse Response

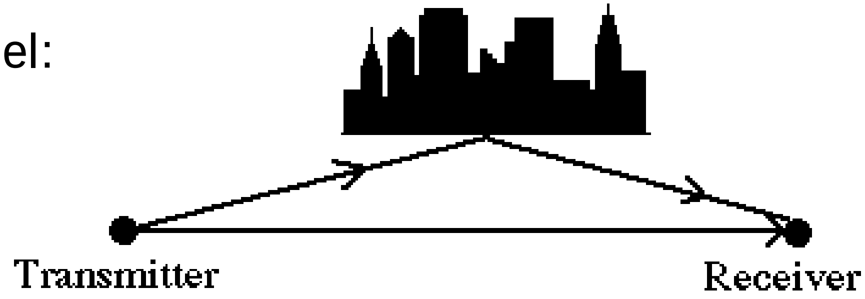


Multipath Fading

- Consider a two-path time-invariant channel:

$$h(t,\tau)=\sqrt{\frac{1}{2}}\delta(\tau)+\sqrt{\frac{1}{2}}e^{j\theta_0}\delta(\tau-\tau_0)$$

$$\theta_o=2\pi f_c\tau_o$$



- Now at the receiver we see (after convolution)

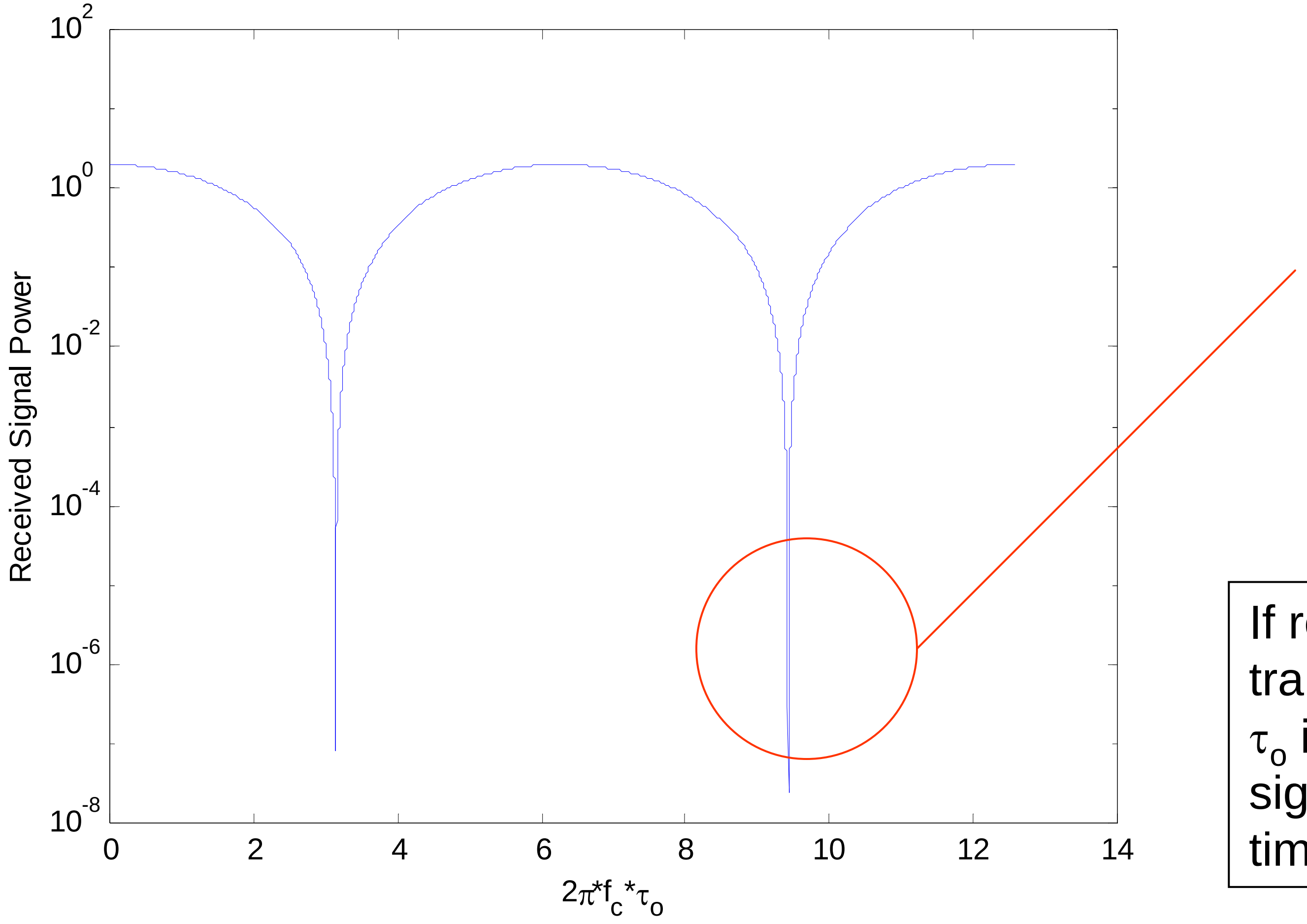
$$r(t)=s(t)+e^{j\theta_o}s(t-\tau_o)$$

- Now, let us assume that $\tau_o << T$ (narrowband channel)

$$r(t)\approx s(t)\left[1+e^{j\theta_o}\right]$$

Envelope Fading

- The envelope of the received signal clearly depends on the values of f_c and τ_o.



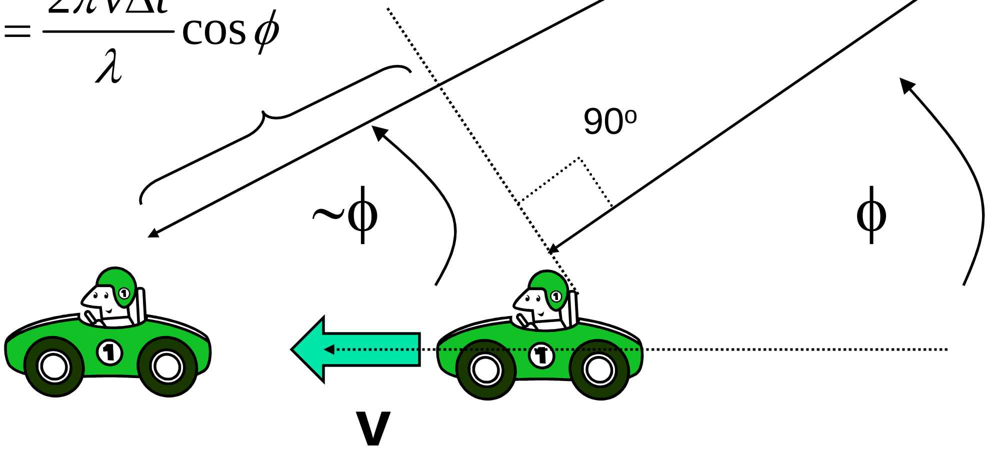
Signal power reduced drastically due to destructive phase combining at certain values of $f_c\tau_o$ (This is termed a “fade”)

If receiver or transmitter is moving, τ_o is changing and the signal envelope is time-varying

Doppler Shift

- Assume that a mobile is moving with velocity *v* and a path is arriving from angle ϕ, the phase change in time Δt is

$$\Delta\theta = \frac{2\pi}{\lambda} d \cos\phi = \frac{2\pi v \Delta t}{\lambda} \cos\phi$$



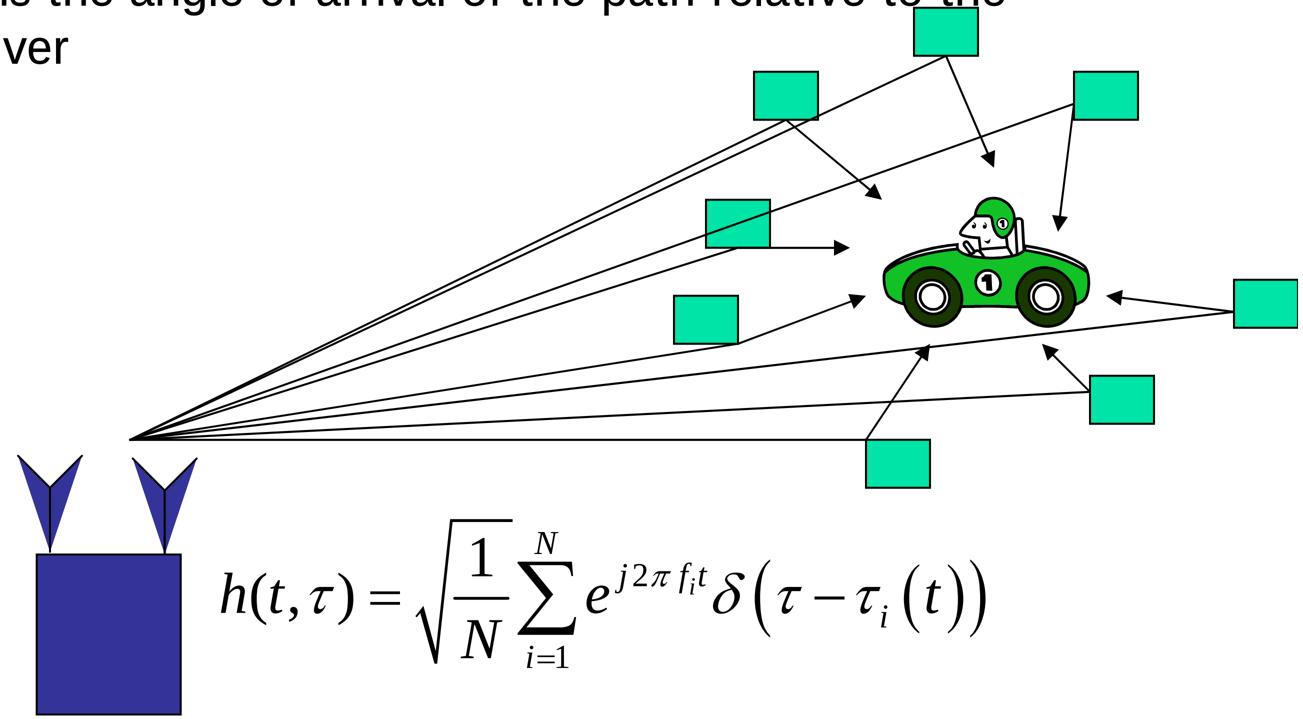
Thus, the mobile experiences a shift in frequency called the *Doppler shift*

$$f_d = \frac{1}{2\pi} \frac{\Delta\theta}{\Delta t} = \frac{v}{\lambda} \cos\phi$$

Time-Varying Fading

- Now consider a large number of multipath components which arrive at the receiver from 360º
- Further, if the receiver is moving each reflection experiences a Doppler shift of $f_i = \frac{v}{\lambda}\cos\Phi_i$ where λ is the wavelength of the carrier, v is the velocity of the receiver, and Φ_i is the angle of arrival of the path relative to the direction of the receiver

If all of the delays are small relative to the symbol duration (narrowband channel), we model each path as experiencing a time varying phase. This time varying phase is related to the Doppler shift experienced by that path.



$$h(t,\tau) = \sqrt{\frac{1}{N}}\sum_{i=1}^{N} e^{j2\pi f_i t}\delta\left(\tau - \tau_i\left(t\right)\right)$$

Flat Fading

- If the delays are all small relative to the symbol duration, we can ignore the difference between delays and model the signal as having only a single time varying impulse

$$h(t,\tau)=\sqrt{\frac{1}{N}}\sum_{i=1}^{N}e^{j2\pi f_i t}\delta\left(\tau-\tau_i\left(t\right)\right)$$

$$=\delta\left(\tau-\tau_0\right)\sqrt{\frac{1}{N}}\sum_{i=1}^{N}e^{j\left(2\pi f_i t+\theta_i\right)}$$

$$=\delta\left(\tau-\tau_0\right)\gamma(t)$$

- The received signal is then (normalizing the delay to τ_o=0)

$$r(t)=s(t)\otimes h(t,\tau)+n(t)$$

$$=s(t)\gamma(t)+n(t)$$

- Thus, the narrowband channel causes only *multiplicative distortion.* This is termed flat fading since the channel transfer function is flat (i.e., the Fourier Transform of an impulse is constant)

Rayleigh Fading

- The sum of a large number of complex sinusoids with random frequencies and phases will tend toward complex Gaussian random process. Thus, the real and imaginary components of γ(t) are zero mean Gaussian random processes

$$\gamma(t) = a(t) + j\,b(t)$$

Gaussian distributed

Gaussian distributed

- Since the real and imaginary components are Gaussian, the amplitude has a Rayleigh distribution. Thus, we term this a flat *Rayleigh fading* channel.

$$\gamma(t) = A(t)e^{j\theta(t)}$$

Phase = uniform random variable

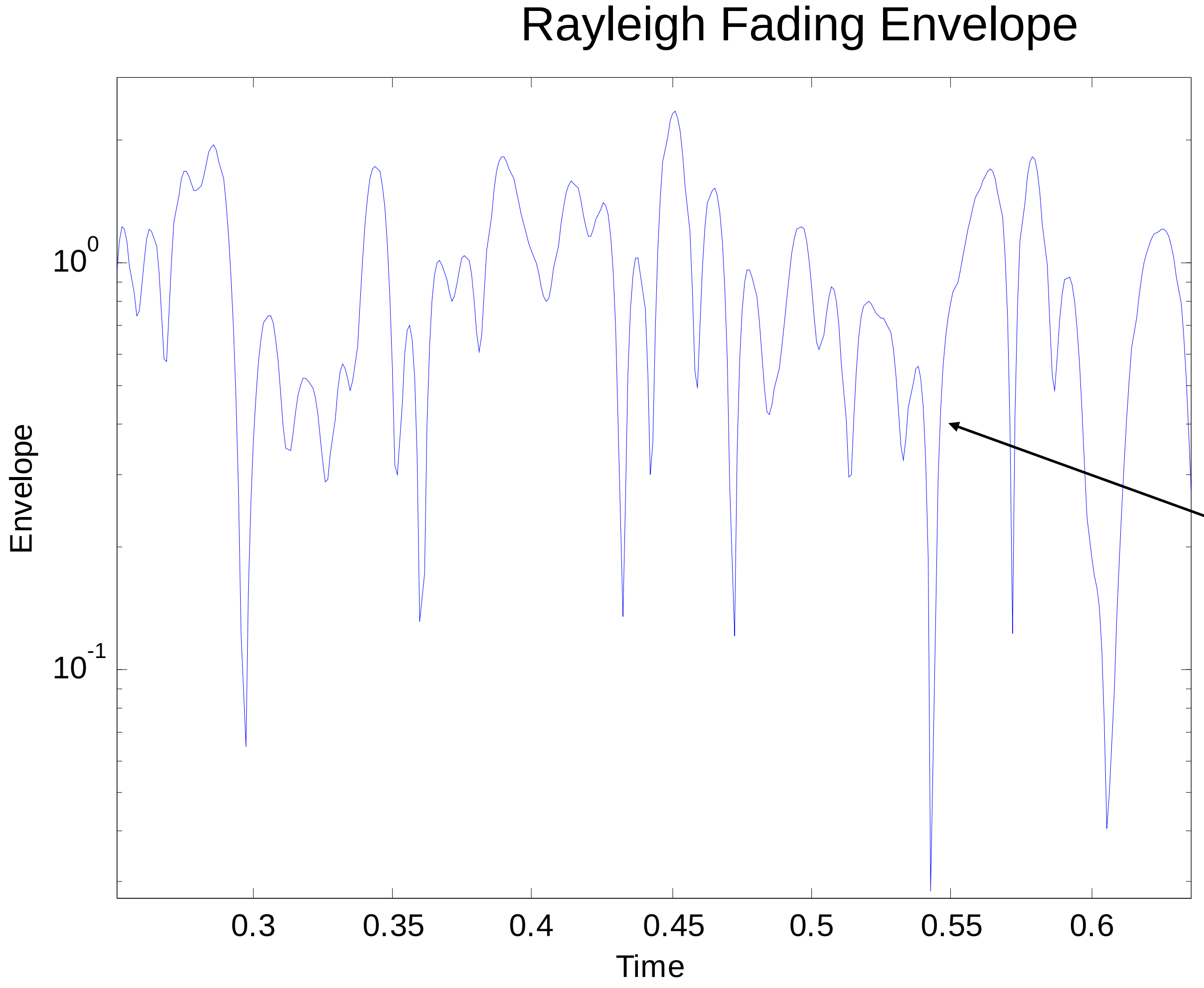
Amplitude = Rayleigh random variable

Rayleigh Fading Channels

- Note however, that unlike thermal noise the Gaussian random processes that model the channel are not *white.*
- They are exhibit a high degree of temporal correlation. Specifically, $R_g(\tau) = J_o(2\pi f_m \tau)$ where $J_o(\ )$ is a zeroth order Bessel function and f_m is the maximum Doppler frequency
- The resulting Doppler spectrum of the flat Rayleigh channel is

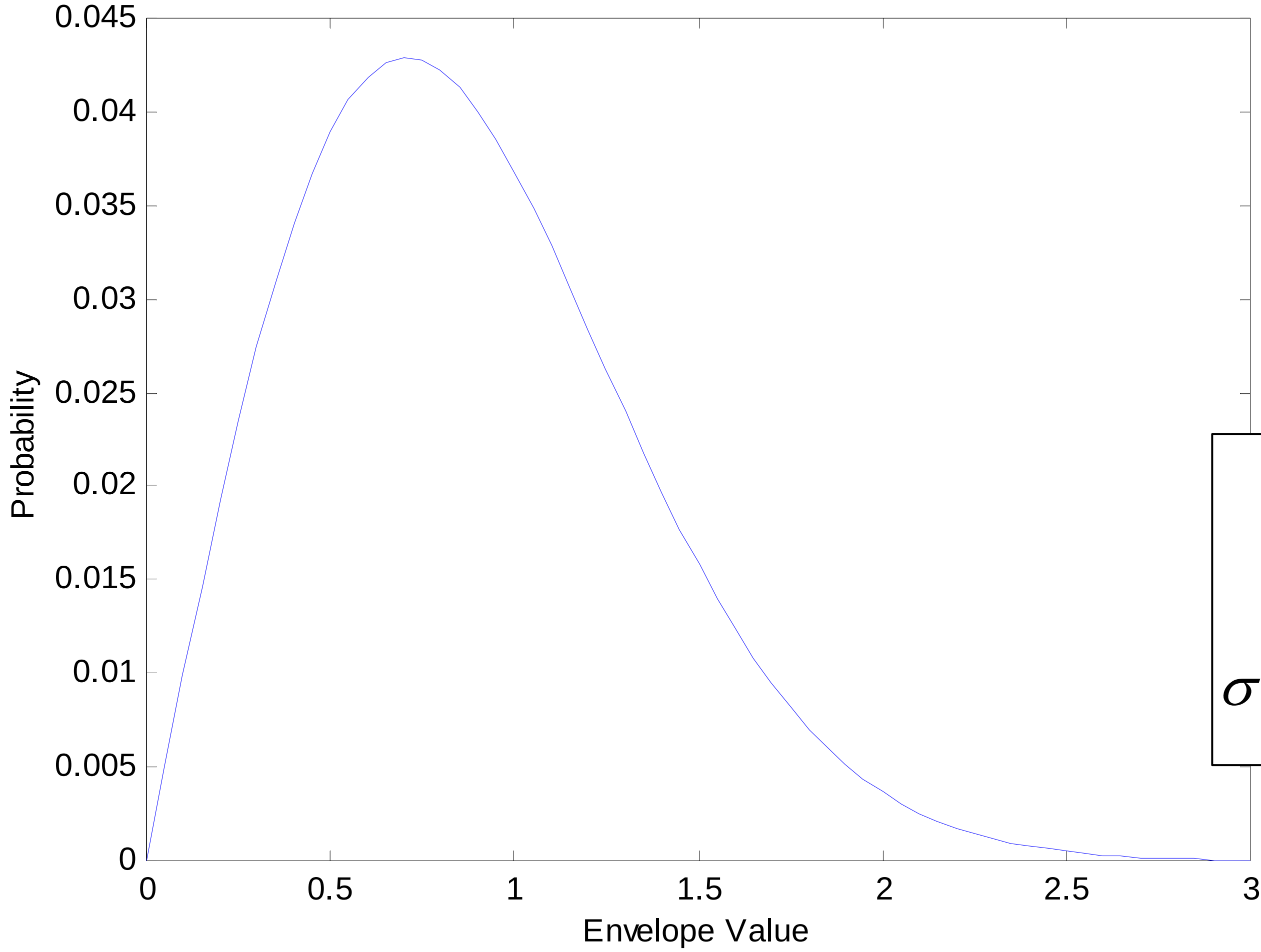
$$S(f) = \begin{cases} \dfrac{1}{\pi f_m \sqrt{1 - \left(\dfrac{f}{f_m} \right)^2}} & |f| \le f_m \\ 0 & else \end{cases}$$

Rayleigh Fading Channels



The temporal correlation (i.e., the rate of fading) is related to the *maximum Doppler frequency* which is due to mobile movement.

Rayleigh Fading Channel



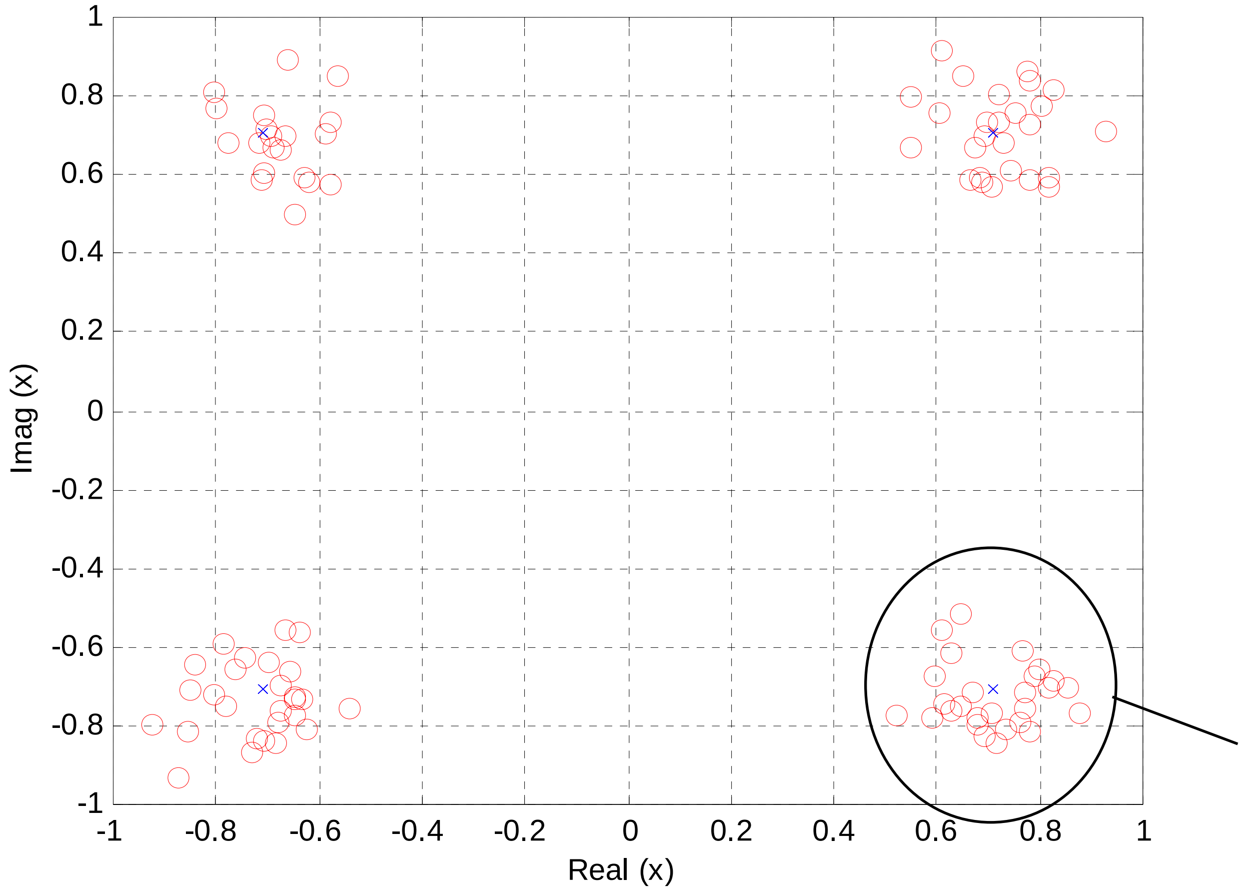
Envelope or Amplitude Distribution

$$p_A(x) = \frac{2}{a} e^{-\frac{x^2}{a}} \qquad x \geq 0$$

$$\sigma^2 = \frac{a(4-\pi)}{4}$$

Usually we assume a = 1;

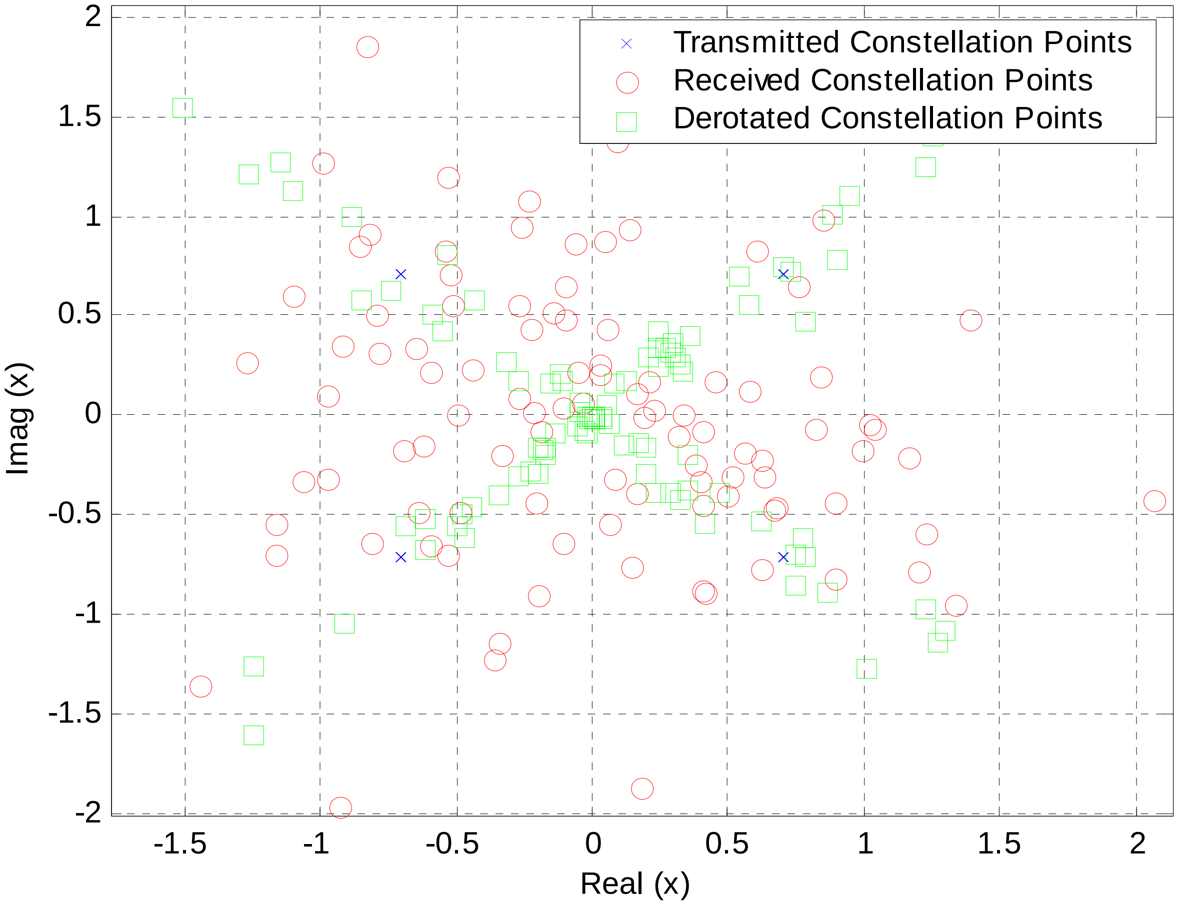
Received Constellation Example: AWGN



g(t) = QPSK signal
x(t) = received signal

Thermal Noise moves points.
Since noise has zero mean, the average value is the same as the transmitted value

Rayleigh Fading Channel – Ex.



g(t) = QPSK signal
x(t) = received signal

- Rayleigh fading and noise move the signal points all over the graph
- Channel compensation “rotates” the points back to approximately the right angle leaving only amplitude distortion.

Simulation Note

- We normally simulate a Rayleigh fading channel by using a summation of complex sinusoids:

$$\gamma(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{j(2\pi f_k t + \theta_k)}$$

- For large *N* (>30) this will tend to a complex Gaussian random process.
- By letting $f_k = f_d \cos\left(\frac{2\pi}{N} k\right)$ where the maximum Doppler frequency is determined by the velocity of the mobile and wavelength of the carrier $f_d = \frac{v}{\lambda}$ we obtain the proper time behavior (correct Doppler spectrum). The phase term θ_k is a uniform random variable over $[0, 2\pi)$.

Ricean Fading Channels

- In Ricean fading we include a line-of-sight path that has considerably more power than the other specular paths. The model is similar to the previous model but the amplitude is a Ricean random variable

$$x(t) = \gamma(t) g(t) + n(t)$$

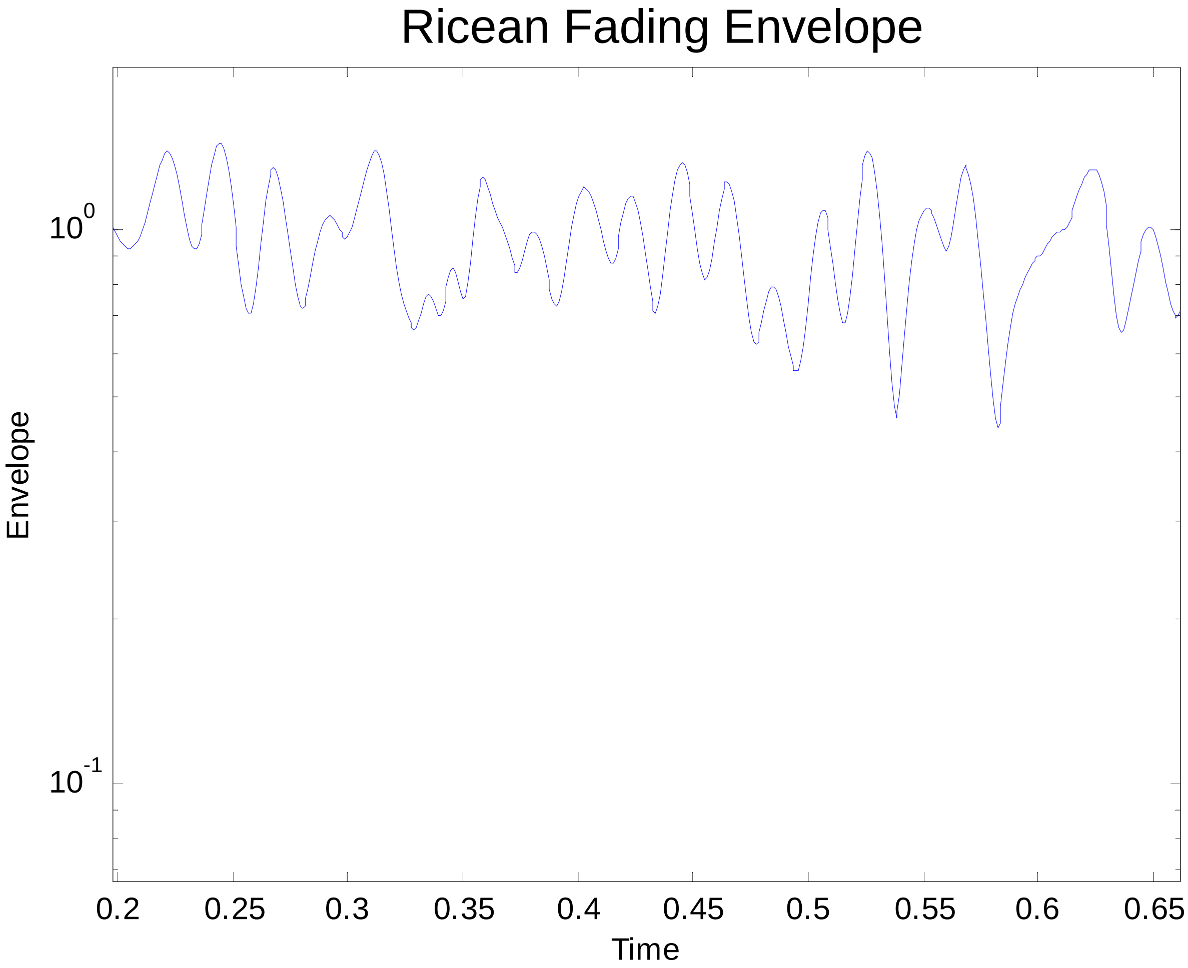
- where

$$\gamma(t) = A(t) e^{j\theta(t)}$$

Phase = uniform random variable

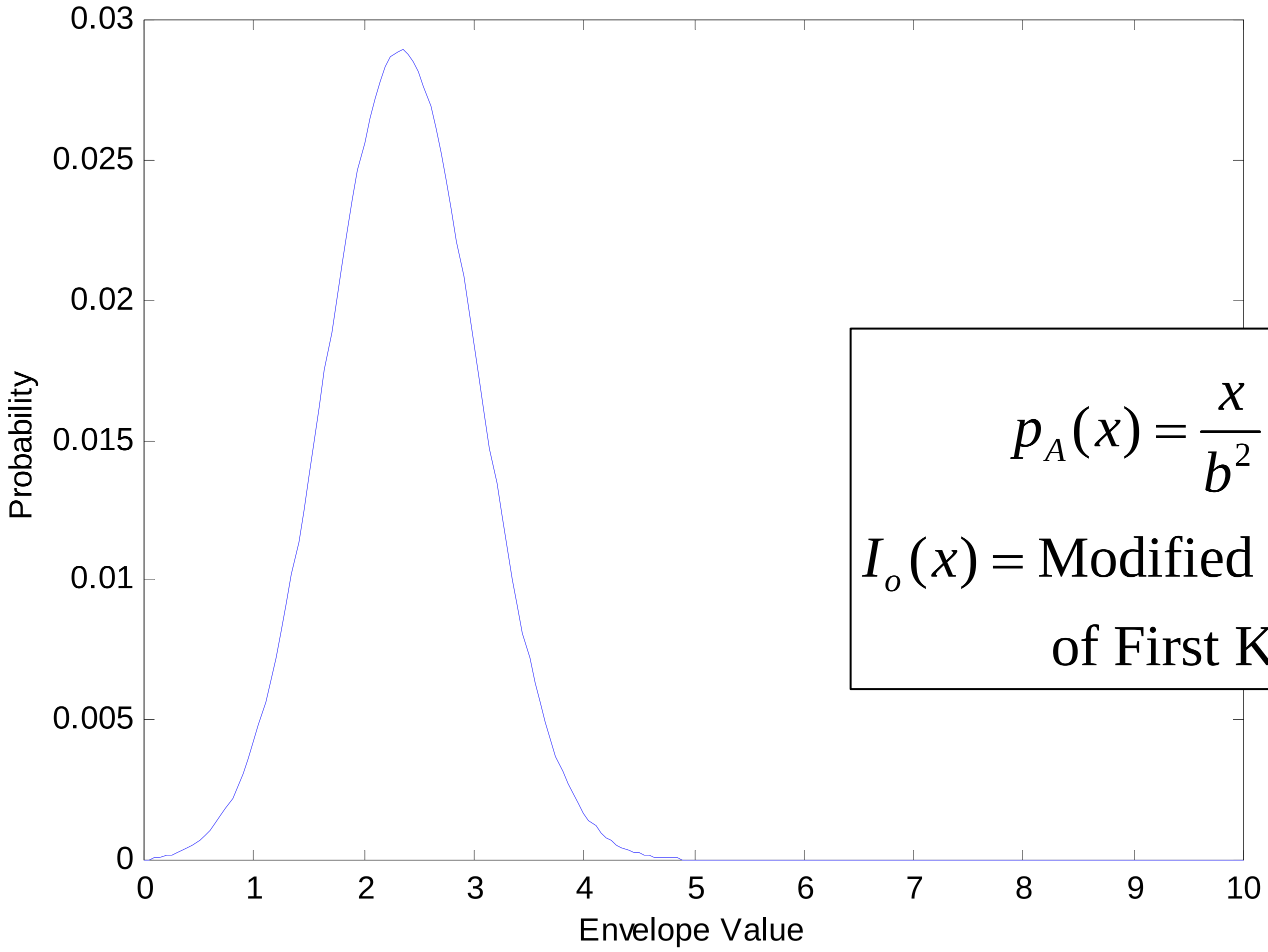
Amplitude = Ricean random variable

Ricean Fading Channels



Fading is less severe than Rayleigh fading

Ricean Fading Channel



Envelope or Amplitude Distribution

$$p_A(x) = \frac{x}{b^2} e^{-\frac{(a^2+x^2)}{2b^2}} I_o\left(\frac{ax}{b^2}\right) \quad x \geq 0$$

$I_o(x) =$ Modified Bessel Function of First Kind (order 0)

Usually we assume $\sigma^2 = 1$;

Performance of BPSK in Rayleigh Fading

- Suppose we transmit a BPSK signal with pulse shape *p(t)*

$$\sqrt{P}b(t)\,p(t)$$

- Through a Flat Rayleigh fading channel with constant fading γ over the symbol duration *T*
- At the receiver we pass the signal through a filter matched to the pulse shape:

$$\begin{aligned}
Z &= \frac{1}{T}\int_0^T r(t)\,p^*(t)dt \\
&= \frac{1}{T}\int_0^T \left(\sqrt{P}b(t)p(t)\gamma + n(t)\right)p^*(t)dt \\
&= \frac{1}{T}\sqrt{P}b\gamma\int_0^T p(t)p^*(t)dt + \frac{1}{T}\int_0^T n(t)p^*(t)dt \\
&= \sqrt{P}\gamma_1 b + N
\end{aligned}$$

Assumes the pulse has unit energy

Performance

- Assuming BPSK modulation, in order to make a decision we must eliminate the phase rotation due to γ. Thus,

$$\begin{aligned}\hat{b} &= \text{sign}\left\{\text{Re}\left[\gamma^* Z\right]\right\}\\ &= \text{sign}\left\{\text{Re}\left[\gamma^*\left(\sqrt{P}\gamma\, b + N\right)\right]\right\}\\ &= \text{sign}\left\{\text{Re}\left[\sqrt{P}\left|\gamma\right|^2 b + \gamma^* N\right]\right\}\\ &= \text{sign}\left\{\sqrt{P}\left|\gamma\right|^2 b + \text{Re}\left[\gamma^* N\right]\right\}\end{aligned}$$

- The probability of error is then equal to

$$\begin{aligned}P_e &= \Pr\left\{\left\{\sqrt{P}\left|\gamma_1\right|^2 b + \text{Re}\left[\gamma_1^* N\right]\right\} < 0 \middle| b = 1\right\}\\ &= Q\left(\sqrt{\frac{\left(\overline{Z_r}\right)^2}{\sigma_{z_r}^2}}\right)\end{aligned}$$

$$\boxed{Z_r = \sqrt{P}\left|\gamma_1\right|^2 b + \text{Re}\left[\gamma_1^* N\right]}$$

Performance

- We can easily show that

$$\overline{Z_r} = \sqrt{P}\left|\gamma_1\right|^2 b$$

$$\sigma_{Z_r}^2 = \left|\gamma_1\right|^2 \frac{N_o}{2T}$$

- Thus

$$P_e = Q\left(\sqrt{\frac{\left(\overline{Z_r}\right)^2}{\sigma_{z_r}^2}}\right) = Q\left(\sqrt{\frac{\left(\sqrt{P}\left|\gamma_1\right|^2 b\right)^2}{\left|\gamma_1\right|^2 \frac{N_o}{2T}}}\right)$$

$$= Q\left(\sqrt{\frac{P\left|\gamma_1\right|^4}{\left|\gamma_1\right|^2 \frac{N_o}{2T}}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b\left|\gamma_1\right|^2}{N_o}}\right)$$

Performance

- For a given channel realization the performance is then

$$P_e = Q\left(\sqrt{\frac{2E_b\left|\gamma_1\right|^2}{N_o}}\right)$$

$$= Q\left(\sqrt{2\beta}\right)$$

$$\boxed{\beta = \frac{E_b\left|\gamma_1\right|^2}{N_o}}$$

- However, we desire the performance over all channel realizations. Thus, we require the distribution of the signal-to-noise ratio β
- Since the γ is a complex Gaussian random variable, β is a central Chi-Square random variable with two degrees of freedom. The underlying GRV has variance equal to the average signal-to-noise ratio of the channel $\overline{\beta}$.

$$p(\beta) = \frac{1}{\overline{\beta}} e^{-\frac{\beta}{\overline{\beta}}} \qquad \beta \geq 0$$

Performance (cont.)

- Substituting

$$P_e = \int_0^\infty p(\beta) Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \int_0^\infty \frac{1}{\bar{\beta}} e^{-\frac{\beta}{\bar{\beta}}} Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\beta}}{1+\bar{\beta}}}\right)$$

- In high SNR situations $\left(\bar{\beta} >> 1\right)$

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\beta}}{1+\bar{\beta}}}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\frac{1}{\bar{\beta}}}}\right)$$

$$\approx \frac{1}{2}\left(1 - \left(1 - \frac{1}{2\bar{\beta}}\right)\right)$$

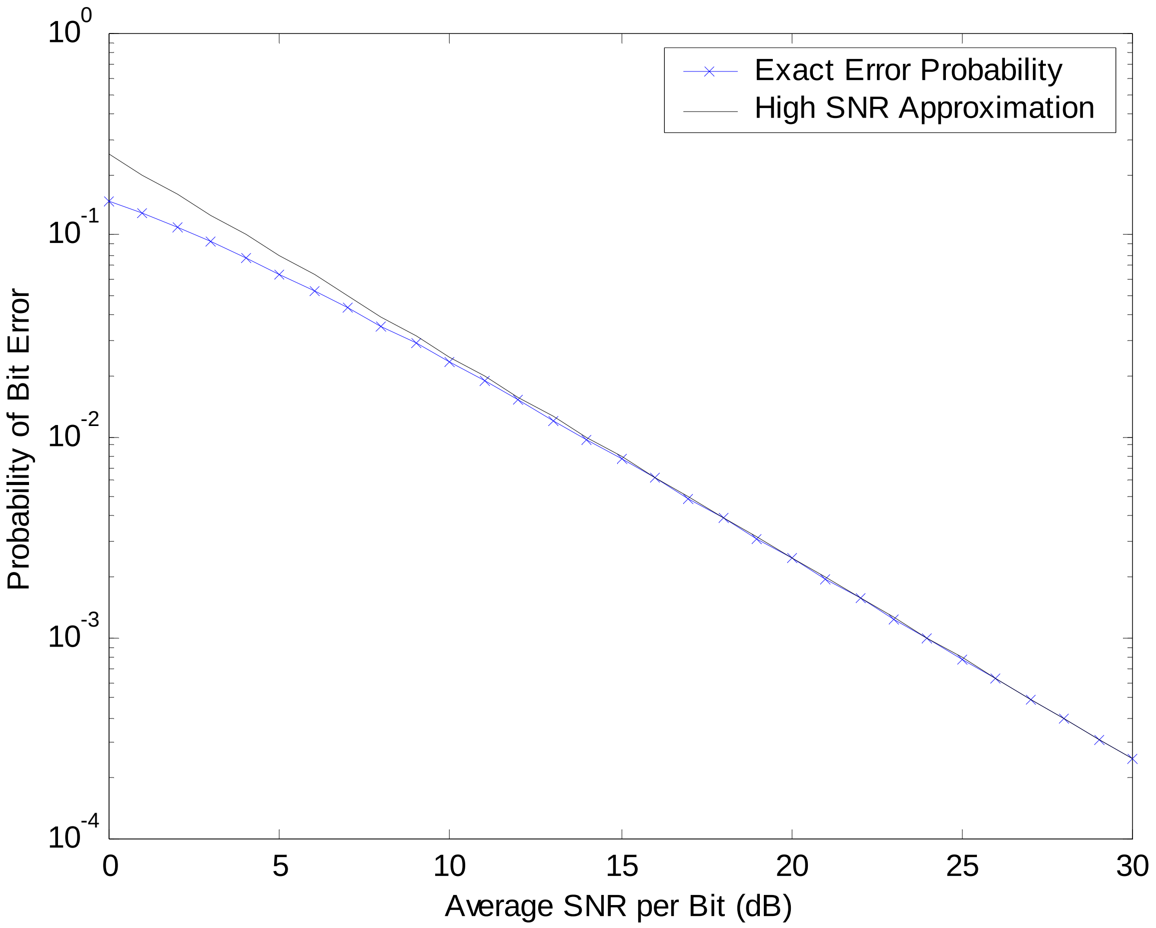
$$= \frac{1}{4\bar{\beta}}$$

Recall the binomial series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \ldots$$

$$\frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \ldots$$

Performance (cont.)



- Flat Rayleigh fading with perfect channel knowledge (we needed to know γ)
- Note that the performance is significantly worse than simple AWGN

Diversity

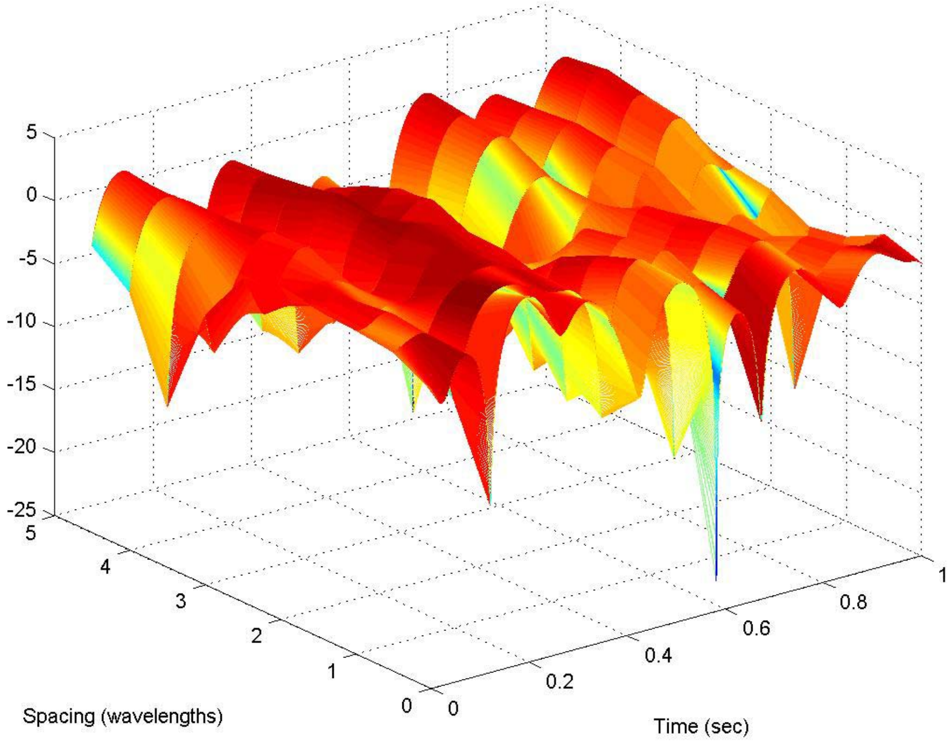
- Diversity techniques provide the receiver several replicas of the received signal (spatial, temporal, frequency diversity possible)
- A receiver with multiple receive antennas achieves diversity by combining information from several (hopefully) independent copies of the received signal (one from each antenna).
- If one multipath component fades, it is very likely that one of the other paths will not be in a fade.
- We typically combine the signals to form a single decision statistic

$$Z = \sum_{k=1}^{L} w_k Z_k$$

- The rule that we use to combine different diversity signals is called the *combining method.*
  - There are several combining methods that we will briefly mention.

Diversity

- Fading changes envelope varies with space and time



Diversity Combining Techniques

1) Equal Gain Combining

$$|w_l| = 1, \quad l = 1, \ldots, L$$

$$\arg(w_l) = \arg(Z_l^*)$$

- This is a good choice only if we know that all components are equal in magnitude.
- Otherwise, weak components can contribute a lot of noise but very little signal to the overall decision statistic.
- If we are using coherent demodulation we must also align the phases of all of the paths.

Diversity Combining Techniques (continued)

2) Selection "Combining"

$$\left|w_l\right| = \begin{cases} 1, & \text{if } \left|Z_l\right| > \left|Z_j\right|, \forall j \neq l \\ 0, & \text{else} \end{cases}$$

- Uses only the strongest decision statistic to compute answer
- If signals fade independently, chances are good that at least one signal will be usable.
- This type of diversity combining is popular in "switched beam" antenna systems

Diversity Combining Techniques (continued)

3) Maximal Ratio Combining

- Select $\{w_1, w_2, \ldots, w_L\}$ to maximize the SNR of the combined decision statistic $Z = \sum_{i=1}^{L} w_l Z_l$
- This SNR is given by:

$$SNR = \frac{P\left(\sum_{l=1}^{L} w_l \gamma_l\right)^2}{\sum_{l=1}^{L} w_l^2 \left(\frac{N_0 T}{2}\right)}$$

Maximal Ratio Combining (continued)

- Setting $\frac{\partial \mathrm{SNR}}{\partial w_l} = 0$ leads to: $w_l = \gamma_l^*,\ l = 1, \ldots, L$
  - We can interpret this as weighting each decision statistic in direct proportion to the relative strength of the component.
- In practice, we may use: $|w_l| = |Z_l|,\ l = 1, \ldots, L$ or some longer term average of decision statistics
- MRC most commonly applied when coherent demodulation is used

Performance of a diversity Receiver

- Consider Maximal Ratio Combining with perfect knowledge of γ_i

$$\begin{aligned} Z &= \sum_{i=1}^{L} \gamma_i^* Z_l \\ &= \sum_{i=1}^{L} \gamma_i^* \left(\sqrt{P} \gamma_i b + N_i \right) \\ &= \sum_{i=1}^{L} \left(\sqrt{P} \left| \gamma_i \right|^2 b + \gamma_i^* N_i \right) \\ &= \sqrt{P} b \sum_{i=1}^{L} \left| \gamma_i \right|^2 + \sum_{i=1}^{L} \gamma_i^* N_i \end{aligned}$$

Performance of a diversity Receiver (cont.)

- Again, we can calculate the probability of error as

$$
\begin{aligned}
P_e &= \Pr\left\{\mathrm{Re}\{Z\} < 0 \middle| b = 1\right\} \\
&= Q\left(\sqrt{\frac{\left(\overline{Z}_r\right)^2}{\sigma_{z_r}^2}}\right) = Q\left(\sqrt{\frac{P\left(\sum_{i=1}^{L}\left|\gamma_i\right|^2\right)^2}{\sum_{i=1}^{L}\left|\gamma_i\right|^2 \frac{N_o}{2T}}}\right) \\
&= Q\left(\sqrt{\frac{2E_b\left(\sum_{i=1}^{L}\left|\gamma_i\right|^2\right)}{N_o}}\right)
\end{aligned}
$$

Performance of a diversity Receiver (cont.)

- Thus, the probability of error is

$$P_e = Q\left(\sqrt{\frac{2E_b\left(\sum_{i=1}^{L}\left|\gamma_i\right|^2\right)}{N_o}}\right) = Q\left(\sqrt{2\beta}\right)$$

$$\boxed{\beta = \frac{E_b\left(\sum_{i=1}^{L}\left|\gamma_i\right|^2\right)}{N_o}}$$

- If all paths are independent and identically distributed, β is a central chi-square random variable with *2L* degrees of freedom. Thus,

$$p\left(\beta\right) = \frac{1}{\left(L-1\right)!\overline{\beta}^{L}}\beta^{L-1}e^{-\frac{\beta}{\overline{\beta}}}$$

Performance of the diversity receiver (cont.)

- The probability of error is then:

$$P_e = \int_0^\infty p(\beta) Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \int_0^\infty \frac{1}{(L-1)!\,\bar{\beta}^L} \beta^{L-1} e^{-\frac{\beta}{\bar{\beta}}} Q\left(\sqrt{2\beta}\right) d\beta$$

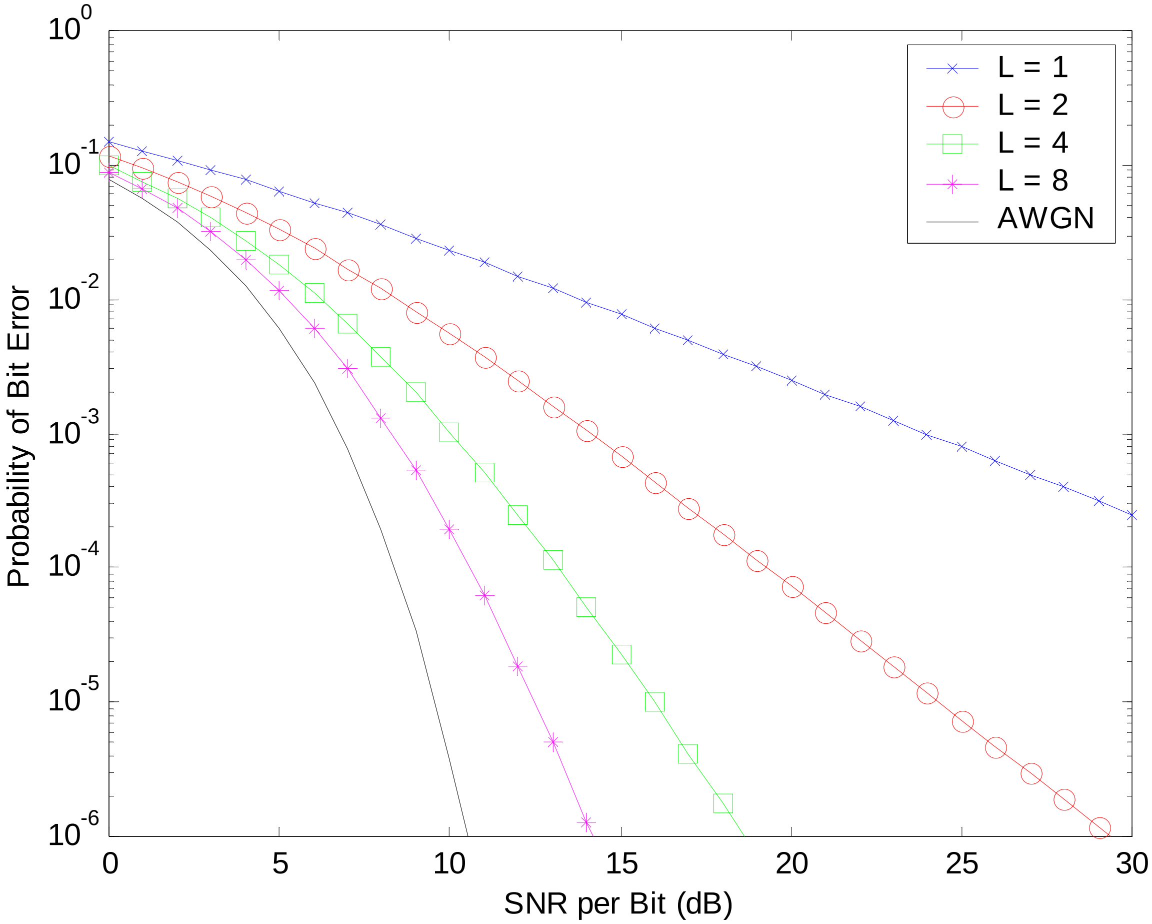
$$= \left[\frac{1}{2}\left(1-\sqrt{\frac{\bar{\beta}}{1+\bar{\beta}}}\right)\right]^{L-1} \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left[\frac{1}{2}\left(1+\sqrt{\frac{\bar{\beta}}{1+\bar{\beta}}}\right)\right]^k$$

- When $\left(\bar{\beta} >> 1\right)$ we obtain

$$P_e = \left(\frac{1}{4\bar{\beta}}\right)^L \binom{2L-1}{L}$$

Note: The exponent (slope on a log-log plot) is equal to the diversity order *L.*

Performance of a diversity Receiver (cont.)



- Diversity receiver can provide substantial improvement in fading environment.
- The performance curve has a slope consistent with the *diversity order L.*
- *Note that this plot does not include average SNR improvement which typically accompanies multiple antenna systems.*

Impact of Unequal Energies

- We have assumed thus far that the energies received on each diversity branch were equal. However, this is not always the case (but typically is).
- We would like to determine the impact that this has on performance. Recall that the probability of error is

$$P_e = Q\left(\sqrt{\frac{2E_b\left(\sum_{i=1}^{L}\left|\gamma_i\right|^2\right)}{N_o}}\right) = Q\left(\sqrt{2\beta}\right)$$

- Thus, we must find the distribution of β. It can be seen that we have the sum of L central chi-square random variables, each with different average SNR values.
- The characteristic function of a sum of random variables is simply the product of the individual characteristic functions. Thus:

$$\psi_\beta(s) = \prod_{i=1}^{L}\frac{1}{1+s\overline{\beta}_i}$$

Impact of Unequal Energies (cont.)

- If all of the SNR values are different we have

$$\psi_{\beta}(s) = \prod_{i=1}^{L} \frac{1}{1 + s\bar{\beta}_i}$$

$$= \sum_{i=1}^{L} \frac{c_i}{1 + s\bar{\beta}_i}$$

- where

$$c_i = \prod_{\substack{k=1 \\ k \neq i}}^{L} \frac{1}{1 + s\bar{\beta}_k} \Bigg|_{s = -\frac{1}{\bar{\beta}_i}}$$

$$= \frac{\left(\bar{\beta}_i\right)^{L-1}}{\prod_{\substack{k=1 \\ k \neq i}}^{L} \left(\bar{\beta}_i - \bar{\beta}_k\right)}$$

Impact of Unequal Energies (cont.)

- Thus, we have

$$\psi_\beta(s)=\sum_{i=1}^{L}\left(\frac{\left(\overline{\beta}_i\right)^{L-1}}{\prod_{\substack{k=1\\k\neq i}}^{L}\left(\overline{\beta}_i-\overline{\beta}_k\right)}\right)\frac{1}{1+s\overline{\beta}_i}$$

- Taking the inverse transform we obtain

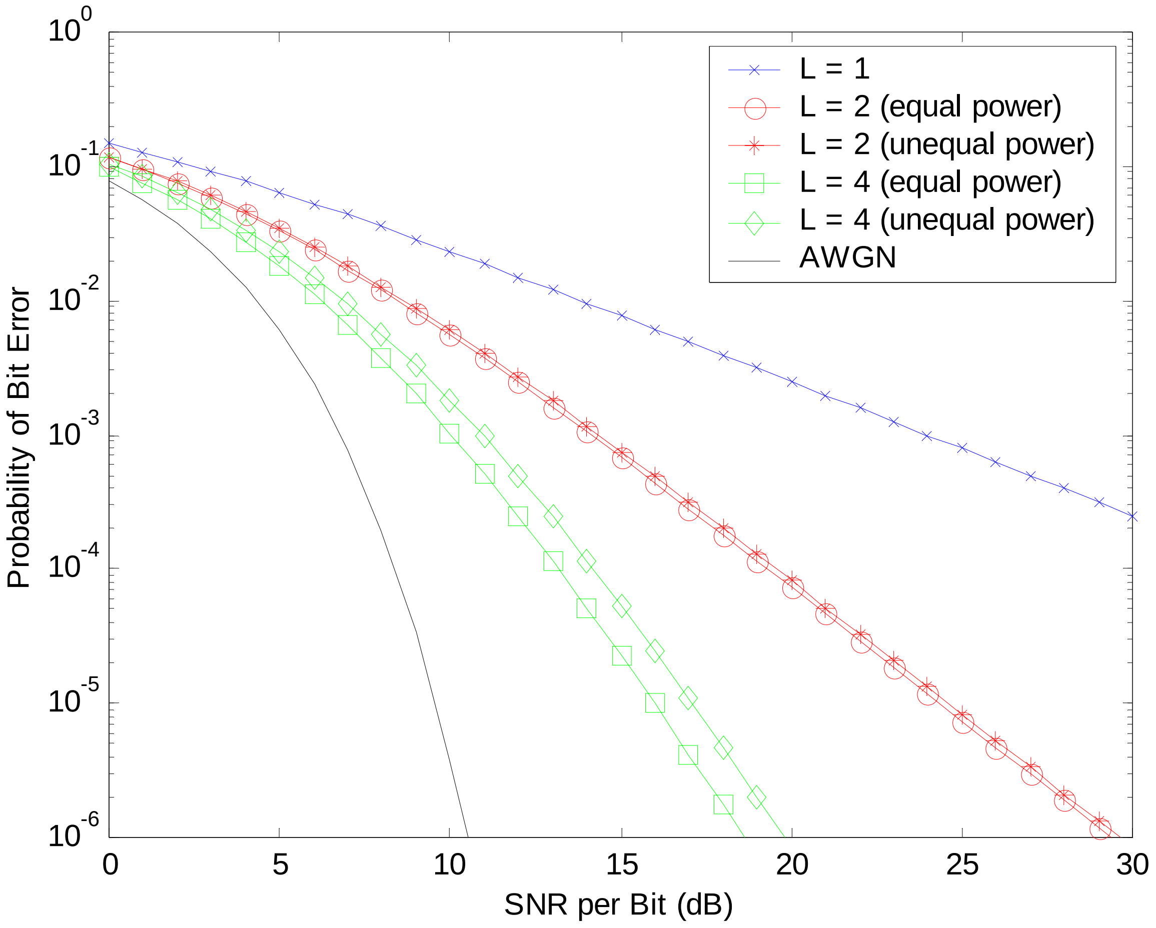
$$p(\beta)=\sum_{i=1}^{L}\left(\frac{\left(\overline{\beta}_i\right)^{L-1}}{\prod_{k\neq i}\left(\overline{\beta}_i-\overline{\beta}_k\right)}\right)\exp\left[-\frac{\beta}{\overline{\beta}_i}\right]$$

- The probability of error is then

$$P_e=\int_0^\infty p(\beta)Q\left(\sqrt{2\beta}\right)d\beta=\int_0^\infty\left(\sum_{i=1}^{L}\left(\frac{\left(\overline{\beta}_i\right)^{L-1}}{\prod_{k\neq i}\left(\overline{\beta}_i-\overline{\beta}_k\right)}\right)\exp\left[-\frac{\beta}{\overline{\beta}_i}\right]Q\left(\sqrt{2\beta}\right)\right)d\beta$$

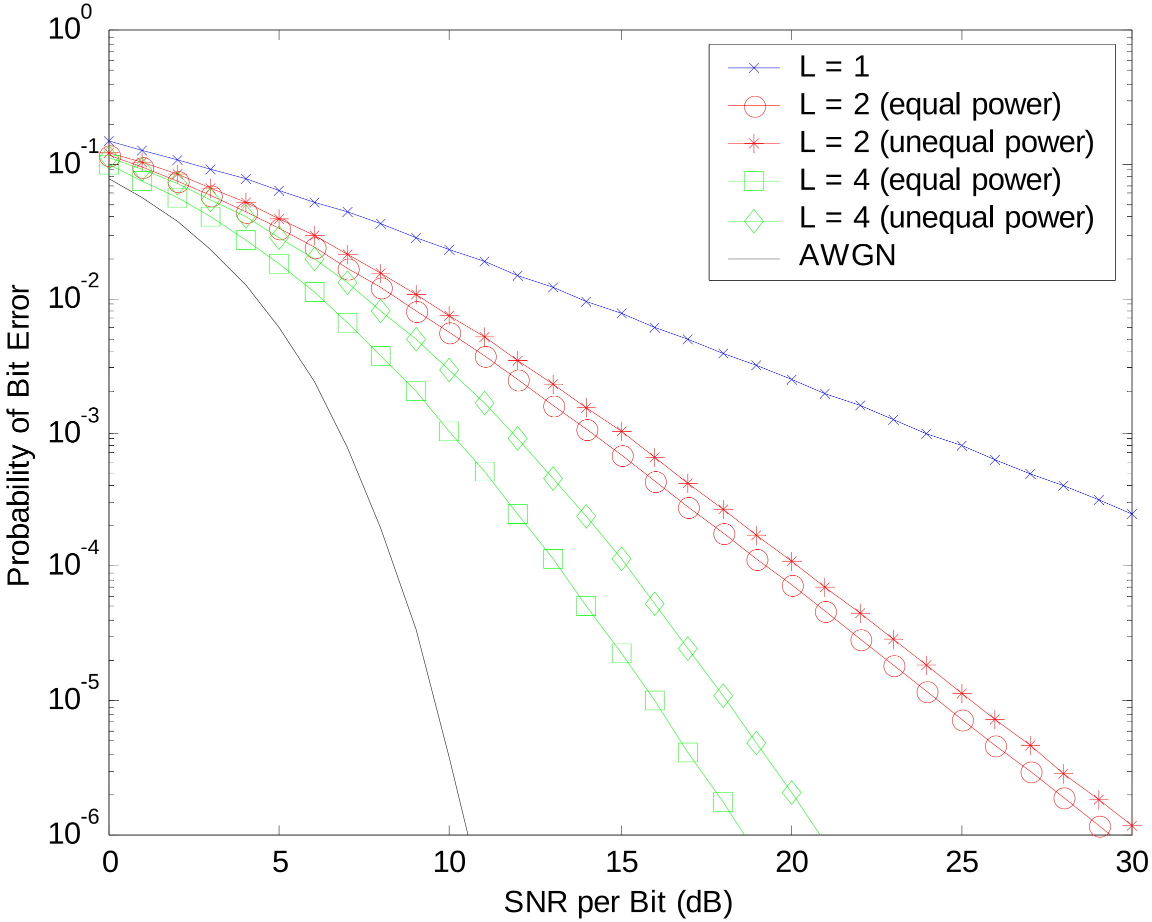
$$=\frac{1}{2}\sum_{i=1}^{L}\left(\frac{\left(\overline{\beta}_i\right)^{L-1}}{\prod_{k\neq i}\left(\overline{\beta}_i-\overline{\beta}_k\right)}\right)\left(1-\sqrt{\frac{\overline{\beta}_i}{1+\overline{\beta}_i}}\right)$$

Impact of Unequal Energies (cont.)



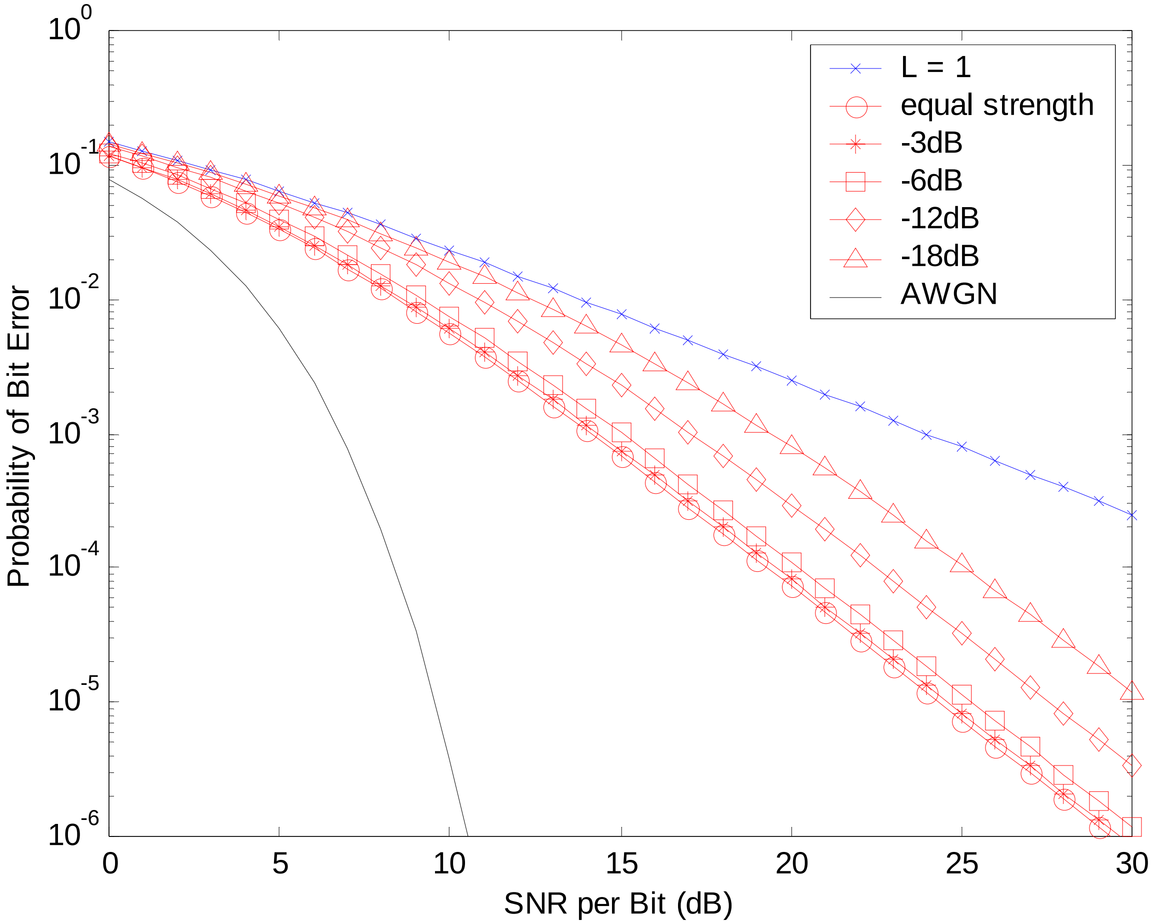
- 2 diversity branches
  - Second branch 3dB down from first
- 4 branches
  - 3, 6, 9 dB down
- Perfect channel knowledge

Impact of Unequal Path Energies (cont.)



- 2 branches
  - Second branch 6dB down from first
- 4 branches
  - 6, 9, 12 dB down
- Perfect channel knowledge

Impact of Unequal Path Energies (cont.)



- 2 path profiles with weak second path
- Perfect channel knowledge
- Even at 6dB below the main branch, there is almost no diversity loss
- A diversity branch that is 12dB down can still provide diversity benefit (channel estimation could be difficult though)