EE 5654 - Digital Communications Spring 2005

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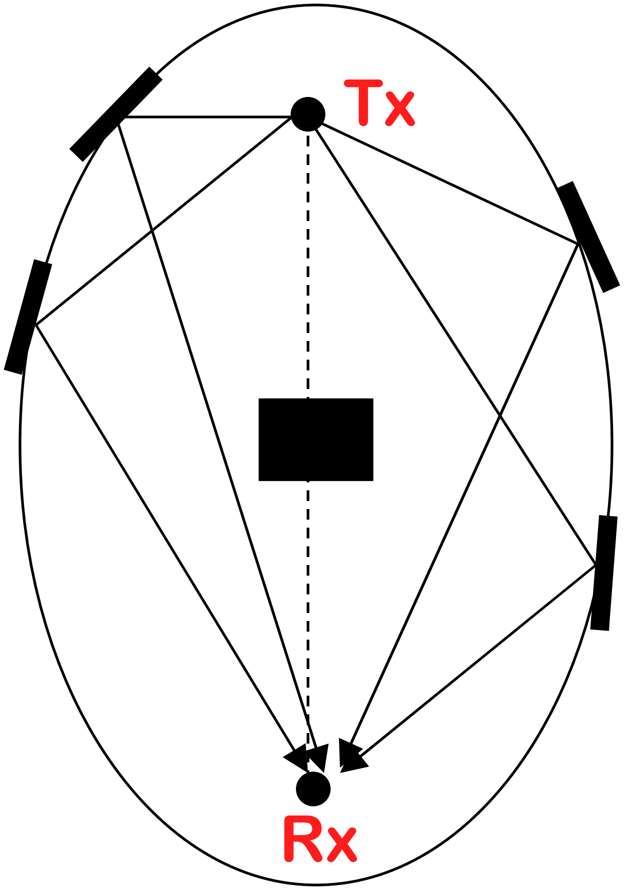
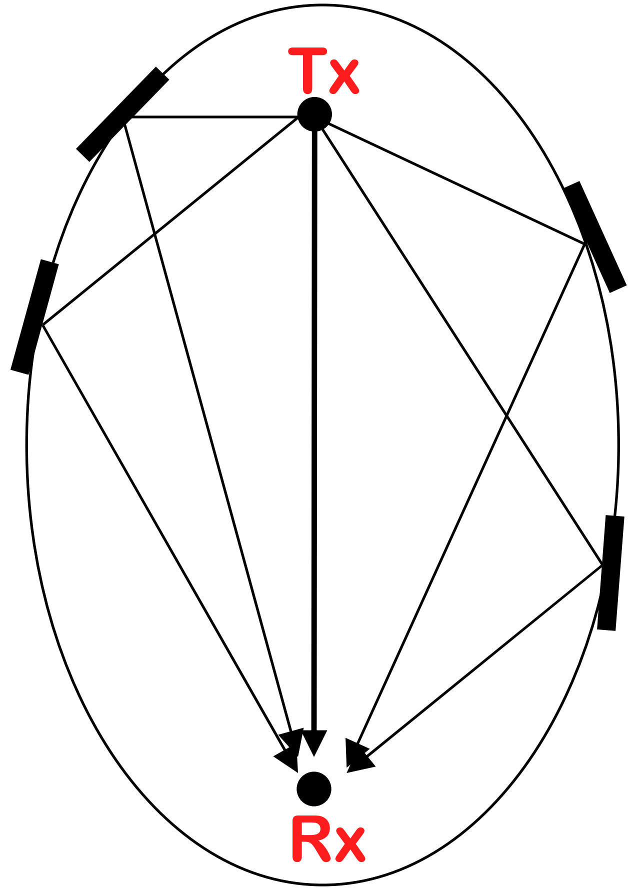
Lecture #13 –Diversity Mechanisms

Geometric Interpretation of Fading Models

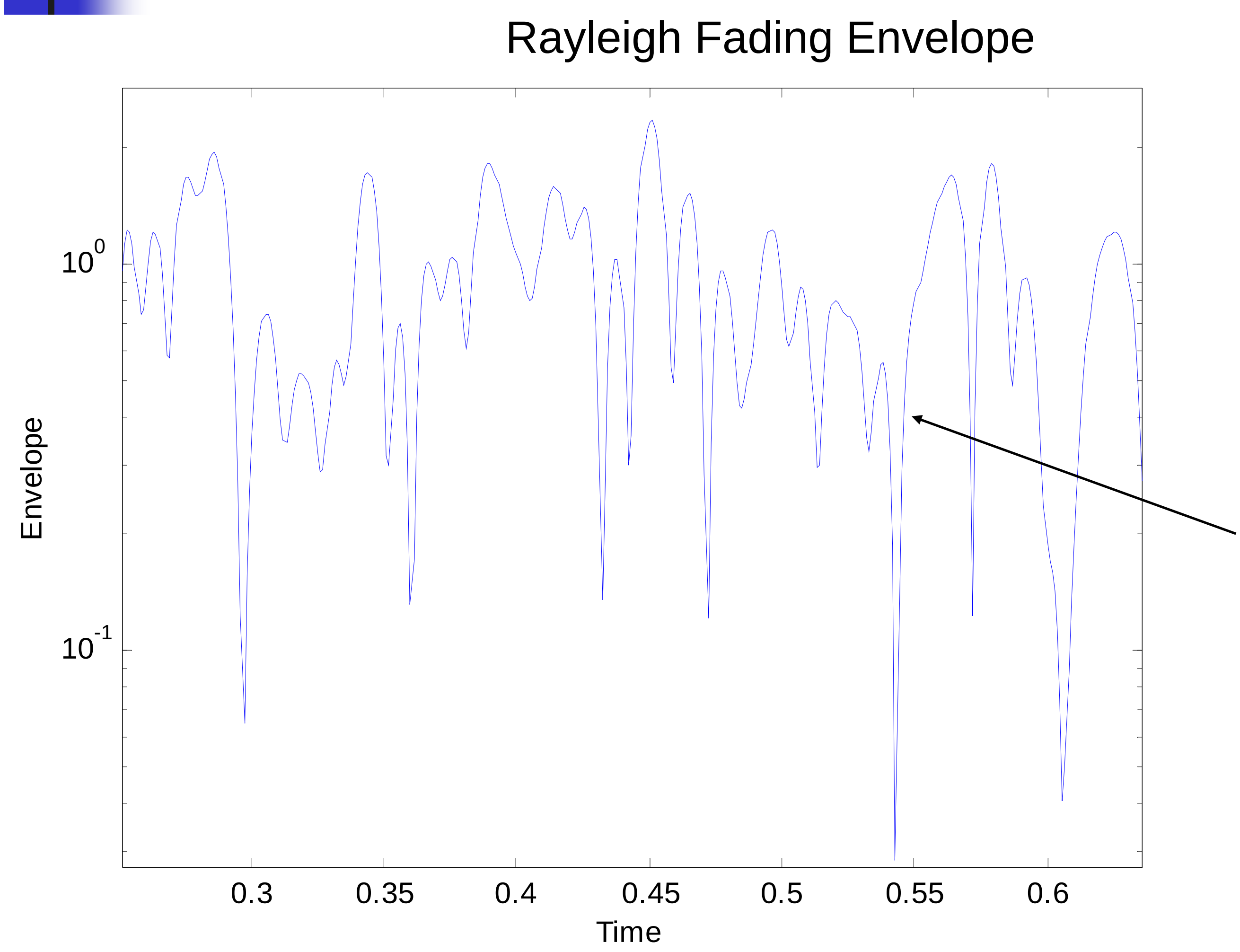
NLOS-Channel (Rayleigh)



LOS-Channel (Rician)

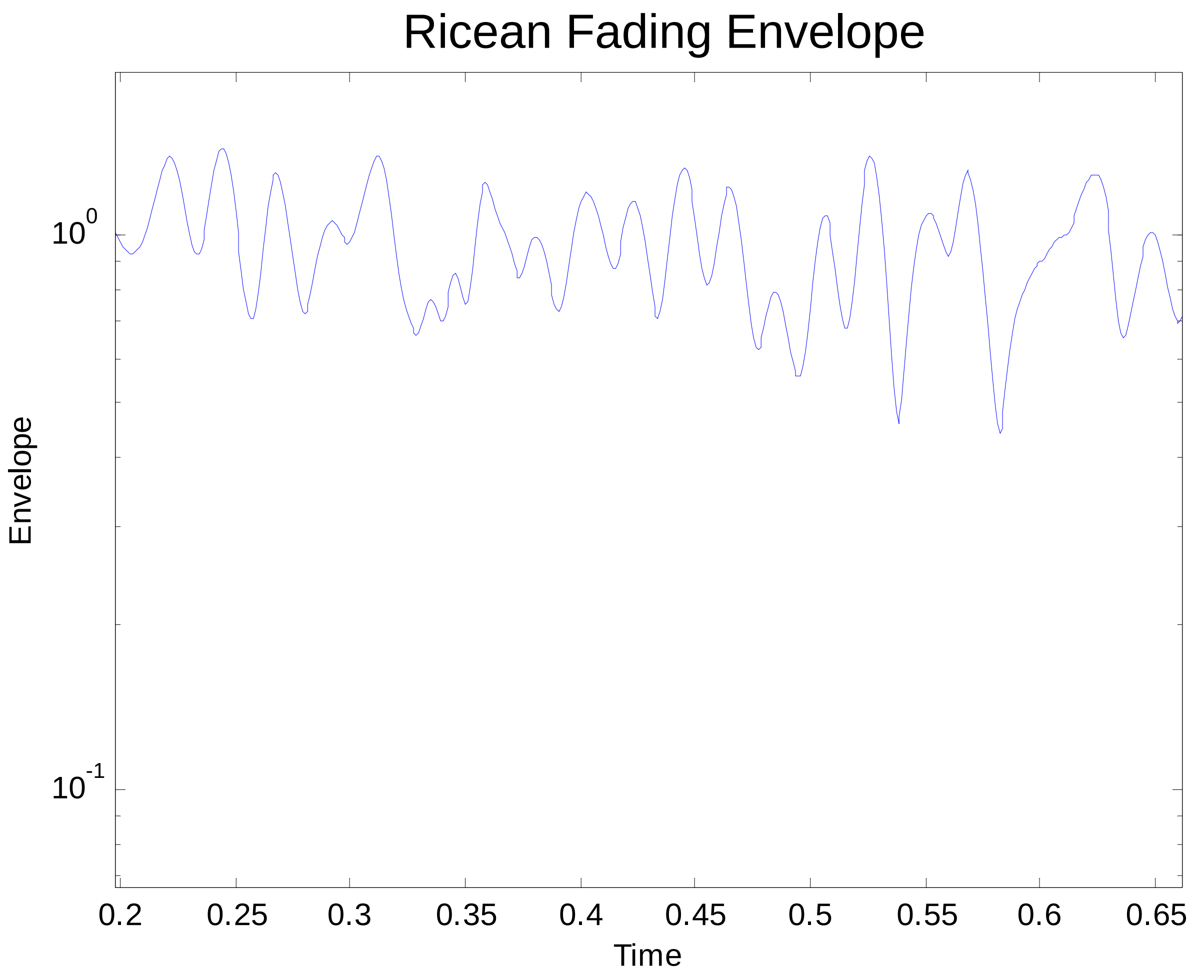


Rayleigh Fading Channels



The temporal correlation (i.e., the rate of fading) is related to the *maximum Doppler frequency* which is due to mobile movement.

Ricean Fading Channels



Fading is less severe than Rayleigh fading

Ricean Fading

- If there is a direct (LOS) path present between the transmitter and the receiver, the signal envelope is no longer Rayleigh and the distribution of the signal amplitude is Ricean.
- The Ricean distribution is often defined in terms of the Ricean **K-factor**, which is the ratio of the power in the mean component of the channel to the power in the scattered component. The Ricean PDF of the envelope of the received signal is given by

$$p_R(r) = \frac{2r(K+1)}{\Omega} e^{-K-(K+1)r^2/\Omega} I_0\left(2r\sqrt{\frac{K(K+1)}{\Omega}}\right), \qquad r > 0$$

Ω is the mean received power,

$I_0(x)$ is the zero-order Bessel function of the first kind

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-x\cos\theta} d\theta$$

Simulation of Rician Fading

- The fading amplitude γ at any time instant i can be represented as

$$\gamma = \sqrt{\left(x_i + \xi\right)^2 + \left(y_i + \xi\right)^2}$$

- where ξ is the amplitude of the specular component and x_i, y_i ~ N(0, σ^2) are samples of zero-mean stationary Gaussian random processes each with variance σ^2.
- The ratio of specular to diffuse energy defines the so-called Rician K-factor, which is given by

$$K = \frac{\xi^2}{\sigma^2}$$

- The Rician *K*-factor is therefore a measure of the reliability of the link, since it determines how strong the LOS path is relative to the diffuse multipath.

Nakagami-*m* Distribution

- The PDF of the Nakagami-*m* distribution is given by

$$p_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega}$$

- The parameter Ω is defined by

$$\Omega = E\left(R^2\right)$$

- The parameter *m* is defined as the ratio of moments, called *the fading figure*:

$$m = \frac{\Omega^2}{E\left[\left(R^2 - \Omega^2\right)\right]}, \qquad m \geq \frac{1}{2}$$

- The *n*th moment of *R* is given by

$$E\left[R^n\right] = \frac{\Gamma\left(m + \frac{1}{2}n\right)}{\Gamma\left(m\right)}\left(\frac{\Omega}{m}\right)^{n/2}$$

A Special case — *m* =1

- For $m = 1$, the expression for the PDF of the Nakagami-*m* distribution is:
- This is exactly the expression for the PDF of the Rayleigh distribution.

$$p_R(r) = \left(\frac{r}{\Omega}\right) e^{-r^2/\Omega}$$

$$\Omega = E\left(R^2\right)$$

$$E\left[R^n\right] = \frac{\Gamma\left(m + \frac{1}{2}n\right)}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{n/2}$$

- The *n*th moment of *R* can be computed using the general expression:

$$\Rightarrow E\left[R^n\right] = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{2} \Omega^{n/2}$$

Another Special case — *m* =1/2

- For *m* = 1/2, the expression for the PDF of the Nakagami-*m* distribution is:
- This is exactly the expression for the PDF of the zero-mean Gaussian random variable with variance Ω.

$$p_R(r) = \frac{2}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{1}{2\Omega}\right)^{1/2} e^{-r^2/2\Omega} = \left(\frac{1}{2\pi\Omega}\right)^{1/2} e^{-r^2/2\Omega}$$

$$\Omega = E\left(R^2\right)$$

- The *n*th moment of *R* can be computed using the general expression:

$$E\left[R^n\right] = \frac{\Gamma\left(m + \frac{1}{2}n\right)}{\Gamma(m)}\left(\frac{\Omega}{m}\right)^{n/2}$$

$$\Rightarrow E\left[R^n\right] = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}(2\Omega)^{n/2}$$

Notes on the Gamma function

- Definition:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

- Properties
  - If *n* is an integer $\Gamma(n) = (n-1)!$
  - For a general *x* $\Gamma(x) = x\Gamma(x-1)$
  - For x = 1/2 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Rician-K Factor and Nakagami-m factor

- The amount of fading of a channel, (which can be defined as the variance of the received power to the square of the mean of the received energy) is controlled by the *m*-factor. The Nakagami-*m* distribution therefore spans a range of fading environments via the *m* parameter.
- As *m* increases, the fading amount decreases, and in the limit as $m \rightarrow \infty$, the Nakagami fading channel converges to an AWGN channel.
- When $m \neq 1$, a one-to-one mapping between the Rician-*K* factor and the Nakagami-*m* parameter allows the Nakagami-*m* distribution to closely approximate the Rice distribution.

$$m = \frac{(K+1)^2}{2K+1} \Rightarrow \frac{1}{m} = 1 - \frac{K^2}{(K+1)^2}$$

Comparison of Nakagami and Rician Distributions

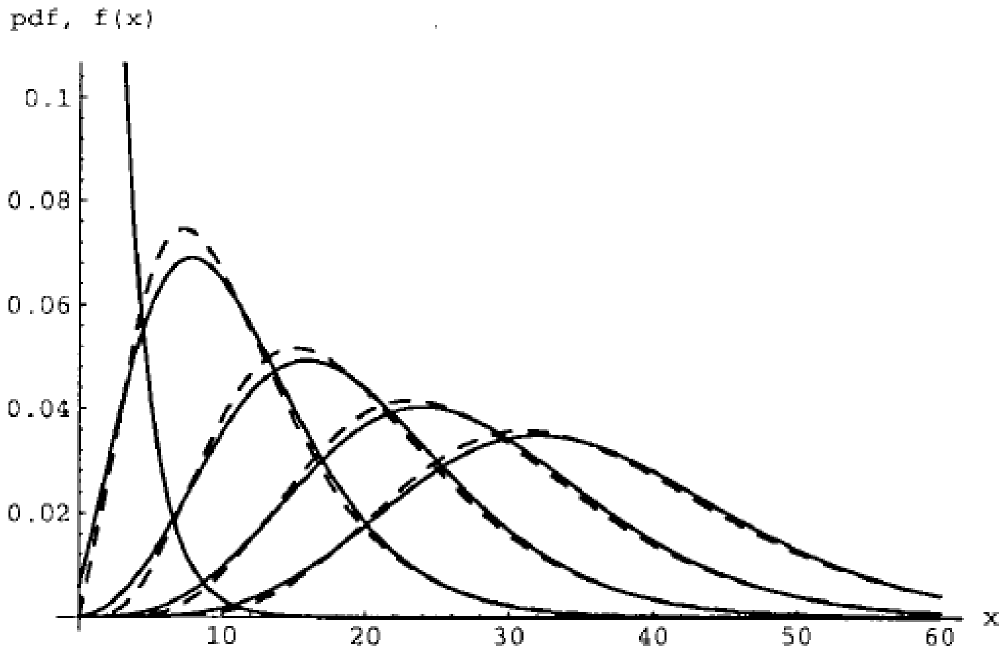


Fig. 1. Comparison of Rician fading model (noncentral Chi-square distribution with 2 d.f., solid lines) and Nakagami fading model (Gamma distribution, dashed lines), where $m_d = 1, 3, 5, 7, 9$; the scattering power is equal to two.

Co-channel interference analysis of shadowed Rician channels
Wang, L.-C.; Lea, C.-T.;
Communications Letters, IEEE , Volume: 2 , Issue: 3 , March 1998
Pages:67 - 69

Performance of BPSK in Rician Fading

- Suppose we transmit a BPSK signal with pulse shape *p(t)*

$$\sqrt{P}b(t)p(t)$$

- Through a Rician flat fading channel with constant fading γ over the symbol duration *T.*
- At the receiver we pass the signal through a filter matched to the pulse shape:

$$\begin{aligned} Z &= \frac{1}{T}\int_0^T r(t)\,p^*(t)dt \\ &= \frac{1}{T}\int_0^T \left(\sqrt{P}b(t)p(t)\gamma + n(t)\right) p^*(t)dt \\ &= \frac{1}{T}\sqrt{P}b\gamma\int_0^T p(t)p^*(t)dt + \frac{1}{T}\int_0^T n(t)p^*(t)dt \\ &= \sqrt{P}\gamma b + N \end{aligned}$$

Performance

- Assuming BPSK modulation, in order to make a decision we must eliminate the phase rotation due to γ. Thus,

$$\begin{aligned}\hat{b} &= \text{sign}\left\{\text{Re}\left[\gamma^* Z\right]\right\} \\ &= \text{sign}\left\{\text{Re}\left[\gamma^*\left(\sqrt{P}\gamma\, b + N\right)\right]\right\} \\ &= \text{sign}\left\{\text{Re}\left[\sqrt{P}|\gamma|^2 b + \gamma^* N\right]\right\} \\ &= \text{sign}\left\{\sqrt{P}|\gamma|^2 b + \text{Re}\left[\gamma^* N\right]\right\}\end{aligned}$$

- The probability of error is then equal to

$$\begin{aligned}P_e &= \Pr\left\{\left\{\sqrt{P}|\gamma|^2 b + \text{Re}\left[\gamma^* N\right]\right\} < 0 \,\middle|\, b = 1\right\} \\ &= Q\left(\sqrt{\frac{\left(\overline{Z_r}\right)^2}{\sigma_{Z_r}^2}}\right)\end{aligned}$$

$$\boxed{Z_r = \sqrt{P}|\gamma|^2 b + \text{Re}\left[\gamma^* N\right]}$$

Performance

- We can easily show that

$$\overline{Z_r} = \sqrt{P}\,|\gamma|^2\, b$$

$$\sigma^2_{Z_r} = |\gamma|^2 \frac{N_o}{2T}$$

- Thus

$$P_e = Q\left(\sqrt{\frac{\left(\overline{Z_r}\right)^2}{\sigma^2_{z_r}}}\right) = Q\left(\sqrt{\frac{\left(\sqrt{P}|\gamma|^2 b\right)^2}{|\gamma|^2 \frac{N_o}{2T}}}\right)$$

$$= Q\left(\sqrt{\frac{P|\gamma|^4}{|\gamma|^2 \frac{N_o}{2T}}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b|\gamma_1|^2}{N_o}}\right)$$

Performance

- For a given channel realization the performance is then

$$P_e = Q\left(\sqrt{\frac{2E_b|\gamma|^2}{N_o}}\right)$$
$$= Q\left(\sqrt{2\beta}\right)$$

$$\beta = \frac{E_b|\gamma|^2}{N_o} = \frac{E_b\left(\gamma_R^2 + \gamma_I^2\right)}{N_o}$$

- However, we desire the performance over all channel realizations. Thus, we require the distribution of the signal-to-noise ratio β
- Since the γ is a complex Gaussian random variable, β is a **non-central Chi-Square random variable with two degrees of freedom** with **non-centrality parameter** $\lambda = 2\xi^2$. The underlying GRV has variance equal to the average signal-to-noise ratio of the channel $\bar{\beta} = 2\sigma^2$

$$p(\beta) = \frac{1}{2\sigma^2}\left(\frac{\beta}{2\xi^2}\right)\exp\left(-\frac{\beta + 2\xi^2}{2\sigma^2}\right)I_0\left(\frac{2\xi^2\sqrt{\beta}}{\sigma^2}\right), \quad \beta \geq 0$$

Performance (cont.)

- Substituting

$$P_e = \int_0^\infty p(\beta) Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \frac{1}{2\sigma^2} \int_0^\infty \left(\frac{\beta}{2\xi^2} \right) \exp\left(-\frac{\beta + 2\xi^2}{2\sigma^2} \right) I_0\left(\frac{2\xi^2 \sqrt{\beta}}{\sigma^2} \right) Q\left(\sqrt{2\beta}\right) d\beta$$

- This integral is hard to compute. Therefore, we use a χ^2 approximation for the non-central χ^2 distribution.

$$f_{\bar{\chi}^2}(x|n) \approx \frac{1}{c} f_{\chi^2}\left(\frac{x}{c}\middle| n^*\right)$$

$$c = \frac{n + 2\lambda}{n + \lambda}$$

$$n^* = \frac{(n + \lambda)^2}{n + 2\lambda}$$

Using the approximation ...

$$f_{\bar{\chi}^2}(x|n) \approx \frac{1}{c} f_{\chi^2}(\frac{x}{c}\Big| n^*) = \frac{1}{\left(2\sigma^2 c\right)^{\frac{n^*}{2}} \Gamma\left(\frac{n^*}{2}\right)} \exp\left(-\frac{x}{2c\sigma^2}\right) x^{\frac{n^*}{2}-1}$$

$$c = \frac{n+2\lambda}{n+\lambda} = \frac{2+4\xi^2}{2+2\xi^2} > 1$$

$$n^* = \frac{(2+2\xi^2)^2}{2+4\xi^2} = 2 \bullet \frac{(1+\xi^2)^2}{1+2\xi^2} = 2\left[1+\frac{\xi^4}{1+2\xi^2}\right] \geq 2 = n$$

- **Mean of a χ^2 random variable with *n* degrees of freedom = $n\sigma^2$.**

- **Therefore, the mean of the NC- χ^2 random variable with n^* degrees of freedom is more than that of a χ^2 random variable with *n* degrees of freedom.**

- **This suggests that the average SNR in the Rician case is higher than the Rayleigh case, and we should expect better performance**

Performance (cont.)

- Substituting $P_e = \int_0^\infty p(\beta) Q\left(\sqrt{2\beta}\right) d\beta$

$$\approx \frac{1}{\left(2\sigma^2 c\right)^{\frac{n^*}{2}} \Gamma\left(\frac{n^*}{2}\right)} \int_0^\infty \exp\left(-\frac{\beta}{2c\sigma^2}\right) \beta^{\frac{n^*}{2}-1} Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \frac{1}{\left(2\sigma^2 c\right)^{\frac{n^*}{2}} \Gamma\left(\frac{n^*}{2}\right)} \int_0^\infty \exp\left(-\frac{\beta}{2c\sigma^2}\right) \beta^{\frac{n^*}{2}-1} Q\left(\sqrt{2\beta}\right) d\beta$$

$$= \left[\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\beta}^*}{1+\overline{\beta}^*}}\right)\right]^{n^*-1} \sum_{k=0}^{L-1} \binom{n^*-1+k}{k} \left[\frac{1}{2}\left(1 + \sqrt{\frac{\overline{\beta}^*}{1+\overline{\beta}^*}}\right)\right]^k$$

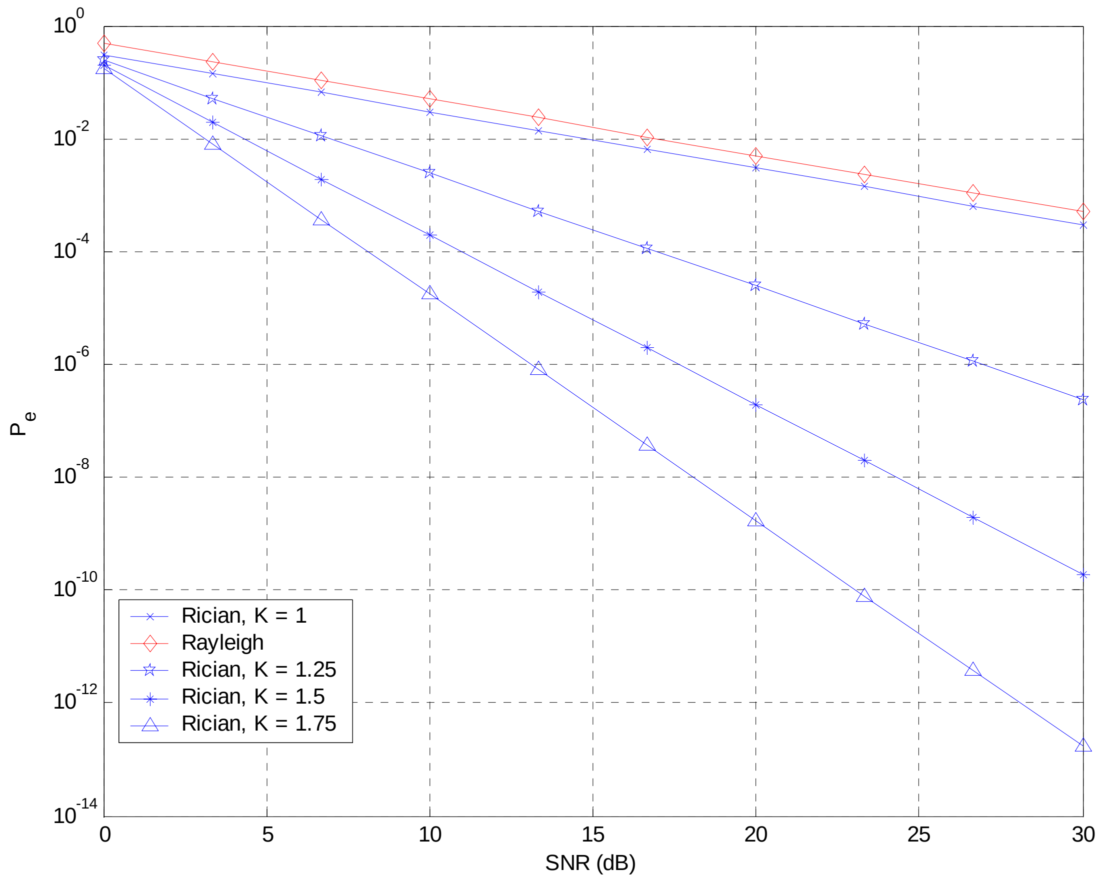
High SNR approximation ...

- When $\left(\overline{\beta} >> 1\right)$ we obtain

$$P_e \approx \left(\frac{1}{4\overline{\beta^*}}\right)^{n^*} \binom{2n^* - 1}{n^*}$$

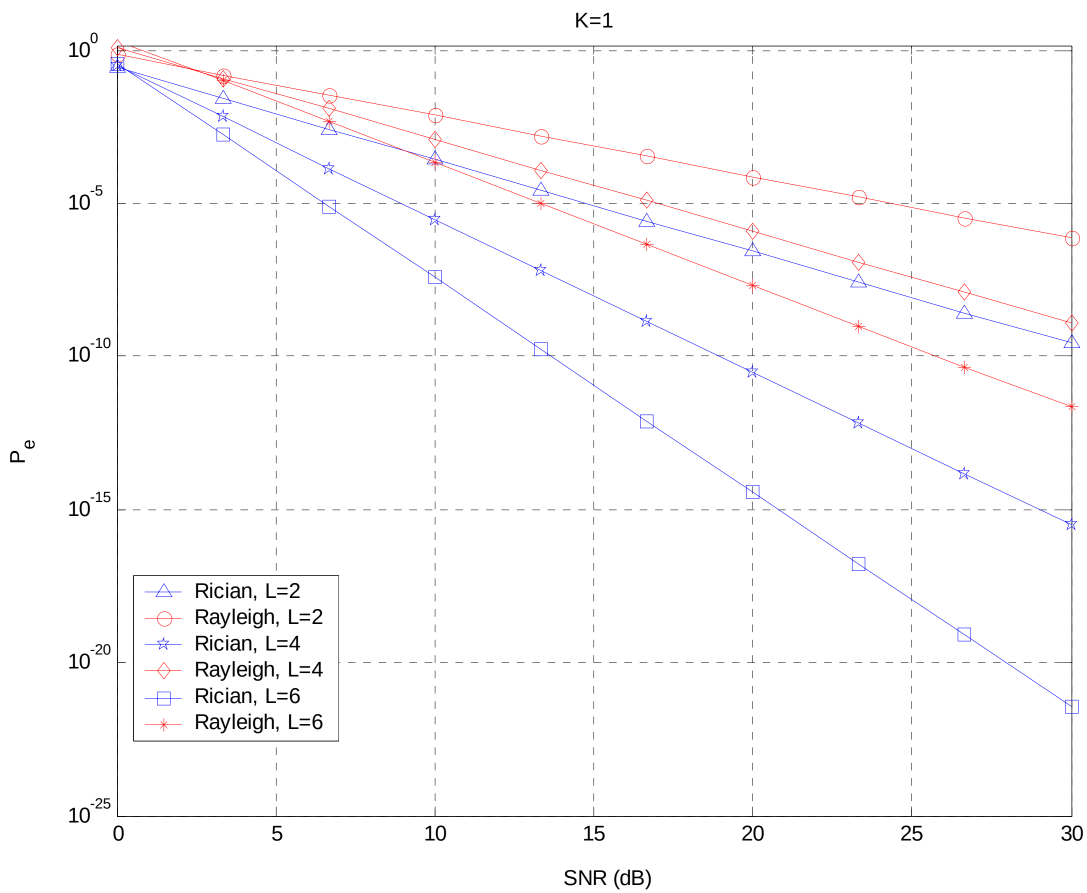
- We see that the diversity order is higher than the Rayleigh case.
- We will also see some gain because $\overline{\beta}^* = c\overline{\beta} > \overline{\beta}$

Results- Rayleigh versus Rician



- **We see that the performance in the Rician fading case is better than the Rayleigh fading case for all values of *K*.**
- **As *K* increases, the performance improves since the power of the line of sight components is larger than the power of the diffuse components.**

Results- Diversity in the Rician Case



- As L increases, the improvement in the performance for the Rician case is larger than for the Rayleigh case.
- This is the result of presence of the LOS component in the Rician case.