EE 5654 - Digital Communications Spring 2005

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Lecture #3 –

Introduction to Signal-Space Approach to Modulation

Announcements

- HW #1 due today
- HW#2 assigned next week

Modulation Principles

- Almost all communication systems transmit digital data using a sinusoidal carrier waveform.
  - Electromagnetic signals propagate well
  - Choice of carrier frequency allows placement of signal in arbitrary part of spectrum
- Physical system implements modulation by:
  - Processing digital information at baseband
  - Pulse shaping and filtering a digital waveform
  - Baseband signal is mixed with carrier signal from oscillator
  - RF signal is filtered, amplified and coupled with antenna

Representation of Modulation Signals

- We can modify amplitude, phase or frequency.
- Amplitude Shift Keying (ASK) or On/Off Keying (OOK): $1 \Rightarrow A\cos(2\pi f_c t), 0 \Rightarrow 0$
- Frequency Shift Keying (FSK):

  $$1 \Rightarrow A\cos(2\pi f_1 t), 0 \Rightarrow A\cos(2\pi f_0 t)$$

- Phase Shift Keying (PSK):

  $$1 \Rightarrow A\cos(2\pi f_c t)$$

  $$0 \Rightarrow A\cos(2\pi f_c t + \pi) = -A\cos(2\pi f_c t)$$

Representation of Bandpass Signals

Bandpass signals (signals with small bandwidth compared to carrier frequency) can be represented in any of three standard formats:

1. **Quadrature Notation**

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

where *x(t)* and *y(t)* are real-valued baseband signals called the in-phase and quadrature components of *s(t)*

Representation of Bandpass Signals (continued)

2. **Complex Envelope Notation**

$$s(t) = \mathrm{Re}\left[\left(x(t) + jy(t)\right)e^{-j2\pi f_c t}\right] = \mathrm{Re}\left[s_l\left(t\right)e^{-j2\pi f_c t}\right]$$

where $s_l(t)$ is the <u>complex envelope</u> of *s(t)*.

3. **Magnitude and Phase**

$$s(t) = a(t)\cos\left(2\pi f_c t + \theta(t)\right)$$

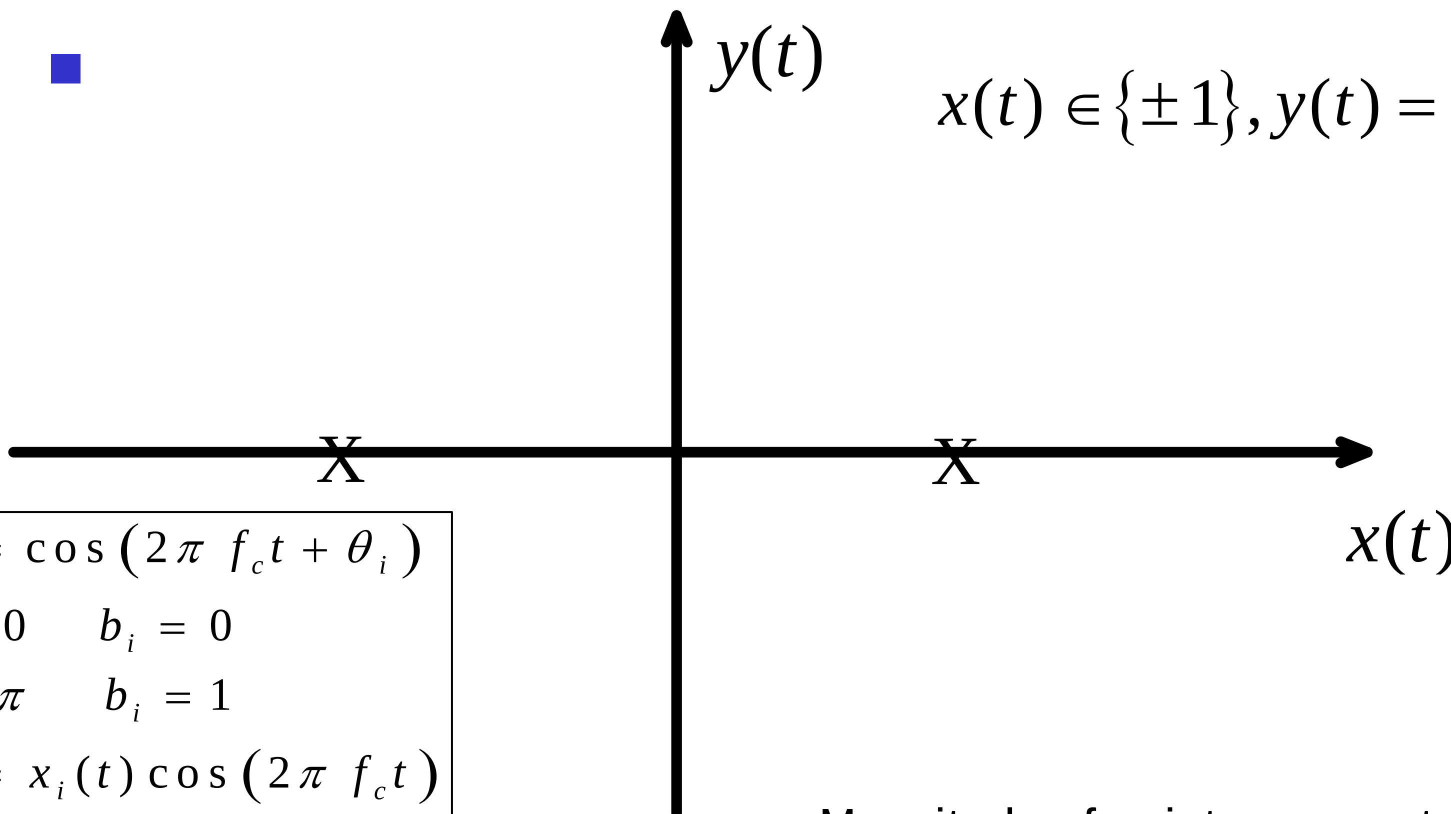
where $a(t) = \sqrt{x^2(t) + y^2(t)}$ is the <u>magnitude</u> of *s(t)*,

and $\theta(t) = \tan^{-1}\left[\frac{y(t)}{x(t)}\right]$ is the <u>phase</u> of *s(t)*.

Key Ideas from I/Q Representation of Signals

- We can represent bandpass signals independent of carrier frequency.
- The idea of quadrature sets up a coordinate system for looking at common modulation types.
- The coordinate system is sometimes called a signal constellation diagram.
- Real part of complex baseband maps to *x*-axis and imaginary part of complex baseband maps to the *y*-axis

Example of Signal Constellation Diagram: BPSK



$$x(t) \in \{\pm 1\}, y(t) = 0$$

$$s_i(t) = \cos\left(2\pi f_c t + \theta_i\right)$$

$$\theta_i = \begin{cases} 0 & b_i = 0 \\ \pi & b_i = 1 \end{cases}$$

$$s_i(t) = x_i(t)\cos\left(2\pi f_c t\right)$$

$$x_i(t) = \begin{cases} 1 & b_i = 0 \\ -1 & b_i = 1 \end{cases}$$

Magnitude of point represents amplitude
Angle of point represents phase

Example of Signal Constellation Diagram: QPSK

$$s_i(t) = \cos\left(2\pi f_c t + \theta_i\right)$$

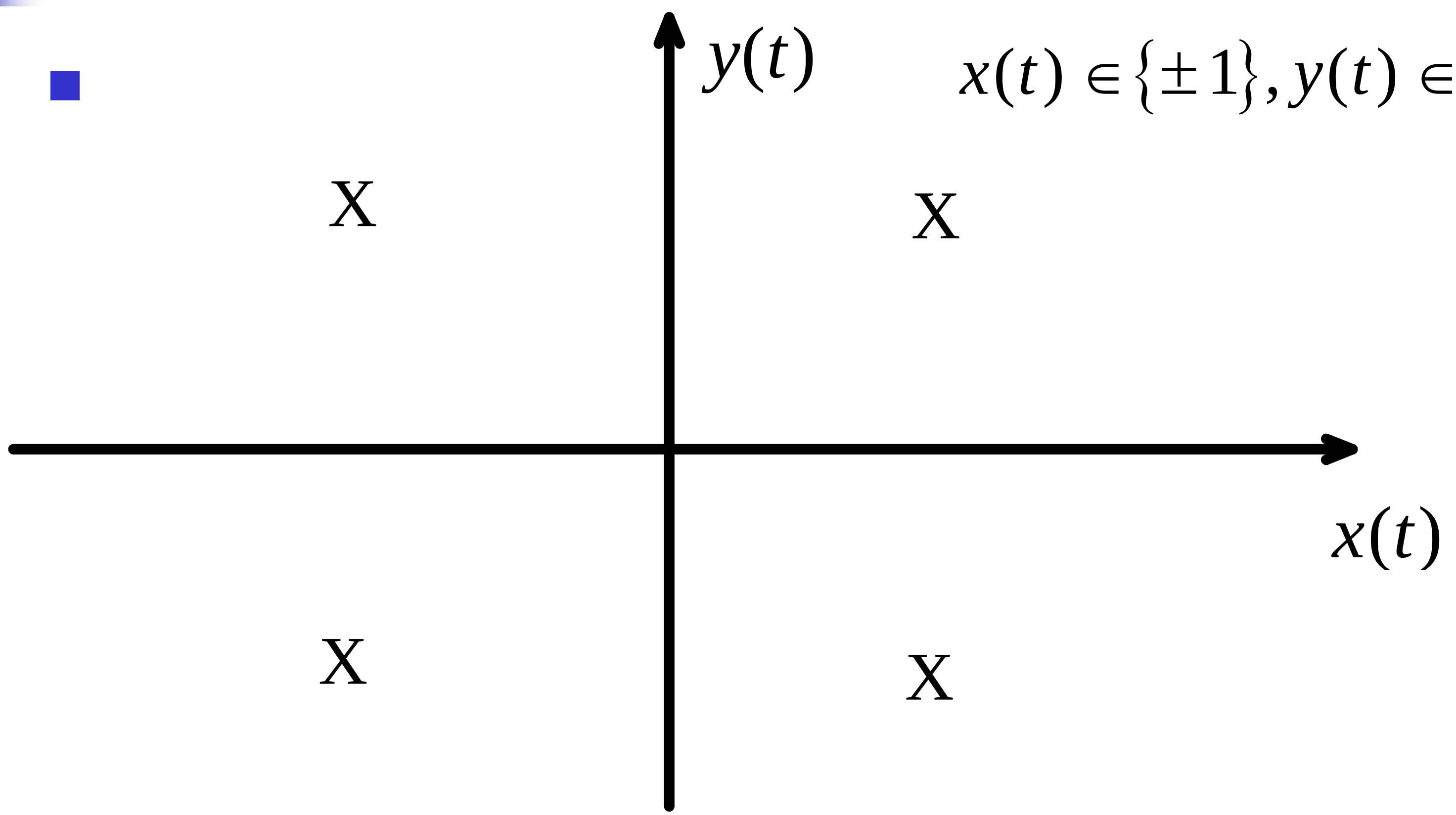
$$\theta_i = \begin{cases} \frac{\pi}{4} & b_i b_{i+1} = 00 \\ \frac{3\pi}{4} & b_i b_{i+1} = 10 \\ -\frac{3\pi}{4} & b_i b_{i+1} = 11 \\ -\frac{\pi}{4} & b_i b_{i+1} = 01 \end{cases}$$

$$s_i(t) = x_i(t)\cos\left(2\pi f_c t\right) - y_i(t)\sin\left(2\pi f_c t\right)$$

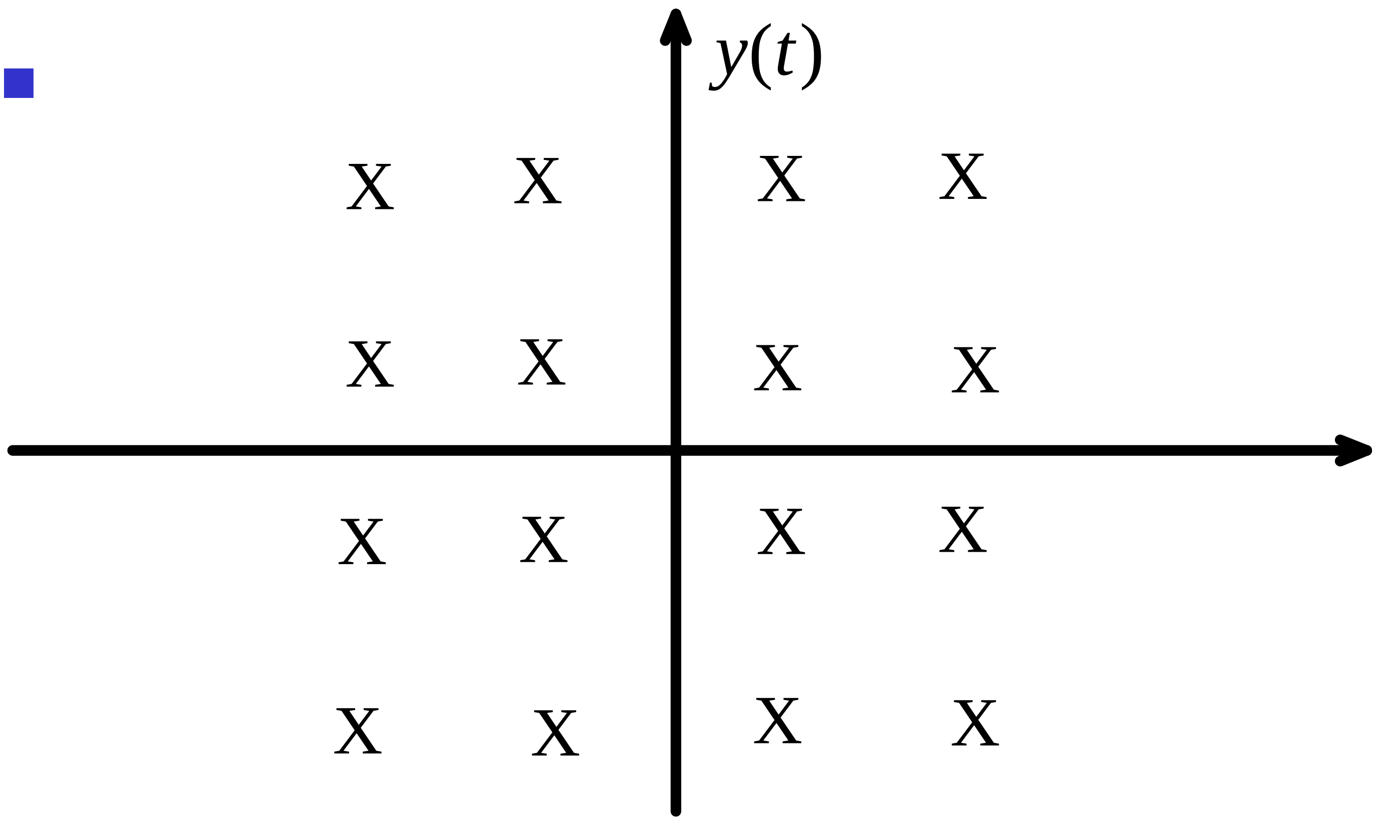
$$x_i(t) = \begin{cases} \frac{\sqrt{2}}{2} & b_i = 0 \\ -\frac{\sqrt{2}}{2} & b_i = 1 \end{cases} \quad y_i(t) = \begin{cases} \frac{\sqrt{2}}{2} & b_{i+1} = 0 \\ -\frac{\sqrt{2}}{2} & b_{i+1} = 1 \end{cases}$$

Example of Signal Constellation Diagram: QPSK

$$x(t) \in \{\pm 1\}, y(t) \in \{\pm 1\}$$



Example of Signal Constellation Diagram: QAM



$$x(t) \in \{-3,-1,+1,+3\},\ y(t) \in \{-3,-1,+1,+3\}$$

Interpretation of Signal Constellation Diagram

- Axis are labeled with *x(t)* and *y(t)*
  - In-phase/quadrature or real/imaginary
- Possible signals are plotted as points
- Symbol amplitude is proportional to distance from origin
- Probability of mistaking one signal for another is related to the distance between signal points
- Decisions are made on the received signal based on the distance of the received signal (in the I/Q plane) to the signal points in the constellation

A New Way of Viewing Modulation

- The I/Q representation of modulation is very convenient for some modulation types.
- We will examine an even more general way of looking at modulation using signal spaces.
- By choosing an appropriate set of axes for our signal constellation, we will be able to:
  - Design modulation types which have desirable properties
  - Construct optimal receivers for a given type of modulation
  - Analyze the performance of modulation types using very general techniques.

Vector Spaces

- An *n*-dimensional vector $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ consists of *n* scalar components $\{v_1, v_2, \ldots, v_n\}$
- The norm (length) of a vector **v** is given by:

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} {v_i}^2}$$

- The inner product of two vectors $\mathbf{v}_1 = [v_{11}, v_{12}, \ldots, v_{1n}]$ and $\mathbf{v}_2 = [v_{21}, v_{22}, \ldots, v_{2n}]$ is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}$$

Basis Vectors

- A vector $\mathbf{v}$ may be expressed as a linear combination of it's <u>basis vectors</u> $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$:

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$$

  where $v_i = \mathbf{e}_i \cdot \mathbf{v}$

- Think of the basis vectors as a coordinate system (x-y-z... axis) for describing the vector $\mathbf{v}$
- What makes a good choice of coordinate system?

A Complete Orthonormal Basis

- The set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ should be complete or span the vector space $\Re^n$. Any vector can be expressed as $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$ for some $\{v_i\}$
- Each basis vector should be orthogonal to all others:
  $\mathbf{e}_i \cdot \mathbf{e}_j = 0, \forall i \neq j$
- Each basis vector should be normalized: $\|\mathbf{e}_i\| = 1, \forall i$
- A set of basis vectors which satisfies these three properties is said to be a complete orthonormal basis.

Signal Spaces

Signals can be treated in much the same way as vectors.

- The norm of a signal $x(t), t \in [a,b]$ is given by:

$$\|x(t)\| = \left(\int_a^b |x(t)|^2 dt \right)^{1/2} = \sqrt{E_x}$$

- The inner product of signals $x_1(t)$ and $x_2(t)$ is:

$$\langle x_1(t), x_2(t) \rangle = \int_a^b x_1(t) x_2^*(t) dt$$

- Signals can be represented as the sum of basis functions:

$$x(t) = \sum_{i=1}^{n} x_k f_k(t), \qquad x_k = \langle x(t), f_k(t) \rangle$$

Basis Functions for a Signal Set

- One of *M* signals is transmitted: $\{s_1(t), \ldots, s_M(t)\}$
- The functions $\{f_1(t), \ldots, f_K(t)\}$ ($K \le M$) form a complete orthonormal basis for the signal set if
  - Any signal can be described by a linear combination:

$$s_i(t) = \sum_{k=1}^{K} s_{i,k} f_k(t), i = 1, \ldots, M$$

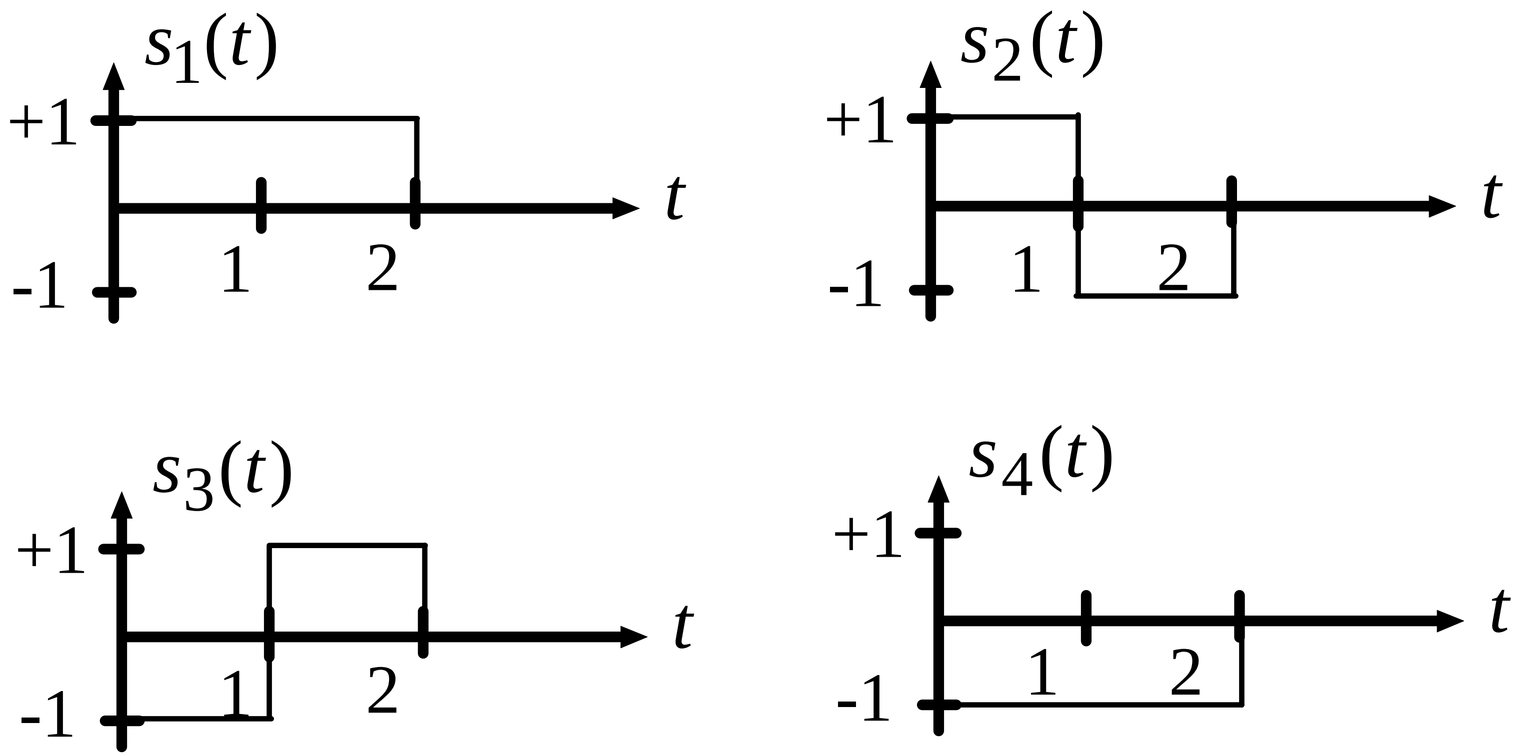
  - The basis functions are orthogonal to each other:

$$\int_a^b f_i(t) f_j^*(t) dt = 0, \forall i \ne j$$

  - The basis functions are normalized: $\int_a^b |f_k(t)|^2 dt = 1, \forall k$

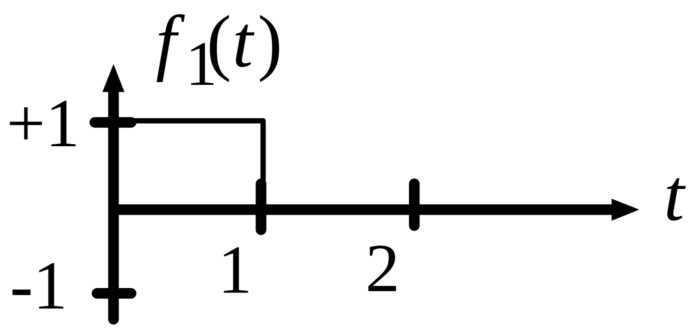
Example of Signal Space

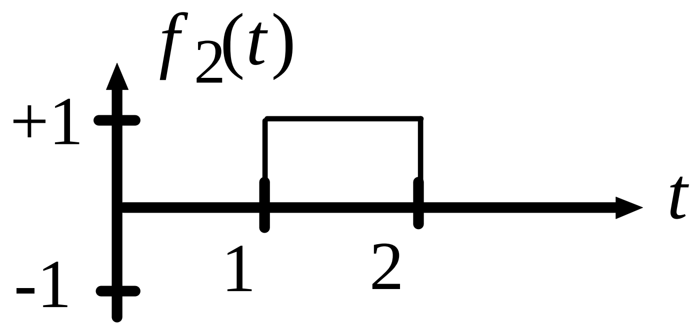
Consider the following signal signal set:



Example of Signal Space (continued)

- We can express each of the signals in terms of the following basis functions:





$$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$$

$$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$$

- Therefore the basis is complete

Example of Signal Space (continued)

- The basis is orthogonal:

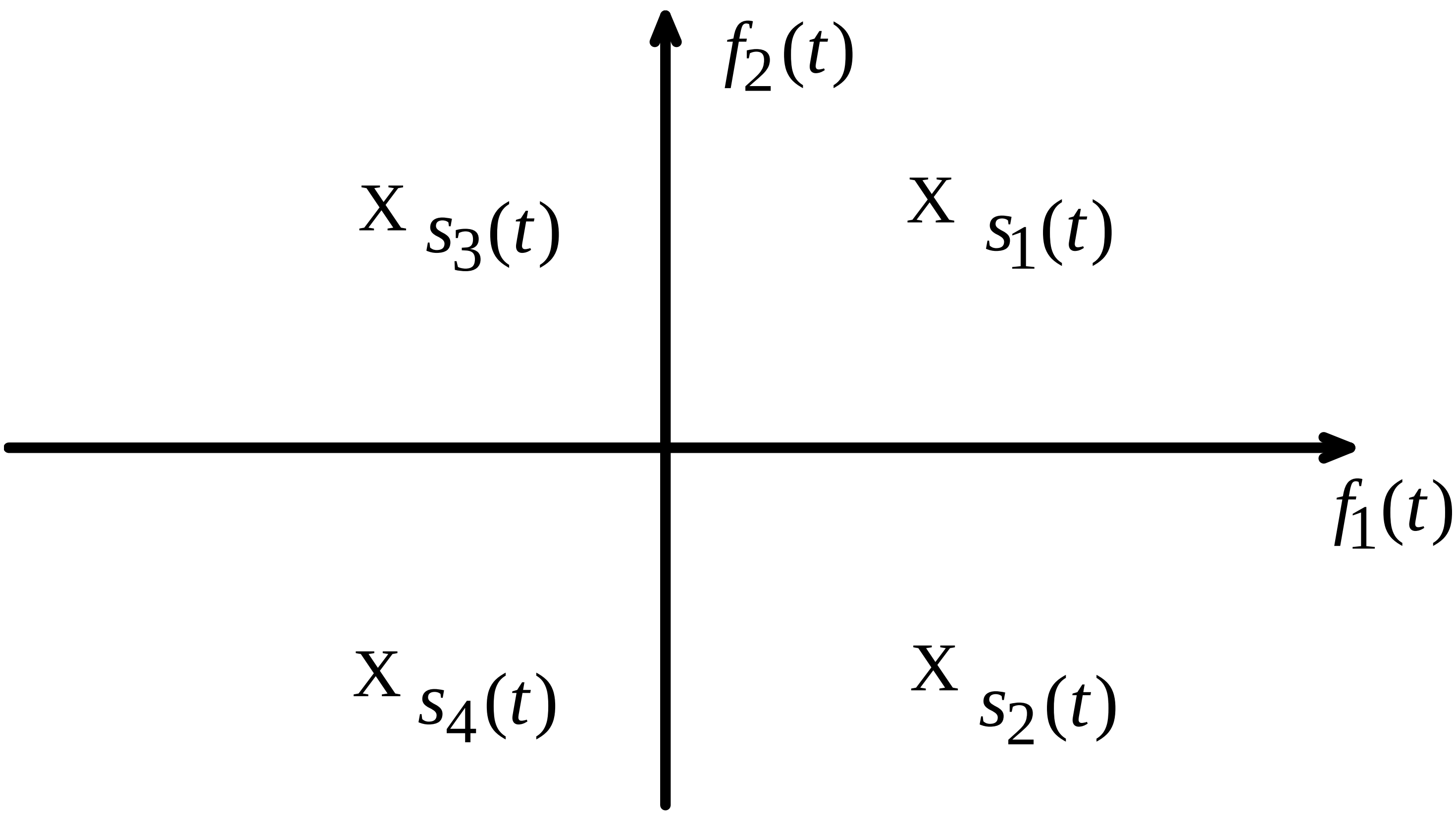
$$\int_{-\infty}^{\infty} f_1(t)\, f_2^{\,*}(t)\,dt = 0$$

- The basis is normalized:

$$\int_{-\infty}^{\infty} \left| f_1(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| f_2(t) \right|^2 dt = 1$$

Signal Constellation for Example

- We've seen this signal constellation before



Another Example

- Suppose our signal set can be represented in I/Q form: $s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)\Big|_0^T$

  where $x(t)$ and $y(t)$ are constants for $t \in [0, T]$

- Then the functions:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Bigg|_0^T, \; f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Bigg|_0^T$$

form a complete orthonormal basis

Proof

- All I/Q signals can be represented by the linear combination of these basis functions.
- These basis functions are orthogonal:

$$\int_0^T f_1(t) f_2^*(t)dt = \int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)dt$$

$$= \frac{2}{T}\int_0^T \frac{1}{2}[\sin(0)+\sin(4\pi f_c t)]dt$$

$$= \frac{-1}{4\pi f_c T}[\cos(4\pi f_c t)]\Big|_0^T \approx 0, \text{ for } f_c T >> 1$$

Proof (continued)

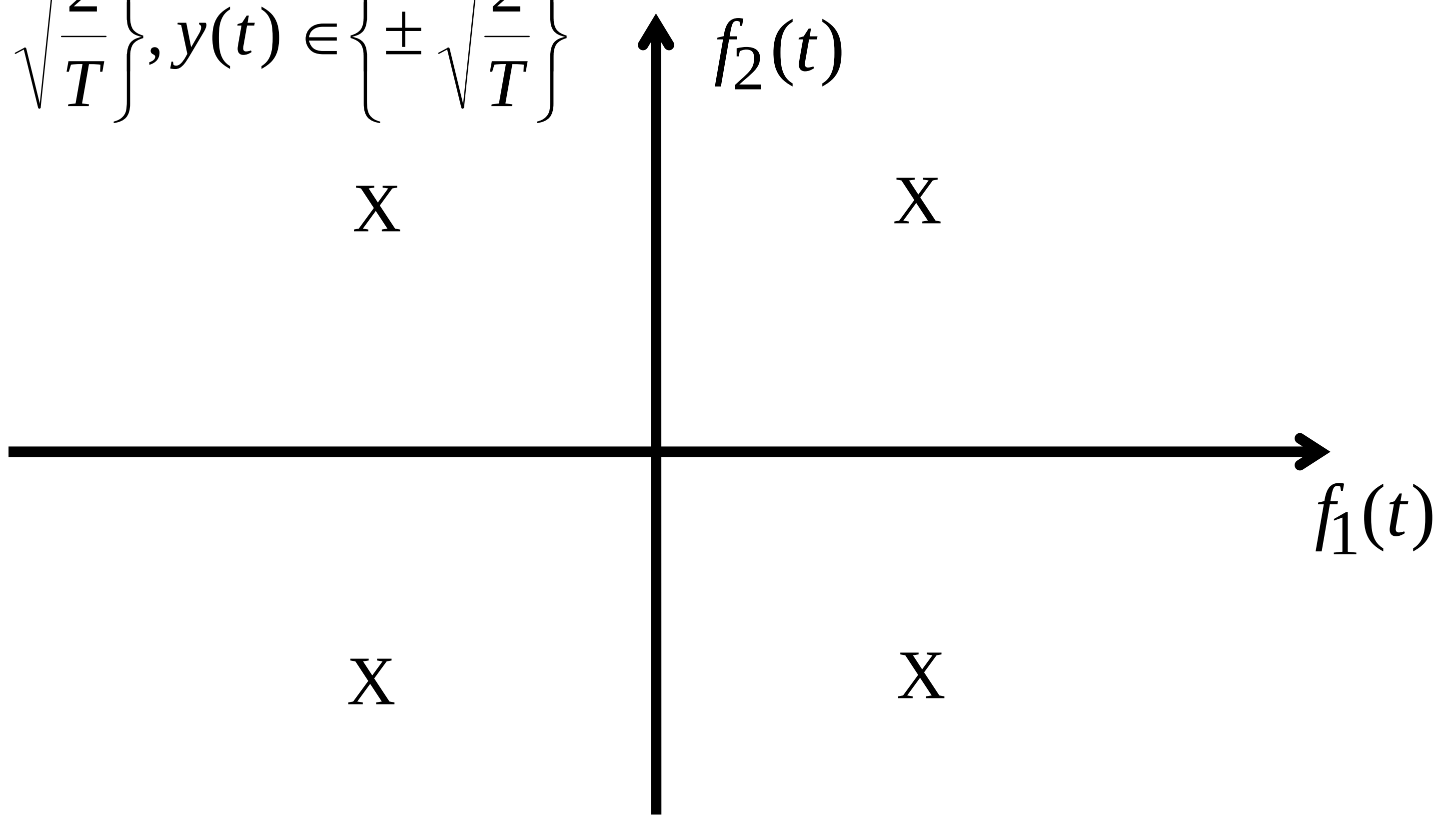
- These basis functions are normalized:

$$\int_0^T |f_1(t)|^2 dt = \int_0^T |f_2(t)|^2 dt = \int_0^T \left(\sqrt{\frac{2}{T}} \cos(2\pi f_c t) \right)^2 dt$$

$$= \frac{2}{T} \int_0^T \frac{1}{2} [\cos(0) + \cos(4\pi f_c t)] dt \approx \frac{1}{T} [1]_0^T = 1$$

Signal Constellation for QPSK

$$x(t) \in \left\{ \pm \sqrt{\frac{2}{T}} \right\}, y(t) \in \left\{ \pm \sqrt{\frac{2}{T}} \right\}$$



Constellation Diagrams

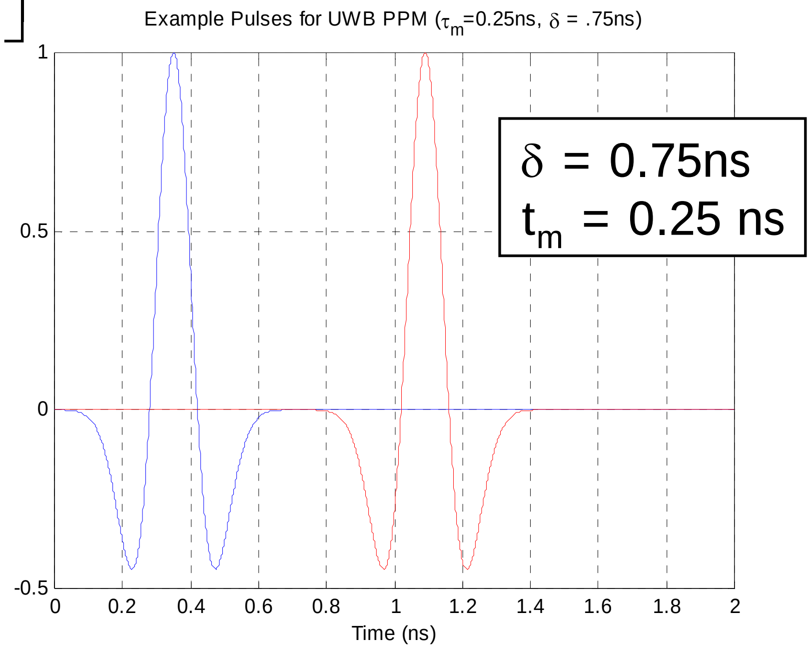
- Thus, constellation diagrams are simply signal space plots for modulation schemes that have only two basis functions.
- Specifically, basis functions are $\cos(2\pi ft)$ and $\sin(2\pi ft)$
- Only good for phase modulation or amplitude modulation
- Other modulation formats require larger number of basis functions.
- Any modulation scheme can be represented using signal space approach – even those that don’t use sinusoidal carrier

Example

- Assume that Pulse Position Modulation is used with a pulse defined as

$$w_r(t) = \left[1 - 4\pi\left(\frac{t - t_d}{t_m}\right)^2\right]\exp\left[-2\pi\left(\frac{t - t_d}{t_m}\right)^2\right]$$

  - $w_r(t)$ → binary 0
  - $w_r(t-\delta)$ → binary 1



- What is the optimal choice for δ ?

Example (cont.)

- The ML detector is simply one which correlates the received signal *r(t)* with each of the two received signals and chooses the largest:

$$\int_0^T r(t)w_r(t)dt \underset{<_1}{\overset{>_0}{}} \int_0^T r(t)w_r(t-\delta)dt$$

$$\int_0^T r(t)w_r(t)dt - \int_0^T r(t)w_r(t-\delta)dt \underset{<_1}{\overset{>_0}{}} 0$$

$$\int_0^T r(t)\left(w_r(t) - w_r(t-\delta)dt\right) \underset{<_1}{\overset{>_0}{}} 0$$

Q: What choice of δ gives us the best performance using this metric? →

$$\int_0^T r(t)v(t) \underset{<_1}{\overset{>_0}{}} 0$$

→ A: The value of δ which maximizes the distance between symbols.

Example (cont.)

- We need to first determine a set of basis functions. Since there are two functions we need at most two basis functions.
- We can choose the first as $f_1(t) = \frac{1}{\sqrt{E_s}} w_r(t)$
- And the second as $f_2(t) = \frac{1}{\sqrt{E_2}} \left(w_r(t-\delta) - c_{12} f_1(t) \right)$

where $E_2 = \int_0^T \left(w_r(t-\delta) - c_{12} f_1(t) \right)^2 dt$

$$c_{12} = \int_0^T w_r(t-\delta) f_1(t) dt$$

Example (cont.)

- We can then plot the two symbol vectors as

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix} \qquad c_{12} = \int_0^T w_r(t-\delta)\, f_1(t)\,dt$$

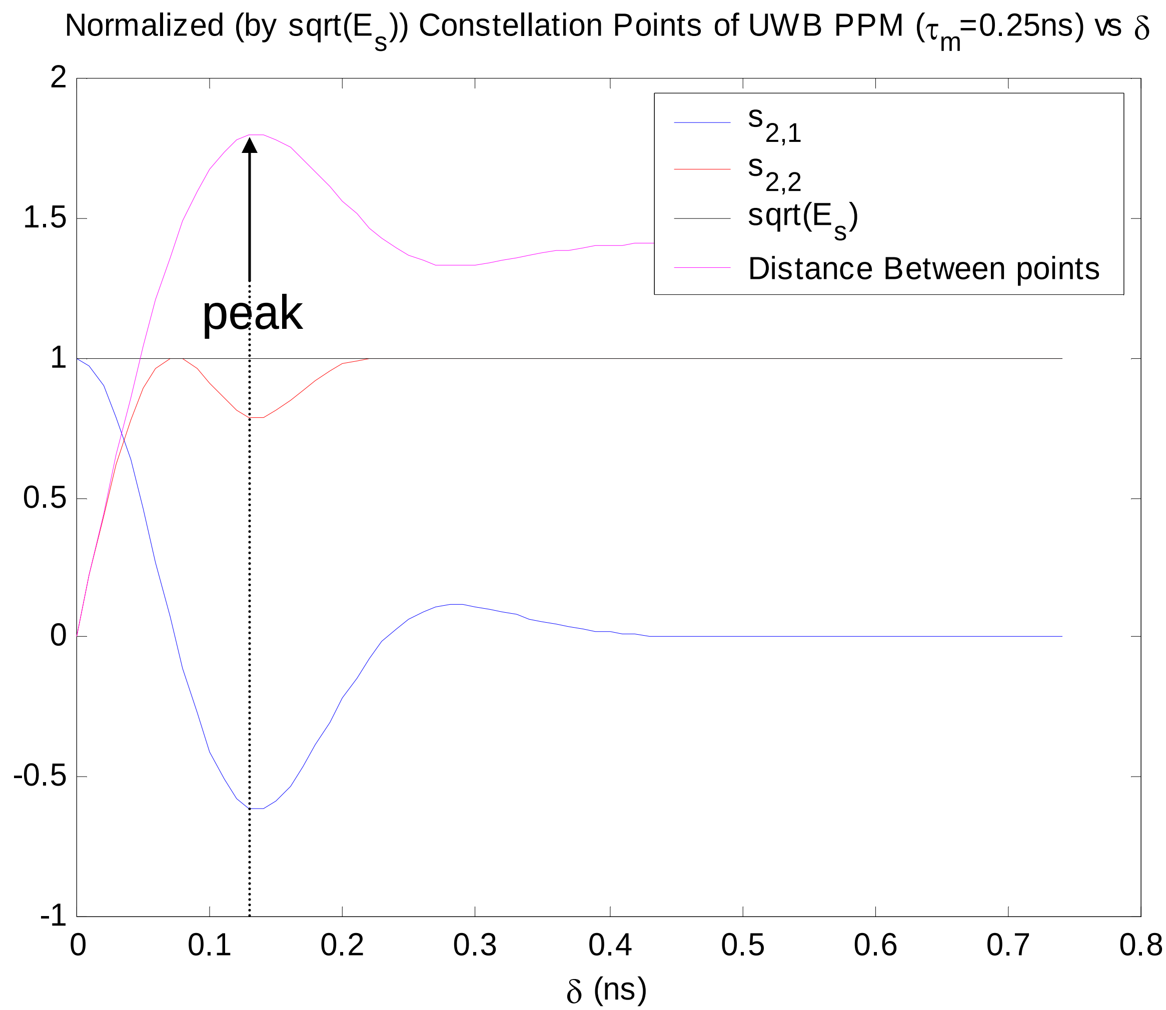
$$\mathbf{s}_2 = \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} \qquad c_{22} = \int_0^T w_r(t-\delta)\, f_2(t)\,dt$$

- The probability of error is determined by the distance between symbol points

$$d = \sqrt{\left(\sqrt{E_s} - c_{12}\right)^2 + \left(c_{22}\right)^2}$$

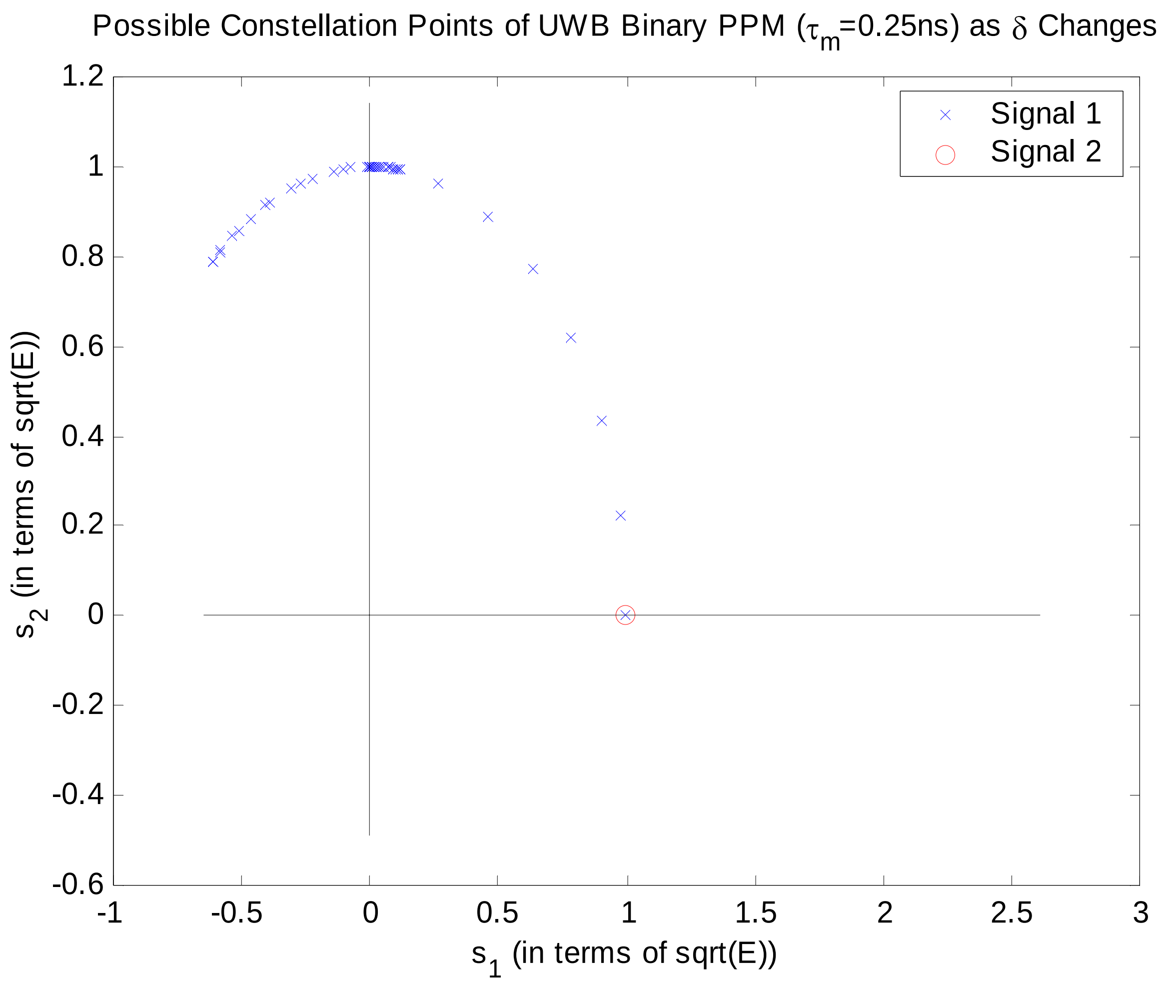
Example (cont.)

- As δ is varied, the point $\mathbf{s}_2$ moves changing the distance between points
- Peak in distance occurs at 1<δ <2 (note that s_{21} is negative).



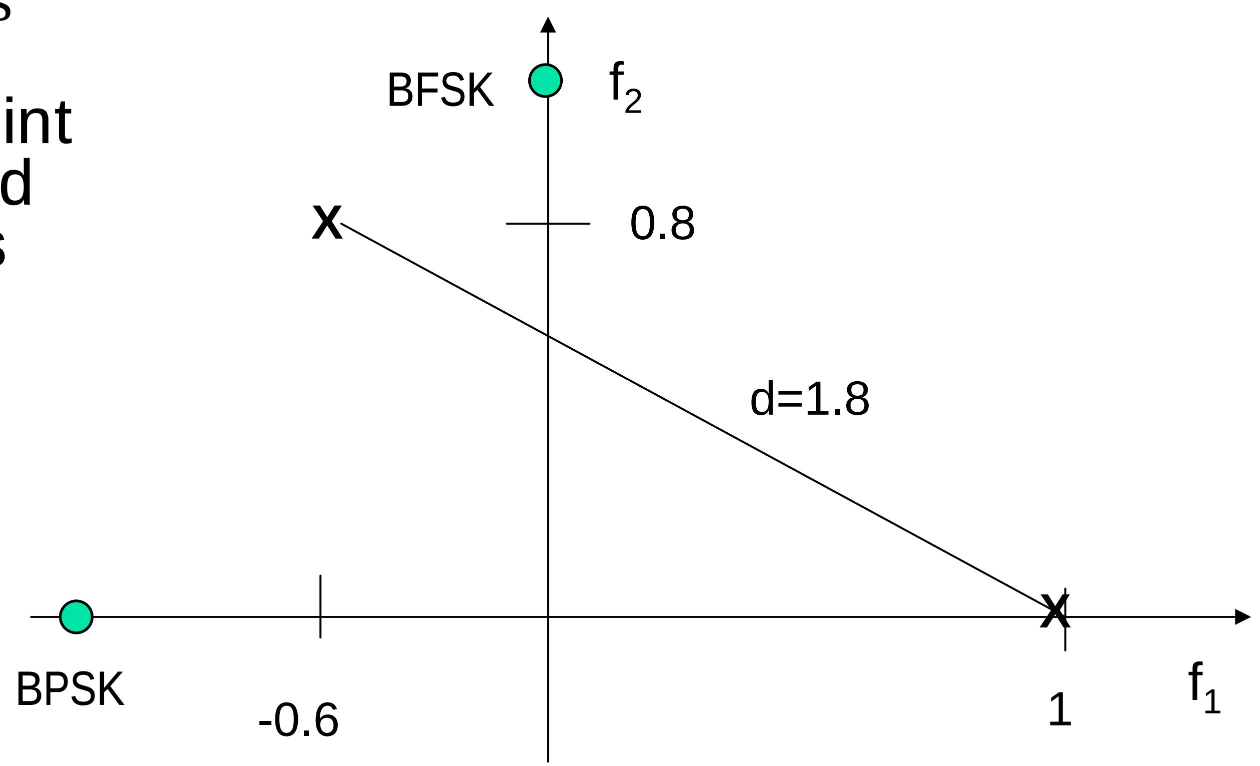
Example (cont.)

- As δ is varied, the point $\mathbf{s}_2$ moves changing the distance between points
- For δ=0, the two points are the same.
- When δ > 0.4ns, the two symbols are orthogonal

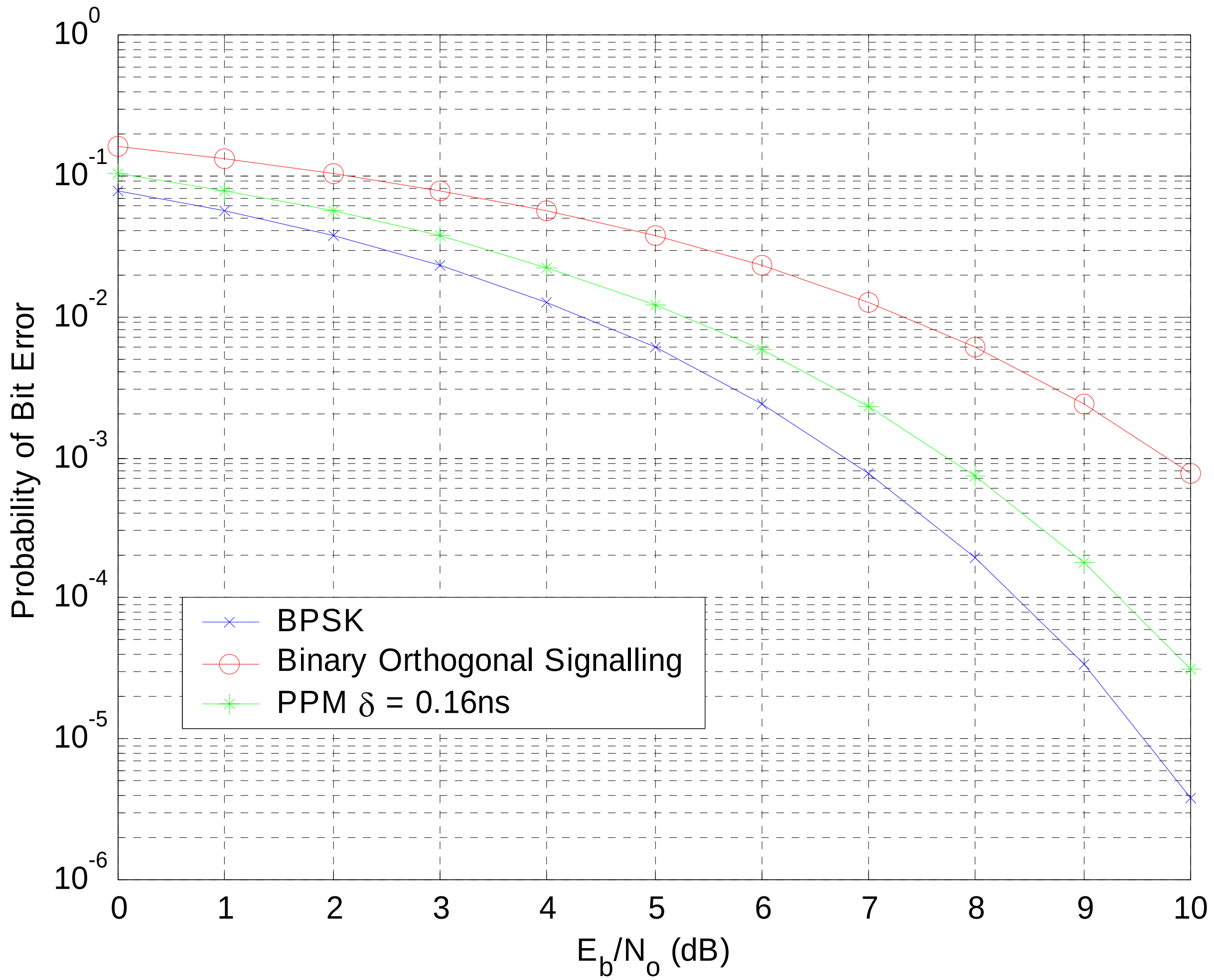


Example (cont.)

- When δ=0.16ns the distance between $\mathbf{s}_1$ point $\mathbf{s}_2$ is maximized and the BER is minimized



Example (cont.)



Notes on Signal Spaces

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In both of these cases we were fortunate enough to guess the correct basis functions.
- Is there a general method to find a complete orthonormal basis for an arbitrary signal set?
  - ***Gram-Schmidt Procedure***