EE 5654 - Digital Communications Spring 2005

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Lecture #4 –

Signal Spaces/Gram-Schmidt Procedure for Obtaining Basis Functions

A New Way of Viewing Modulation

- In previous courses we have examined modulation by looking at the constellation diagram (I/Q representation)
- The I/Q representation of modulation is very convenient for some modulation types.
- We will examine an even more general way of looking at modulation using signal spaces.
- By choosing an appropriate set of axes for our signal constellation, we will be able to:
  - Design modulation types which have desirable properties
  - Construct optimal receivers for a given type of modulation
  - Analyze the performance of modulation types using very general techniques.

Basis Functions for a Signal Set

- One of M signals is transmitted: $\{s_1(t),\ldots,s_M(t)\}$
- The functions $\{f_1(t),\ldots,f_K(t)\}$ ($K \le M$) form a complete orthonormal basis for the signal set if
  - Any signal can be described by a linear combination:

$$s_i(t) = \sum_{k=1}^{K} s_{i,k} f_k(t), i = 1,\ldots,M$$

  - The basis functions are orthogonal to each other:

$$\int_a^b f_i(t) f_j^*(t)dt = 0, \forall i \ne j$$

  - The basis functions are normalized: $\int_a^b \left| f_k(t) \right|^2 dt = 1, \forall k$

Notes on Signal Spaces

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In both of the cases evaluated last time we were fortunate enough to guess the correct basis functions.
- Is there a general method to find a complete orthonormal basis for an arbitrary signal set?
  - ***Yes: The Gram-Schmidt Procedure***

Gram-Schmidt Procedure

- Suppose we are given a signal set: $\{s_1(t), \ldots, s_M(t)\}$
- We would like to find a complete orthonormal basis $\{f_1(t), \ldots, f_K(t)\}$ $K \leq M$ for this signal set.

The Gram-Schmidt procedure is an iterative procedure for creating an orthonormal basis.

Step 1: Construct the First Basis Function

- Compute the energy in signal 1:

$$E_1 = \int_{-\infty}^{\infty} |s_1(t)|^2 \, dt$$

- The first basis function is just a normalized version of $s_1(t)$:

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

Note: We can enumerate the signals arbitrarily

Step 1: Construct the Second Basis Function

- Compute correlation with 1st basis:

$$c_{12} = \int_{-\infty}^{\infty} s_2(t)\, f_1(t)\, dt$$

- Subtract off correlated portion:

$$f_2'(t) = s_2(t) - c_{12} f_1(t)$$

- Compute energy in remaining portion:

$$\mathrm{E}_2 = \int_{-\infty}^{\infty} \left| f_2'(t) \right|^2 dt$$

- Normalize basis function: $f_2(t) = \dfrac{f_2'(t)}{\sqrt{\mathrm{E}_2}}$

Procedure for Successive Signals

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i(t) dt, \quad i = 1, \ldots, k-1$$

$$f_k{}'(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ik} f_i(t)$$

$$\mathrm{E}_k = \int_{-\infty}^{\infty} \left| f_k{}'(t) \right|^2 dt$$

$$f_k(t) = \frac{f_k{}'(t)}{\sqrt{\mathrm{E}_k}}$$

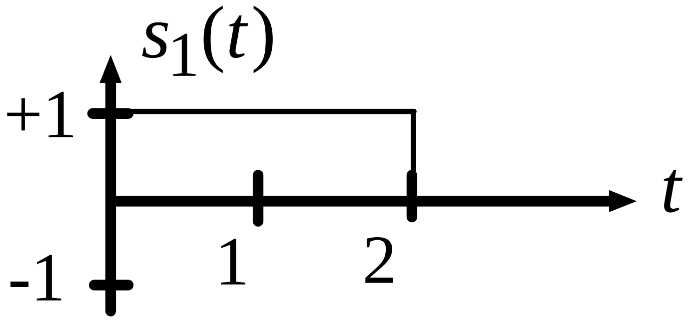
-

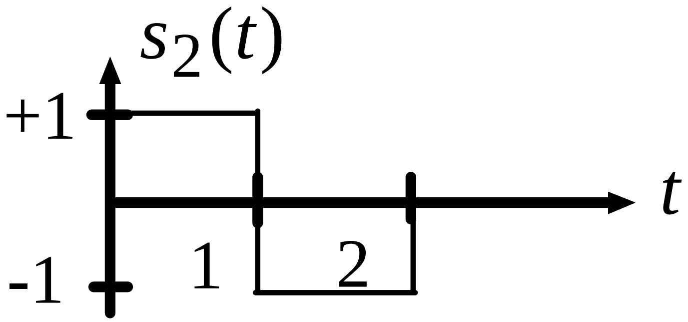
Summary of Gram-Schmidt Procedure

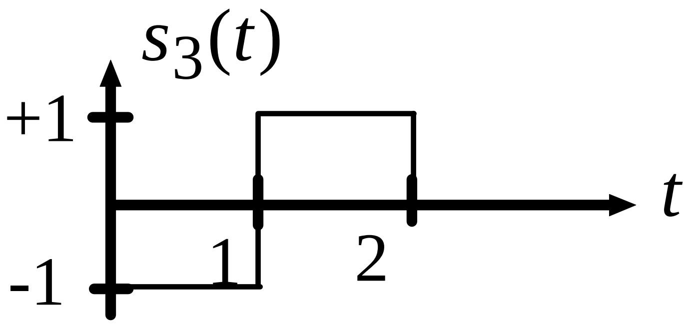
- 1st basis function is normalized version of 1st signal.
- Successive basis functions are found by removing portions of signals which are correlated to previous basis functions, and normalizing the result.
- This procedure is repeated until all symbols are exhausted.
- If $f_k'(t) = 0$, then no new basis function is added.
- The order in which signals are considered is arbitrary.
- A final useful step: *Redefine the points in terms of the average symbol or bit energy*
- What is the maximum number of basis functions?
  - When does the maximum occur?

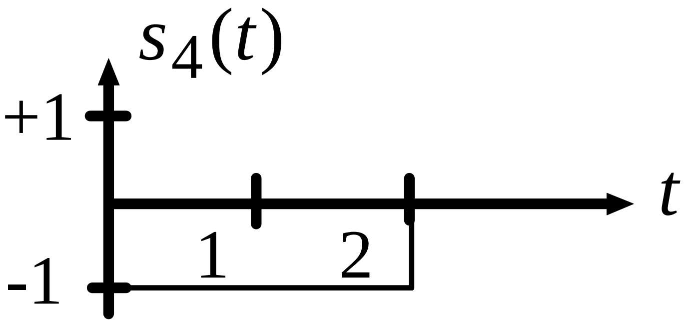
Example of Gram-Schmidt Procedure

- Use the signal set from last time:





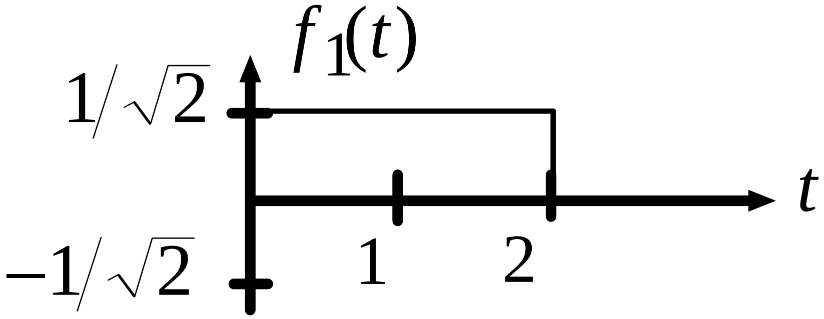




Step 1

$$E_1 = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = 2$$

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}}$$

- 

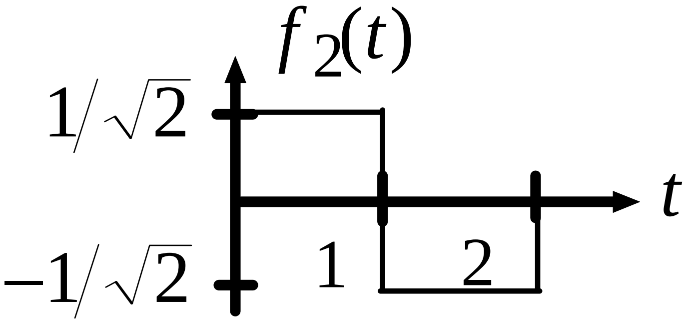
Step 2

$$c_{12} = \int_{-\infty}^{\infty} f_1(t) s_2(t) dt = 0$$

$$f_2{}'(t) = s_2(t) - c_{12} f_1(t) = s_2(t)$$

$$E_2 = \int_{-\infty}^{\infty} \left| s_2(t) \right|^2 dt = 2$$

$$f_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{2}}$$



Step 3

$$c_{13} = \int_{-\infty}^{\infty} f_1(t) s_3(t) dt = 0$$

$$c_{23} = \int_{-\infty}^{\infty} f_2(t) s_3(t) dt = -\sqrt{2}$$

$$\begin{aligned} f_3'(t) &= s_3(t) - c_{13} f_1(t) - c_{23} f_2(t) \\ &= s_3(t) + \sqrt{2} f_2(t) = 0 \end{aligned}$$

- No new basis function

Step 4

$$c_{14} = \int_{-\infty}^{\infty} f_1(t) s_4(t) dt = -\sqrt{2}$$

$$c_{24} = \int_{-\infty}^{\infty} f_2(t) s_4(t) dt = 0$$

$$\begin{aligned} f_4{}'(t) &= s_4(t) - c_{14} f_1(t) - c_{24} f_2(t) \\ &= s_4(t) + \sqrt{2} f_1(t) = 0 \end{aligned}$$

- No new basis function. Procedure Complete

Final Step

- Redefine steps in terms of average symbol (bit) energy

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix}$$

$$\mathbf{s}_3 = \begin{bmatrix} 0 & -\sqrt{2} \end{bmatrix} \quad \mathbf{s}_4 = \begin{bmatrix} -\sqrt{2} & 0 \end{bmatrix}$$

- Average energy

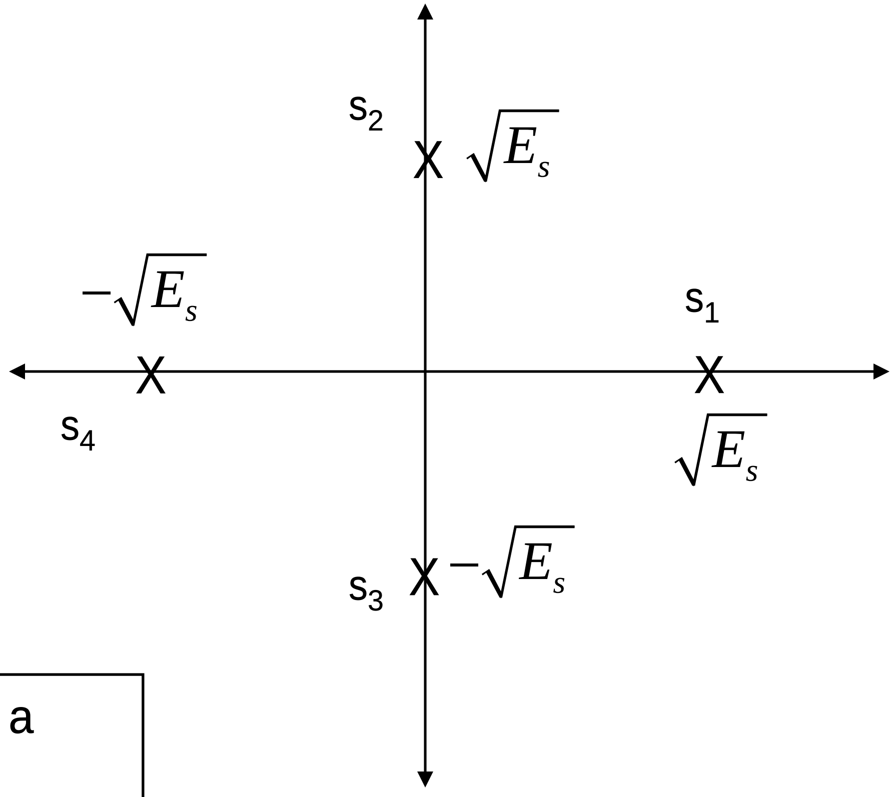
$$\begin{aligned} \overline{E}_s &= \frac{1}{4}\left\{E_{s1} + E_{s2} + E_{s3} + E_{s4}\right\} \\ &= 2 \end{aligned}$$

- Thus

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_s} & 0 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 0 & \sqrt{E_s} \end{bmatrix}$$

$$\mathbf{s}_3 = \begin{bmatrix} 0 & -\sqrt{E_s} \end{bmatrix} \quad \mathbf{s}_4 = \begin{bmatrix} -\sqrt{E_s} & 0 \end{bmatrix}$$

Signal Constellation Diagram



Note that we obtained a different constellation diagram than before. Does this change the performance?

Example: *M* -PSK

Symbols are defined as:

$$s_i(t) = \cos\left(2\pi f_c t + \frac{2\pi}{M} i\right)\Bigg|_0^T, \; i = 0, \ldots, M-1$$

First basis function:

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}\Bigg|_0^T$$

$$= \frac{\cos(2\pi f_c t)}{\sqrt{E_1}}\Bigg|_0^T$$

$$= \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\Bigg|_0^T$$

$$\boxed{\begin{aligned} E_1 &= \int_0^T \cos^2(2\pi f_c t)\, dt \\ &= \frac{T}{2} \end{aligned}}$$

Example: *M* –PSK (cont.)

Now, find the correlation coefficient of the second symbol with the first basis function:

$$
\begin{aligned}
c_{12} &= \int_0^T \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right) \cos\left(2\pi f_c t + \frac{2\pi}{M}\right) dt \\
&= \sqrt{\frac{2}{T}} \int_0^T \frac{1}{2} \cos\left(\frac{2\pi}{M}\right) dt \\
&= \sqrt{\frac{2}{T}} \frac{T}{2} \cos\left(\frac{2\pi}{M}\right) \\
&= \sqrt{\frac{T}{2}} \cos\left(\frac{2\pi}{M}\right)
\end{aligned}
$$

Example: *M* –PSK (cont.)

- Now find the second basis function:

$$
\begin{aligned}
f_2(t) &= \frac{1}{\sqrt{E_2}}\left(s_2(t) - c_{12} f_1(t)\right) \\
&= \left(\cos\left(2\pi f_c t + \frac{2\pi}{M}\right) - \sqrt{\frac{T}{2}}\cos\left(\frac{2\pi}{M}\right)\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t\right)\right) \\
&= \frac{1}{\sqrt{E_2}}\left(\cos\left(2\pi f_c t + \frac{2\pi}{M}\right) - \cos\left(\frac{2\pi}{M}\right)\cos\left(2\pi f_c t\right)\right) \\
&= \frac{1}{\sqrt{E_2}}\left(\cos\left(\frac{2\pi}{M}\right)\cos\left(2\pi f_c t\right) - \sin\left(\frac{2\pi}{M}\right)\sin\left(2\pi f_c t\right) - \cos\left(\frac{2\pi}{M}\right)\cos\left(2\pi f_c t\right)\right) \\
&= -\frac{1}{\sqrt{E_2}}\sin\left(\frac{2\pi}{M}\right)\sin\left(2\pi f_c t\right)
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E_2 &= \int_0^T \left(-\sin\left(\frac{2\pi}{M}\right)\sin\left(2\pi f_c t\right)\right)^2 dt \\
&= \sin^2\left(\frac{2\pi}{M}\right)\int_0^T \sin^2\left(2\pi f_c t\right) dt \\
&= \sin^2\left(\frac{2\pi}{M}\right)\frac{T}{2}
\end{aligned}
}
$$

Example: *M*–PSK (cont.)

- The next basis function is then:

$$\boxed{\begin{aligned} f_2(t) &= -\frac{1}{\sqrt{\sin^2\left(\frac{2\pi}{M}\right)\frac{T}{2}}}\sin\left(\frac{2\pi}{M}\right)\sin\left(2\pi f_c t\right) \\ &= \sqrt{\frac{2}{T}}\sin\left(2\pi f_c t\right) \end{aligned}}$$

- Continuing we find that $s_i(t) - c_{i1} f_1(t) - c_{i2} f_2(t) = 0 \quad i > 2$
- Thus, there are two basis functions.

Example: *M* –PSK (cont.)

$$s_i(t) = \cos\left(2\pi f_c t + \frac{2\pi}{M} i\right)\Bigg|_0^T, \; i = 0, \ldots, M-1$$

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\Bigg|_0^T \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)\Bigg|_0^T$$

$$c_{1i} = \sqrt{\frac{T}{2}} \cos\left(\frac{2\pi}{M} i\right) \qquad c_{2i} = \sqrt{\frac{T}{2}} \sin\left(\frac{2\pi}{M} i\right)$$

$$s_i(t) = c_{1i} f_1(t) + c_{2i} f_2(t) = c_{1i} \cos(2\pi f_c t) - c_{2i} \sin(2\pi f_c t)$$

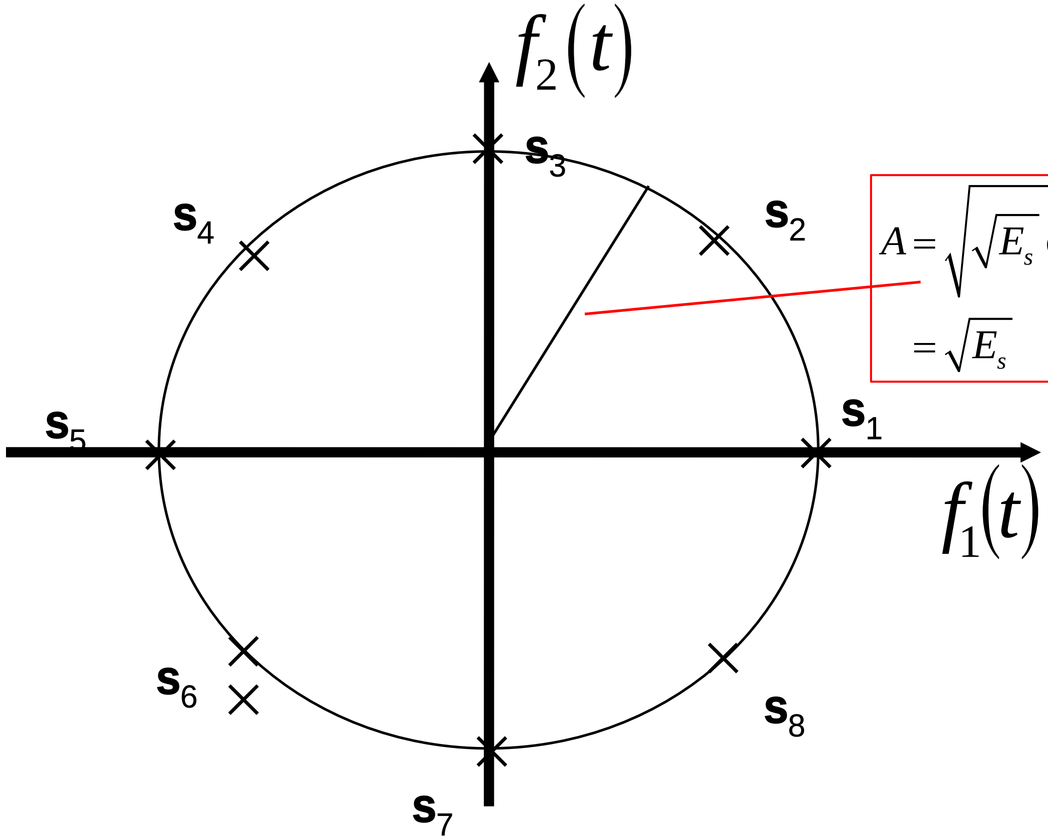
Average Symbol Energy

- All symbols have equal energy

$$
\begin{aligned}
E_s &= \frac{T}{2}\cos^2\left(\frac{2\pi}{M}i\right) + \frac{T}{2}\sin^2\left(\frac{2\pi}{M}i\right) \\
&= \frac{T}{2}
\end{aligned}
$$

$$\mathbf{s}_i = \left[\sqrt{E_s}\cos\left(\frac{2\pi}{M}i\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi}{M}i\right) \right]$$

Ex: 8-ary PSK



$$A=\sqrt{\sqrt{E_s}\cos^2\left(\frac{2\pi}{M}i\right)+\sqrt{E_s}\sin^2\left(\frac{2\pi}{M}i\right)}$$
$$=\sqrt{E_s}$$

$$\mathbf{s}_i=\left[\sqrt{E_s}\cos\left(\frac{2\pi}{M}i\right)\quad\sqrt{E_s}\sin\left(\frac{2\pi}{M}i\right)\right]$$

Example *M*-FSK

- Consider the signal set

$$s_i(t)=\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t+2\pi i\Delta f t\right)$$

- With baseband equivalent

$$s_i(t)=\sqrt{\frac{2}{T}}e^{j2\pi i\Delta f t}$$

- We can show that the correlation between symbols is

$$\rho_{mn}=\frac{1}{T}\int_0^T e^{j2\pi(m-n)\Delta f t}dt$$

$$=\frac{\sin\left[\pi T(m-n)\Delta f\right]}{\pi T(m-n)\Delta f}e^{j\pi T(m-n)\Delta f}$$

M- FSK (cont.)

- If we choose Δf = 1/T, the correlation between symbols is zero thus we will need *M* basis functions (i.e., one for each symbol).

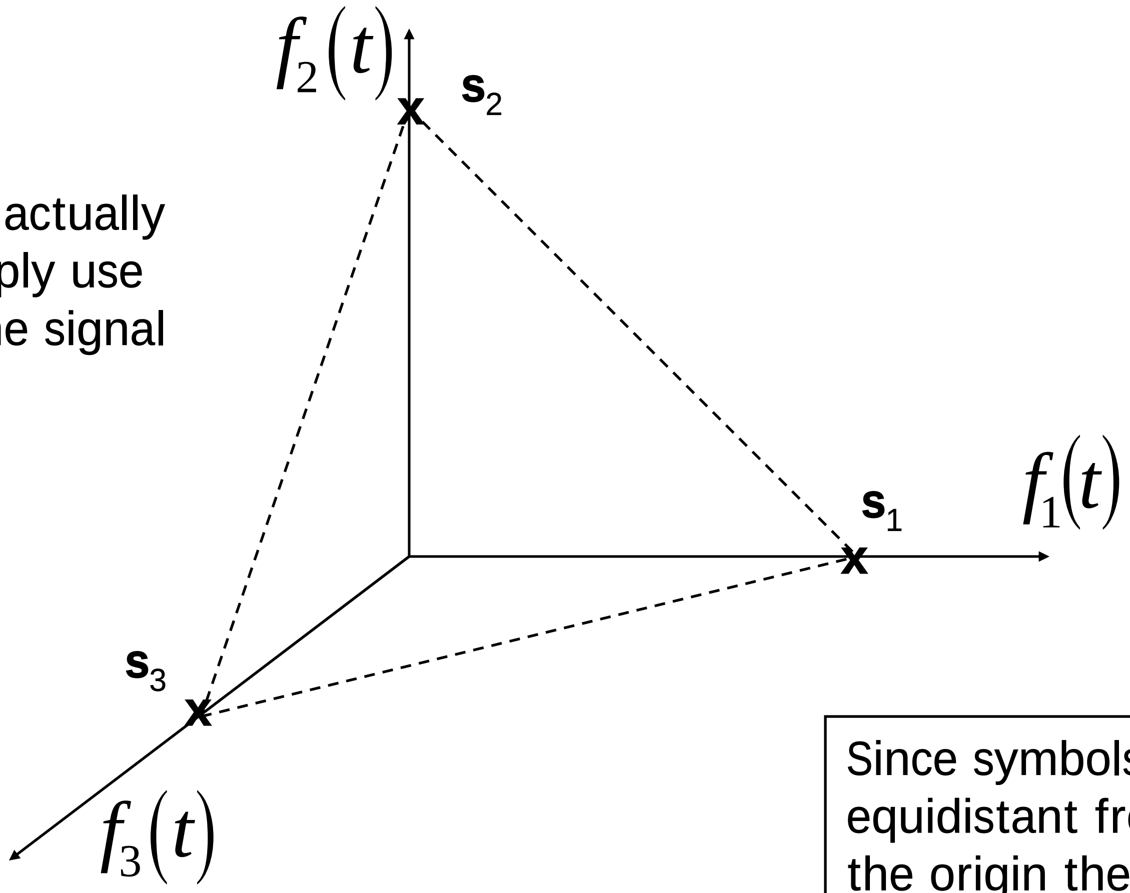
$$f_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t + \frac{2\pi i}{T} t\right)$$

$$\boxed{\overline{E}_s = 1}$$

- Further, we can represent the symbols in signal space as *M*-dimensional vectors:

$$\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} \sqrt{E} & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \\
\mathbf{s}_2 &= \begin{bmatrix} 0 & \sqrt{E} & 0 & 0 & \dots & 0 \end{bmatrix} \\
\mathbf{s}_3 &= \begin{bmatrix} 0 & 0 & \sqrt{E} & 0 & \dots & 0 \end{bmatrix} \\
&\vdots \\
\mathbf{s}_M &= \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \sqrt{E} \end{bmatrix}
\end{aligned}$$

3- FSK

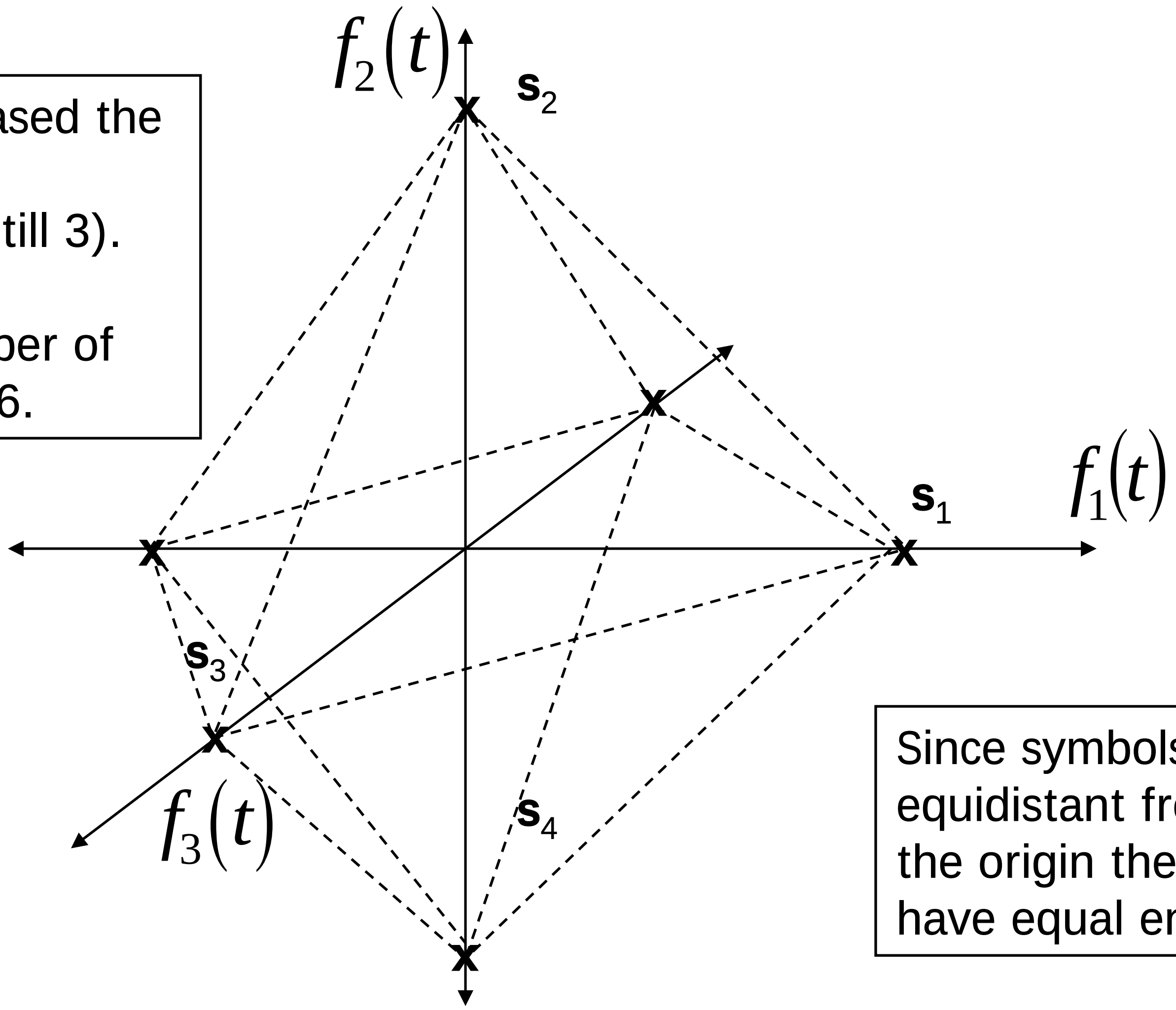


Note that we never actually use 3-FSK. We simply use it to demonstrate the signal space

Since symbols are equidistant from the origin they have equal energy

Bi-orthogonal Modulation Example

We have not increased the signal space (dimensionality is still 3). However we have increased the number of symbols from 3 to 6.



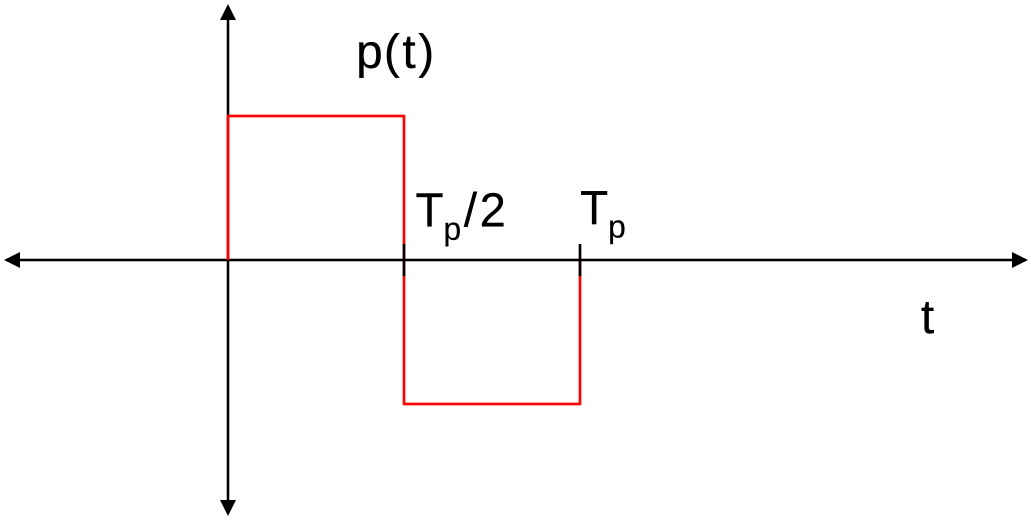
Since symbols are equidistant from the origin they have equal energy

Notes on Gram-Schmidt Procedure

- A signal set may have many different sets of basis functions.
- A change of basis functions is equivalent to rotating coordinates.
- The order in which signals are used in the Gram-Schmidt procedure will affect the resulting basis functions.
- The choice of basis functions does not effect performance.

PPM Example

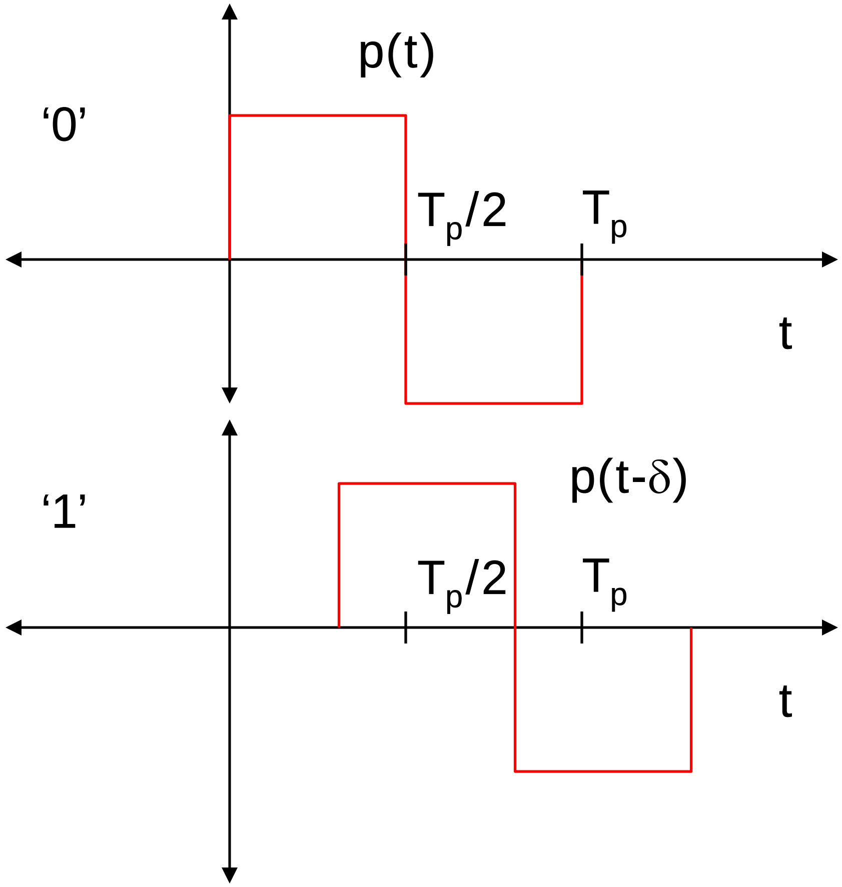
- Let's now return to our Pulse Position Modulation example, but we will use a more amenable pulse shape. Assume the use of the following pulse shape



$$p(t) = u(t) - 2u\left(t - \frac{T_p}{2}\right) + u\left(t - T_p\right)$$

Example (cont.)

- Pulse Position Modulation



$$s(t) = \sum_{n=-\infty}^{\infty} Ap\left(t - nT_s - \delta b_n\right)$$

$$b_n \in \{0,1\}$$

What is the optimal choice for δ given that

$$0 \le \delta \le 2T_p$$

Example (cont.)

- To determine this we will
  - Use the Gram-Schmidt procedure to determine basis functions
  - Plot the resulting points on a signal space diagram
  - Find the value of δ that maximizes the distance between points

Example (cont.)

- First, we determine the energy of signal 1 and set the first basis function to the normalized version of signal 1:

$$
\begin{aligned}
E_1 &= \int_{-\infty}^{\infty} \left| s_1(t) \right|^2 dt \\
&= \int_{0}^{T_p} |A|^2 dt \\
&= A^2 T_p
\end{aligned}
$$

$$
\begin{aligned}
f_1(t) &= \frac{s_1(t)}{\sqrt{E_1}} \\
&= \frac{1}{\sqrt{T_p}} \left(u(t) - 2u\left(t - \frac{T_p}{2} \right) + u\left(t - T_p \right) \right)
\end{aligned}
$$

Example (cont.)

- Next, we compute correlation with 1st basis:

$$c_{12} = \int_{-\infty}^{\infty} A\left(u(t-\delta)-2u\left(t-\frac{T_p}{2}-\delta\right)+u\left(t-T_p-\delta\right)\right)\left(\frac{1}{\sqrt{T_p}}\left(u(t)-2u\left(t-\frac{T_p}{2}\right)+u\left(t-T_p\right)\right)\right)dt$$

for $0 \le \delta \le \dfrac{T_p}{2}$

$$\begin{aligned}
c_{12} &= \int_{\delta}^{T_p/2} \frac{1}{\sqrt{T_p}} A dt - \int_{T_p/2}^{T_p/2+\delta} \frac{1}{\sqrt{T_p}} A dt + \int_{T_p/2+\delta}^{T_p} \frac{1}{\sqrt{T_p}} A dt \\
&= \frac{A}{\sqrt{T_p}}\left(\frac{T_p}{2}-\delta\right) - \frac{A}{\sqrt{T_p}}\left(\delta+\frac{T_p}{2}-\frac{T_p}{2}\right) + \frac{A}{\sqrt{T_p}}\left(T_p-\delta-\frac{T_p}{2}\right) \\
&= \frac{A}{\sqrt{T_p}}\left(T_p-3\delta\right)
\end{aligned}$$

Example (cont.)

for $\frac{T_p}{2} \le \delta \le T_p$

$$c_{12} = -\int_{\delta}^{T_p} \frac{1}{\sqrt{T_p}} A dt$$

$$= -\frac{A}{\sqrt{T_p}} \left(T_p - \delta \right)$$

for $\delta > T_p$

$$c_{12} = 0$$

Example (cont.)

- Now the two signals are represented in vector form as

$$\mathbf{s}_1 = \begin{bmatrix} A\sqrt{T_p} \\ 0 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix}$$

- Further the distance between points is

$$d = \sqrt{\left(A\sqrt{T_p} - c_{12}\right)^2 + \left(c_{22}\right)^2}$$

$$= \sqrt{A^2 T_p - 2A\sqrt{T_p}\,c_{12} + c_{12}^2 + c_{22}^2}$$

Example (cont.)

- The energy of the signals is the same and is independent of δ and signal space representation. Thus the term $c_{12}^2 + c_{22}^2$ is constant
- Thus the distance $d = \sqrt{A^2 T_p - 2A\sqrt{T_p}\,c_{12} + c_{12}^2 + c_{22}^2}$ is maximized by minimizing c_{12}

for $0 \le \delta \le \frac{T_p}{2}$

$$c_{12} = \frac{A}{\sqrt{T_p}}\left(T_p - 3\delta\right) \quad \min_{\delta}\left\{c_{12}\right\} = -\frac{A\sqrt{T_p}}{2}$$

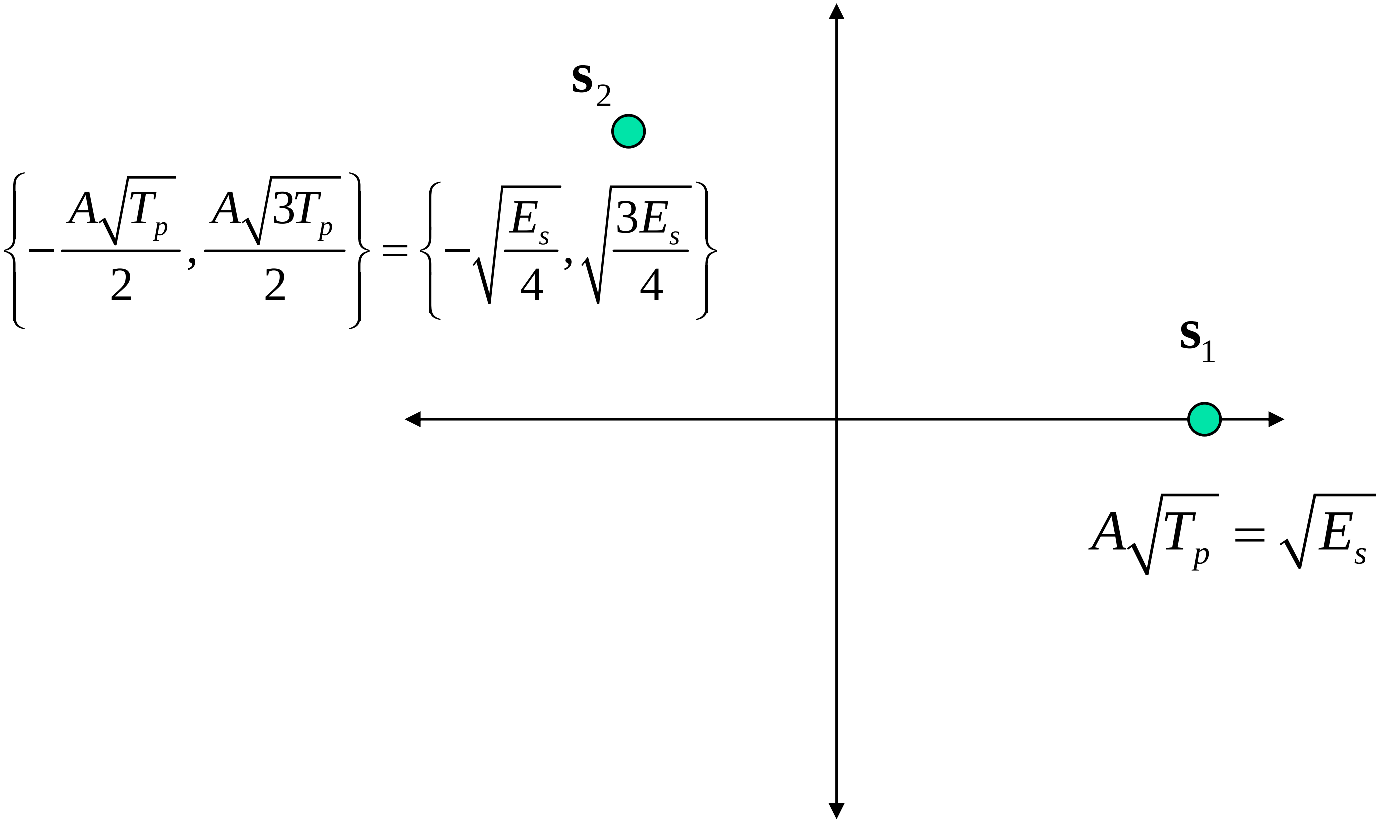
$$\boxed{\delta = \frac{T_p}{2}}$$

for $\frac{T_p}{2} \le \delta \le 2T_p$

$$c_{12} = -\frac{A}{\sqrt{T_p}}\left(T_p - \delta\right) \quad \min_{\delta}\left\{c_{12}\right\} = -\frac{A\sqrt{T_p}}{2}$$

Example (cont.)

- The resulting signal space diagram



$\mathbf{s}_2$

$$\left\{-\frac{A\sqrt{T_p}}{2},\frac{A\sqrt{3T_p}}{2}\right\}=\left\{-\sqrt{\frac{E_s}{4}},\sqrt{\frac{3E_s}{4}}\right\}$$

$\mathbf{s}_1$

$$A\sqrt{T_p}=\sqrt{E_s}$$

Example (cont.)

- For completeness we could find the other basis function

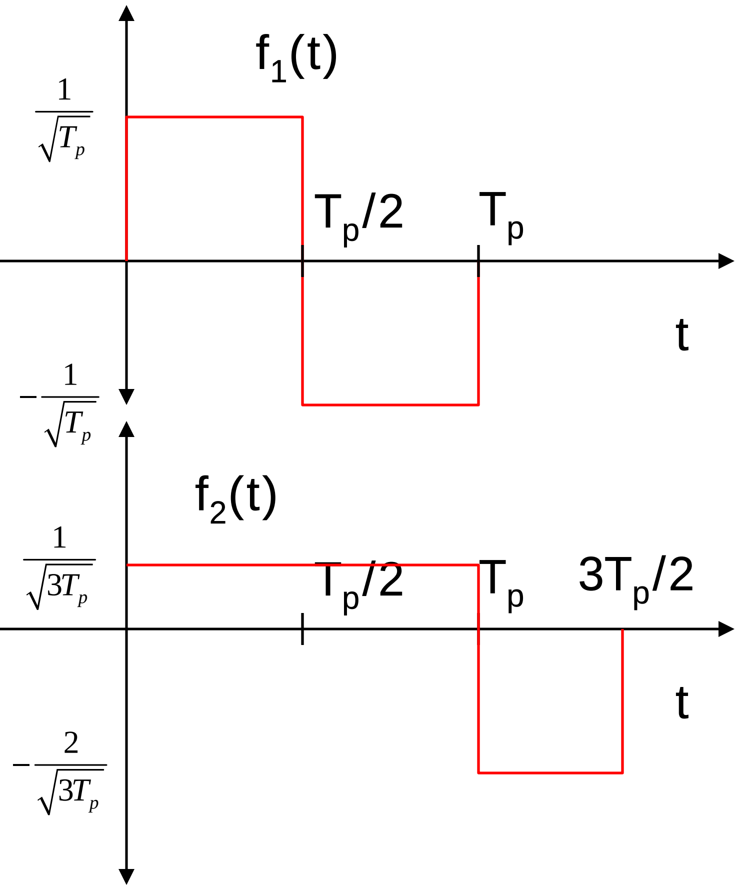
$$
\begin{aligned}
f_2'(t) &= Au\left(t-\frac{T_p}{2}\right)-2Au\left(t-T_p\right)+Au\left(t-\frac{3T_p}{2}\right)-\left(-\frac{A\sqrt{T_p}}{2}\right)\frac{1}{\sqrt{T_p}}\left(u(t)-2u\left(t-\frac{T_p}{2}\right)+u\left(t-T_p\right)\right)\\
&=\frac{A}{2}u(t)+\left(A-A\right)u\left(t-\frac{T_p}{2}\right)+\left(\frac{A}{2}-2A\right)u\left(t-T_p\right)+Au\left(t-\frac{3T_p}{2}\right)\\
&=\frac{A}{2}u(t)-\frac{3A}{2}u\left(t-T_p\right)+Au\left(t-\frac{3T_p}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \int_{-\infty}^{\infty}\left|f_2'(t)\right|^2 dt\\
&=\frac{3A^2T_p}{4}
\end{aligned}
$$

$$
f_2(t)=\frac{f_2'(t)}{\sqrt{\mathrm{E}_2}}
$$

Example (cont.)

- Basis Functions



Do they both have unit energy?
Are they orthogonal?