EE 5654 - Digital Communications Spring 2005

Instructor: R. Michael Buehrer

Lecture #6

Optimal Receiver Design

Modulation

- We want to modulate signals with digital data using signal sets which are:
  - bandwidth efficient – high bits/sec/Hz
  - energy efficient – low SNR needed for BER requirement
- A signal space representation is a convenient form for viewing modulation which allows us to:
  - design energy and bandwidth efficient signal constellations
  - **determine the form of the optimal receiver for a given constellation**
  - evaluate the performance of a modulation type

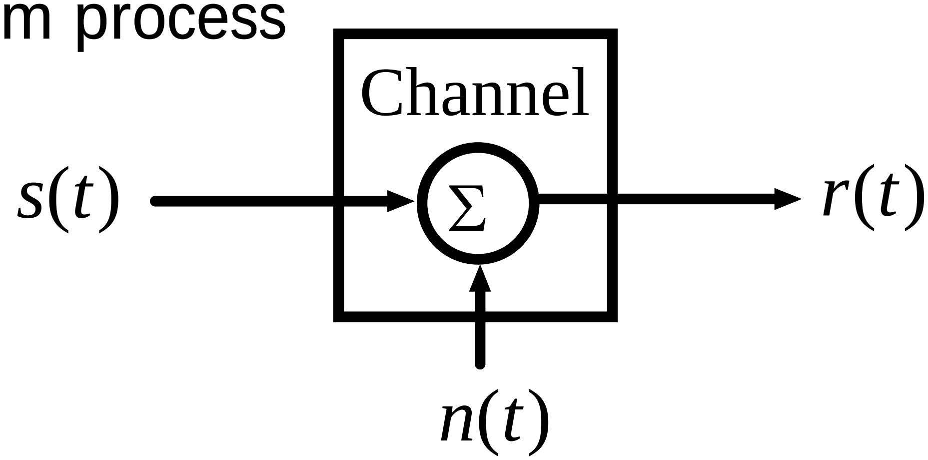
Problem Statement

- We transmit a signal $s(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where $s(t)$ is nonzero only on $t \in [0, T]$.
- Let the various signals be transmitted with probability:
$$p_1 = \Pr[s_1(t)], \ldots, p_M = \Pr[s_M(t)]$$
- The received signal is corrupted by noise:
$$r(t) = s(t) + n(t)$$
- Given $r(t)$, the receiver forms an estimate $\hat{s}(t)$ of the signal $s(t)$ with the goal of minimizing symbol error probability
$$P_s = \Pr[\hat{s}(t) \neq s(t)]$$

Noise Model

- The signal is corrupted by Additive White Gaussian Noise (AWGN) $n(t)$
- The noise $n(t)$ has autocorrelation $\phi_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$

  and power spectral density $\Phi_{nn}(f) = N_0/2$
- Any linear function of $n(t)$ will be a Gaussian random process



Signal Space Representation

- The transmitted signal can be represented as:

$$s_m(t) = \sum_{k=1}^{K} s_{m,k} f_k(t)$$

where

$$s_{m,k} = \int_0^T s_m(t) f_k(t) dt$$

.

In terms of *basis functions*

- The noise can be represented as: $n(t) = n'(t) + \sum_{k=1}^{K} n_k f_k(t)$

where

$$n_k = \int_0^T n(t) f_k(t) dt$$

and

$$n'(t) = n(t) - \sum_{k=1}^{K} n_k f_k(t)$$

Note that the noise has higher dimensionality than *K*

Signal Space Representation (continued)

- The received signal can be represented as:

$$r(t) = \sum_{k=1}^{K} s_{m,k} f_k(t) + \sum_{k=1}^{K} n_k f_k(t) + n'(t) = \sum_{k=1}^{K} r_k f_k(t) + n'(t)$$

where

$$r_k = s_{m,k} + n_k$$

Note: We can represent the coefficients r_k by a K-dimensional vector **r**

$$\mathbf{r} = \mathbf{s} + \mathbf{n} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_K \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_K \end{bmatrix}$$

The Orthogonal Noise: $n'(t)$

- The noise $n'(t)$ can be disregarded by the receiver

$$E\left[n'(t)r_k\right] = E\left[n'(t)\left(s_{m,k} + n_k\right)\right]$$
$$= E\left[n'(t)s_{m,k}\right] + E\left[n'(t)n_k\right]$$
$$= E\left[n'(t)n_k\right]$$
$$= E\left[\left(n(t) - \sum_{i=1}^{K} n_i f_i(t)\right)n_k\right]$$
$$= E\left[\left(n(t)\int_0^T n(\tau) f_k(\tau) d\tau - \sum_{i=1}^{K} n_k n_i f_i(t)\right)\right]$$
$$= \int_0^T E\left[n(t)n(\tau)\right] f_k(\tau) d\tau - \sum_{i=1}^{K} E[n_k n_i] f_i(t)$$
$$= \frac{1}{2} N_o f_k(\tau) - \frac{1}{2} N_o f_k(\tau) = 0$$

Since *n'(t)* and r_k are uncorrelated they are also independent since they are GRV. Thus, we can ignore *n'(t)* since it has no bearing on the decision.

We can formulate the problem in terms of *K*-dimensional vectors

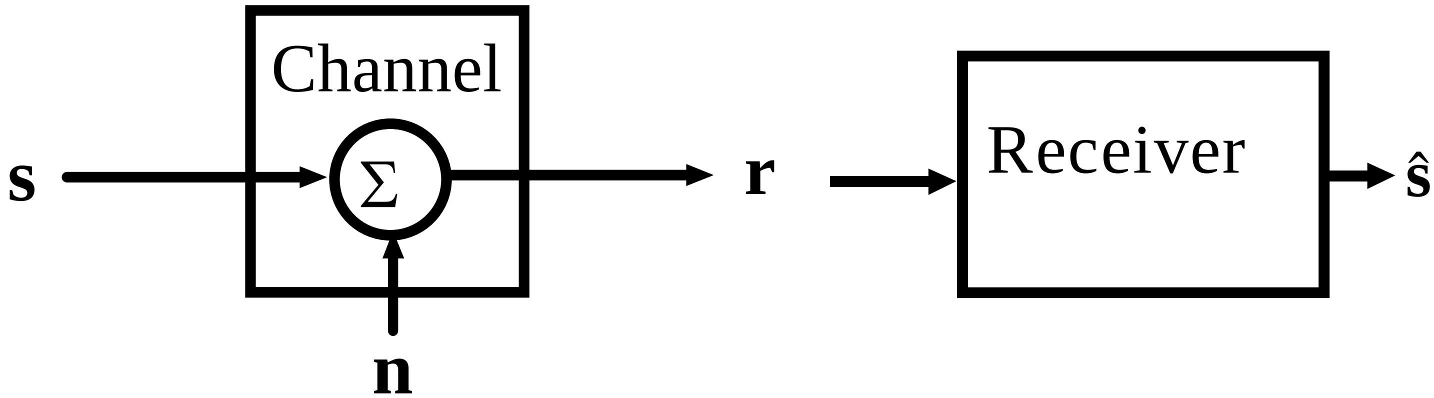
- We transmit a *K* dimensional signal vector:

$$\mathbf{s} = [s_1, s_2, \ldots, s_K] \in \{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$$

- We receive a vector $\mathbf{r} = [r_1, \ldots, r_K] = \mathbf{s} + \mathbf{n}$ which is the sum of the signal vector and noise vector

$$\mathbf{n} = [n_1, \ldots, n_K]$$

- Given $\mathbf{r}$, we wish to form an estimate $\hat{\mathbf{s}}$ of the transmitted signal vector which minimizes $P_S = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}]$



MAP (Maximum a posteriori probability) Decision Rule

- Suppose that signal vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots \mathbf{s}_\mathrm{M}\}$ are transmitted with probabilities $\{p_1, p_2, \ldots p_\mathrm{M}\}$ respectively, and the signal vector $\mathbf{r}$ is received
- We minimize symbol error probability by choosing the signal $\mathbf{s}_\mathrm{m}$ which satisfies: $\Pr(\mathbf{s}_m|\mathbf{r}) \geq \Pr(\mathbf{s}_i|\mathbf{r}), \forall m \neq i$
- Equivalently (using Bayes' rule):

$$\frac{p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m)}{p(\mathbf{r})} \geq \frac{p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i)}{p(\mathbf{r})}, \forall m \neq i$$

or

$$p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i), \forall m \neq i$$

Maximum Likelihood (ML) Decision Rule

- If $p_1 = \cdots = p_m$ or the a priori probabilities are unknown, then the MAP rule simplifies to the ML Rule

- We minimize symbol error probability by choosing the signal $\mathbf{s}_m$ which satisfies

$$p(\mathbf{r}|\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i), \forall m \neq i$$

Evaluation of Probabilities

- In order to apply either the MAP or ML rules, we need to evaluate: $p\left(\mathbf{r}\middle|\mathbf{s}_m\right)$
- Since $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$ where $\mathbf{s}_m$ is constant, it is equivalent to evaluate: $p(\mathbf{n}) = p(n_1, \ldots, n_k)$
- $n(t)$ is a Gaussian random process
  - Therefore $n_k = \int_0^T n(t) f_k(t) dt$ is a Gaussian random variable
  - Therefore $p(n_1, \ldots, n_K)$ will be a joint Gaussian p.d.f.

The Noise p.d.f

$$E[n_i \cdot n_k] = E\left[\int_0^T n(t) f_i(t) dt \cdot \int_0^T n(s) f_k(s) ds\right]$$

$$= E\left[\int_0^T \int_0^T n(t) n(s) f_i(t) f_k(s) ds dt\right] = \int_0^T \int_0^T E[n(t) n(s)] f_i(t) f_k(s) ds dt$$

$$= \int_0^T \int_0^T \phi_{nn}(t-s) f_i(t) f_k(s) ds dt = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-s) f_i(t) f_k(s) ds dt$$

$$= \int_0^T \frac{N_0}{2} f_i(t) f_k(t) dt = \begin{cases} N_0/2, & i = k \\ 0, & i \neq k \end{cases}$$

The Noise p.d.f (continued)

- Since $E[n_i \cdot n_k] = 0, \forall i \neq k$, individual noise components are uncorrelated (and therefore independent)
- Since $E\left[n_k^{\,2}\right] = N_0/2$, each noise component has a variance of $N_0/2$.

$$p(n_1,\ldots,n_K) = p(n_1)\cdots p(n_K)$$

$$= \prod_{k=1}^{K} \frac{1}{\sqrt{\pi N_0}} \exp\left(-n_k^{\,2}/N_0\right)$$

$$= (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^{K} n_k^{\,2} \Big/ N_0\right)$$

Conditional pdf of Received Signal

- Transmitted signal values in each dimension represent the mean values for each signal

$$p\left(\mathbf{r} \mid \mathbf{s}_m\right) = \left(\pi N_0\right)^{-K/2} \exp\left(-\sum_{k=1}^{K} \left(r_k - s_{m,k}\right)^2 \Big/ N_0 \right)$$

Structure of Optimum Receiver

- MAP rule: $\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\; p_m \cdot p(\mathbf{r}|\mathbf{s}_m)$

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\; p_m \cdot (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^{K} (r_k - s_{m,k})^2 \Big/ N_0 \right)$$

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\; \ln\left[p_m \cdot (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^{K} (r_k - s_{m,k})^2 \Big/ N_0 \right)\right]$$

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\; \ln[p_m] - \frac{K}{2}\ln[\pi N_0] - \frac{1}{N_0}\sum_{k=1}^{K} (r_k - s_{m,k})^2$$

Structure of Optimum Receiver (continued)

- $$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\ \ln[p_m] - \frac{K}{2}\ln[\pi N_0] - \frac{1}{N_0}\left(\sum_{k=1}^{K} r_k{}^2 - 2\sum_{k=1}^{K} r_k s_{m,k} + \sum_{k=1}^{K} s_{m,k}{}^2\right)$$

- Eliminating terms which are identical for all choices:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\ \ln[p_m] + \frac{2}{N_0}\sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{N_0}\sum_{k=1}^{K} s_{m,k}{}^2$$

Final Form of MAP Receiver

- Multiplying through by the constant $N_0/2$:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max}\ \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2$$

Interpreting This Result

- $\frac{N_0}{2}\ln\left[p_m\right]$ weights the a priori probabilities
  - If the noise is large, p_m counts a lot
  - If the noise is small, our received signal will be an accurate estimate and p_m counts less
- $\sum_{k=1}^{K} r_k s_{m,k} = \int_0^T s_m(t) r(t) dt$ represents the correlation with the received signal
- $\frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2 = \frac{1}{2}\int_0^T s_m{}^2(t)dt = \frac{E_m}{2}$ represents signal energy

Interpreting This Result: Another View

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \ \ln[p_m] - \frac{1}{N_0}\sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

If equally likely:

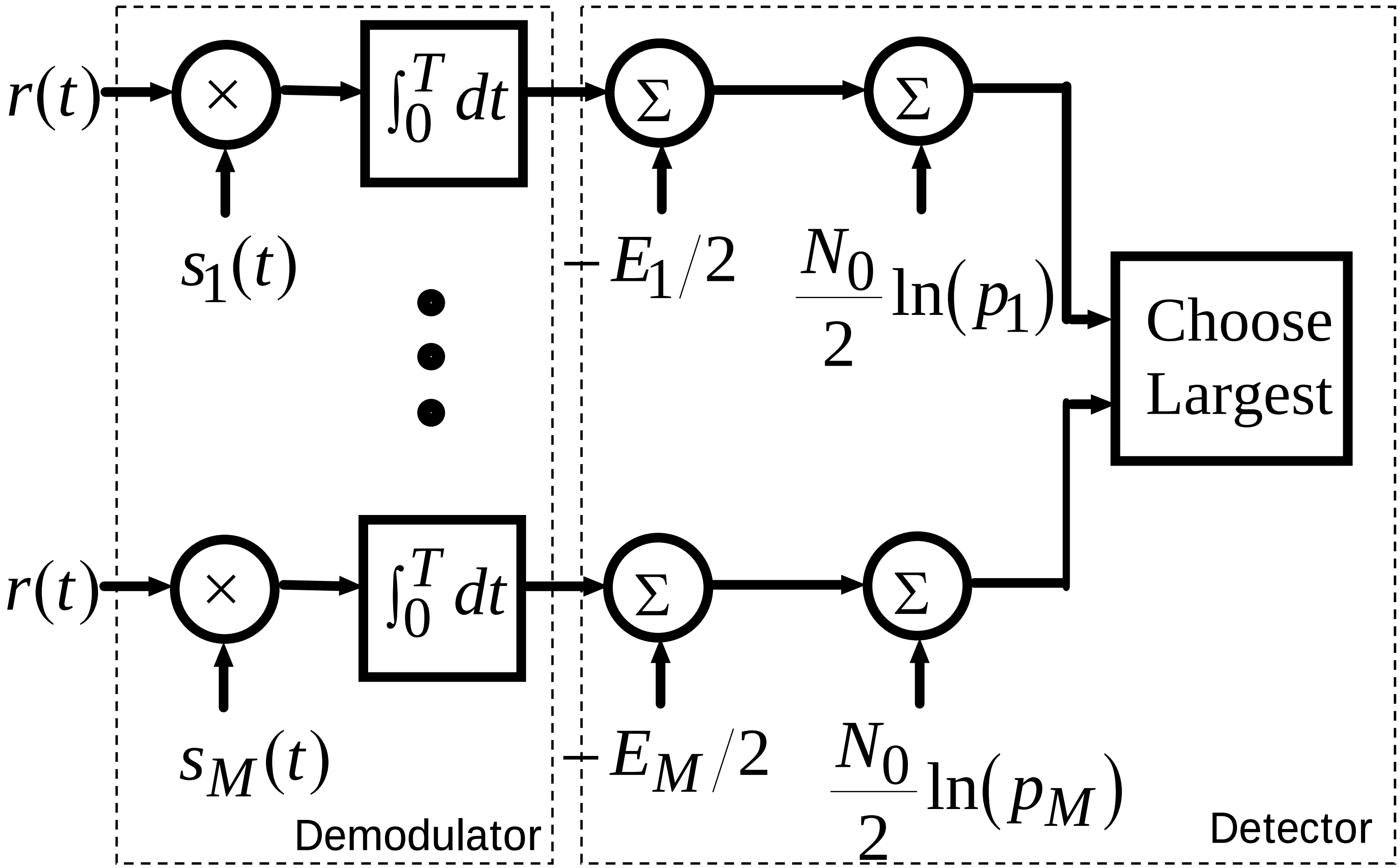
$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \ -\frac{1}{N_0}\sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

In order to maximize the argument, we need to minimize the sum:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\min} \sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

Thus, we wish to find the symbol which is closest in Euclidean distance

An Implementation of the Optimal Receiver - Correlation Receiver



Optimal Receiver Functionality

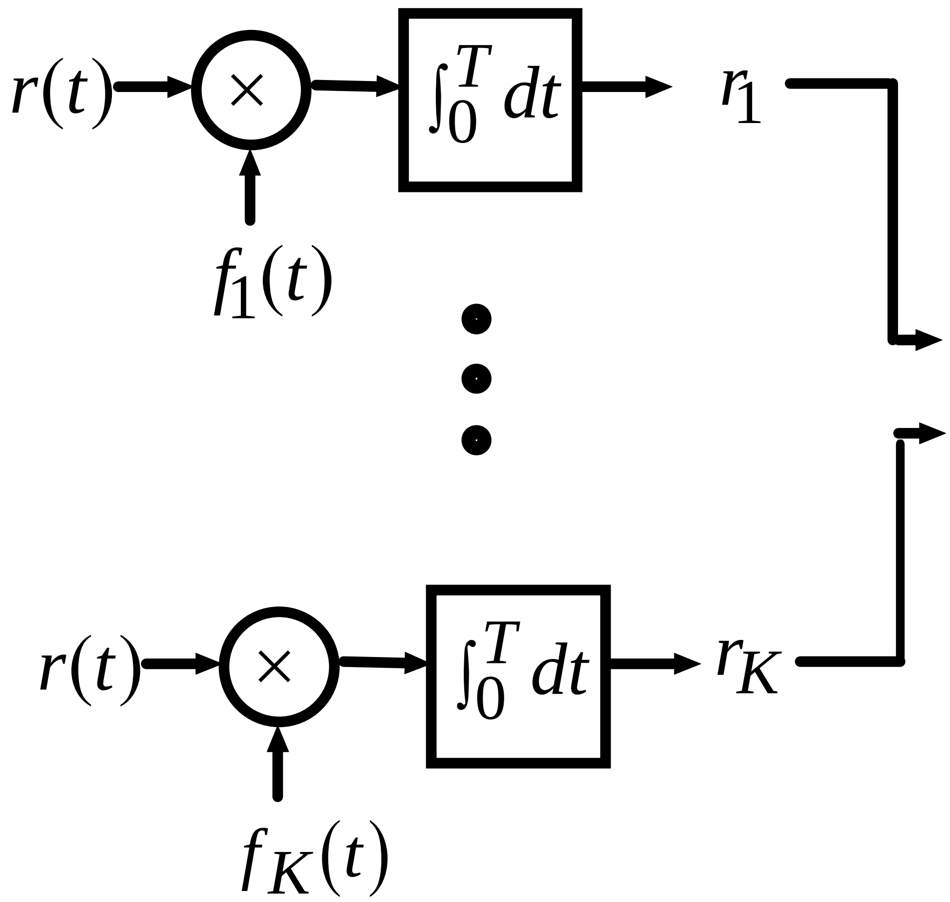
- Demodulator
  - The demodulator converts the received signal into a vector of *decision statistics* which are passed to the detector
- Detector
  - The detector uses the vector of decision statistics to create an estimate of the transmitted signal using either the Maximum A Posteriori or Maximum Likelihood decision rules

Simplifications for Special Cases

- ML case: All signals are equally likely ($p_1 = \cdots = p_M$). A priori probabilities can be ignored.
- All signals have equal energy ($E_1 = \cdots = E_M$). Energy terms can be ignored.
- We can reduce the number of correlations by directly implementing:

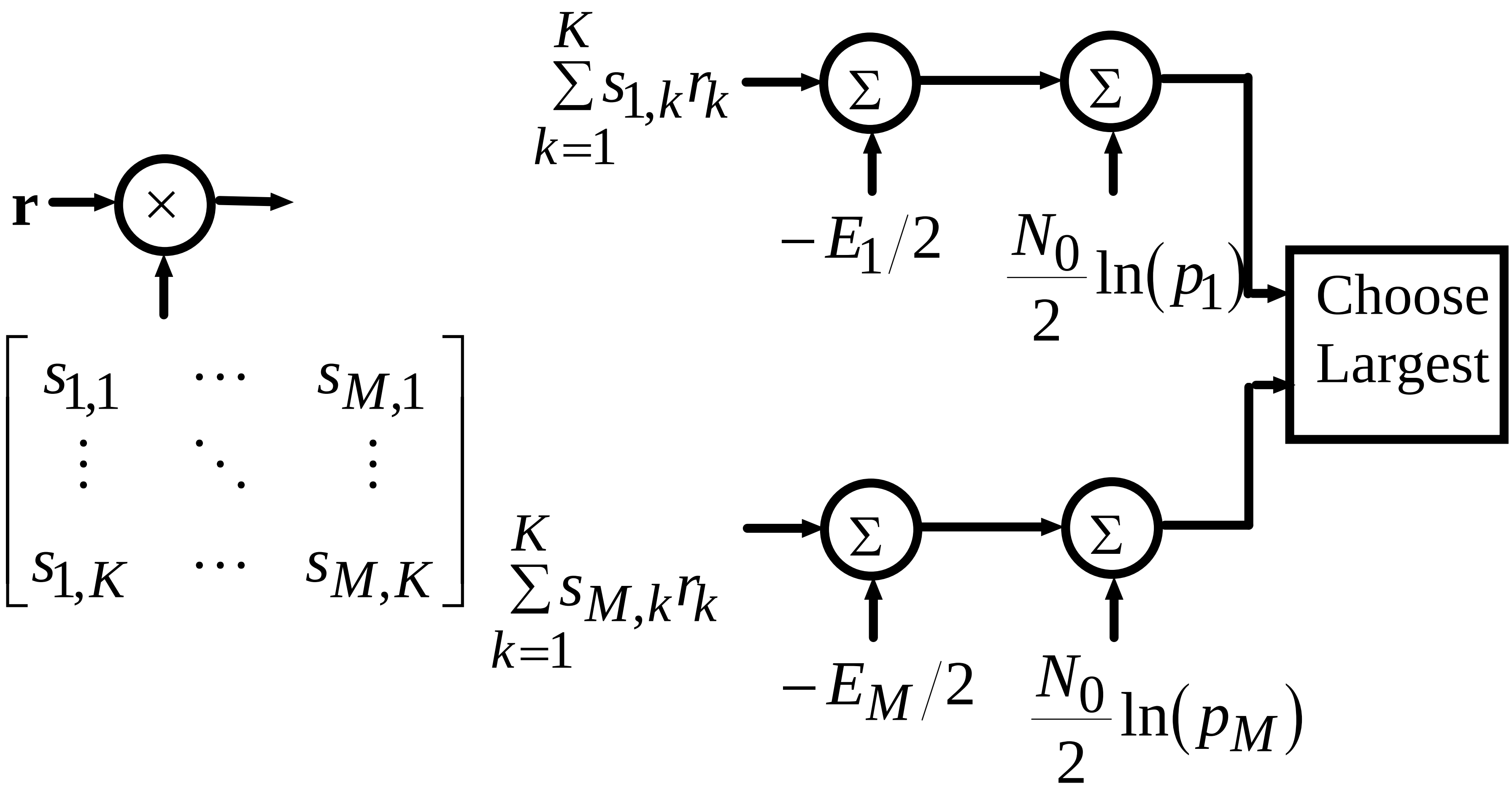
$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \frac{N_0}{2} \ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^{K} s_{m,k}{}^2$$

Reduced Complexity Implementation: Correlation Stage



$$\mathbf{r} = [r_1 \quad \cdots \quad r_K]$$

Reduced Complexity Implementation - Processing Stage



Matched Filter Implementation

- Assume $f_k(t)$ is time-limited to $t \in [0, T]$, and let $h_k(t) = f_k(T - t)$
- Then

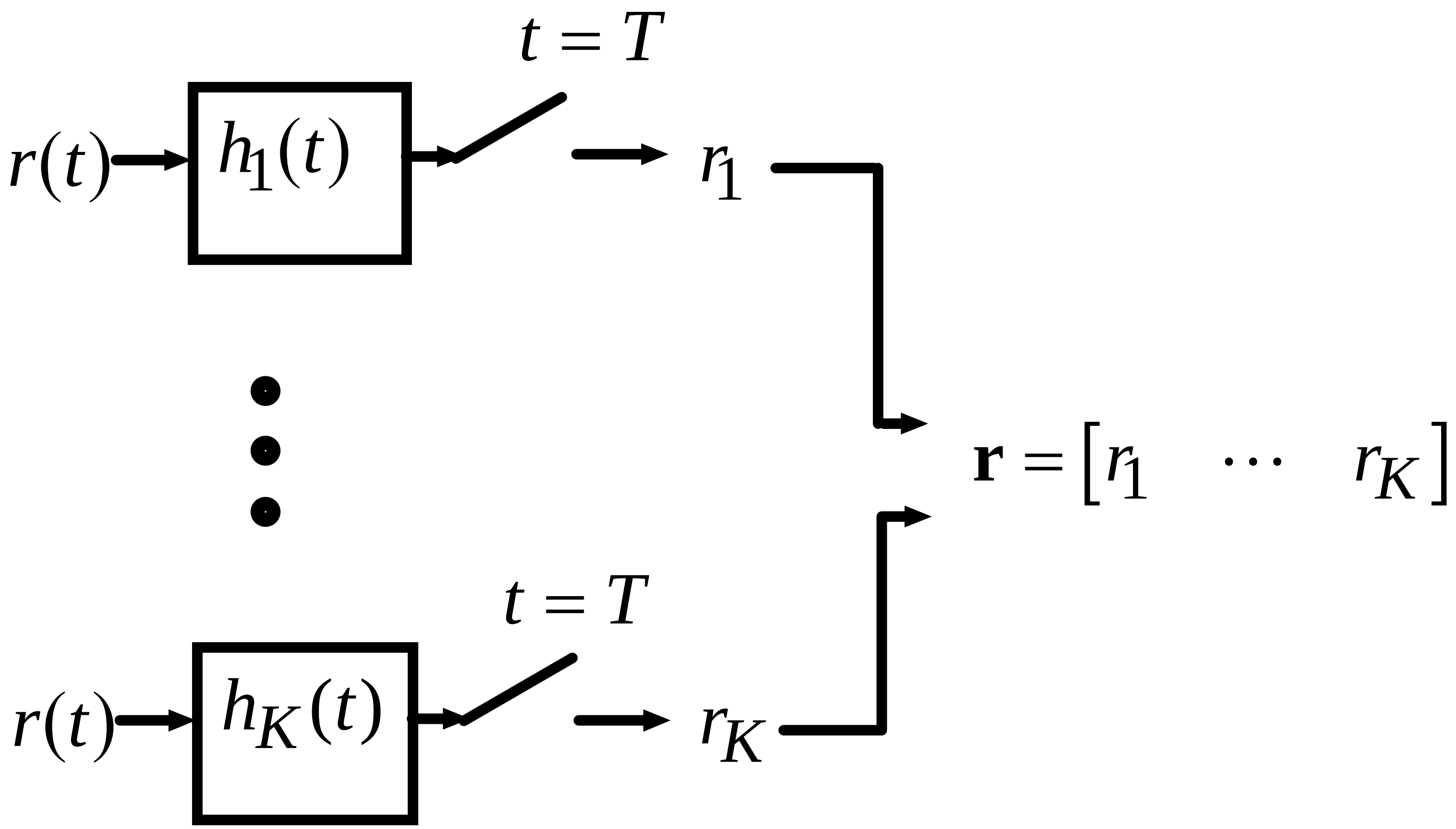
$$r_k = \int_0^T r(t) f_k(t)\,dt = \int_0^T r(t) f_k(T - (T - t))\,dt$$

$$= \int_0^T r(t) h_k(T - t)\,dt = r(t) \otimes h_k(t)\big|_{t=T}$$

where $r(t) \otimes h_k(t)\big|_{t=T}$ denotes the convolution of the signals $r(t)$ and $h_k(t)$ evaluated at time T

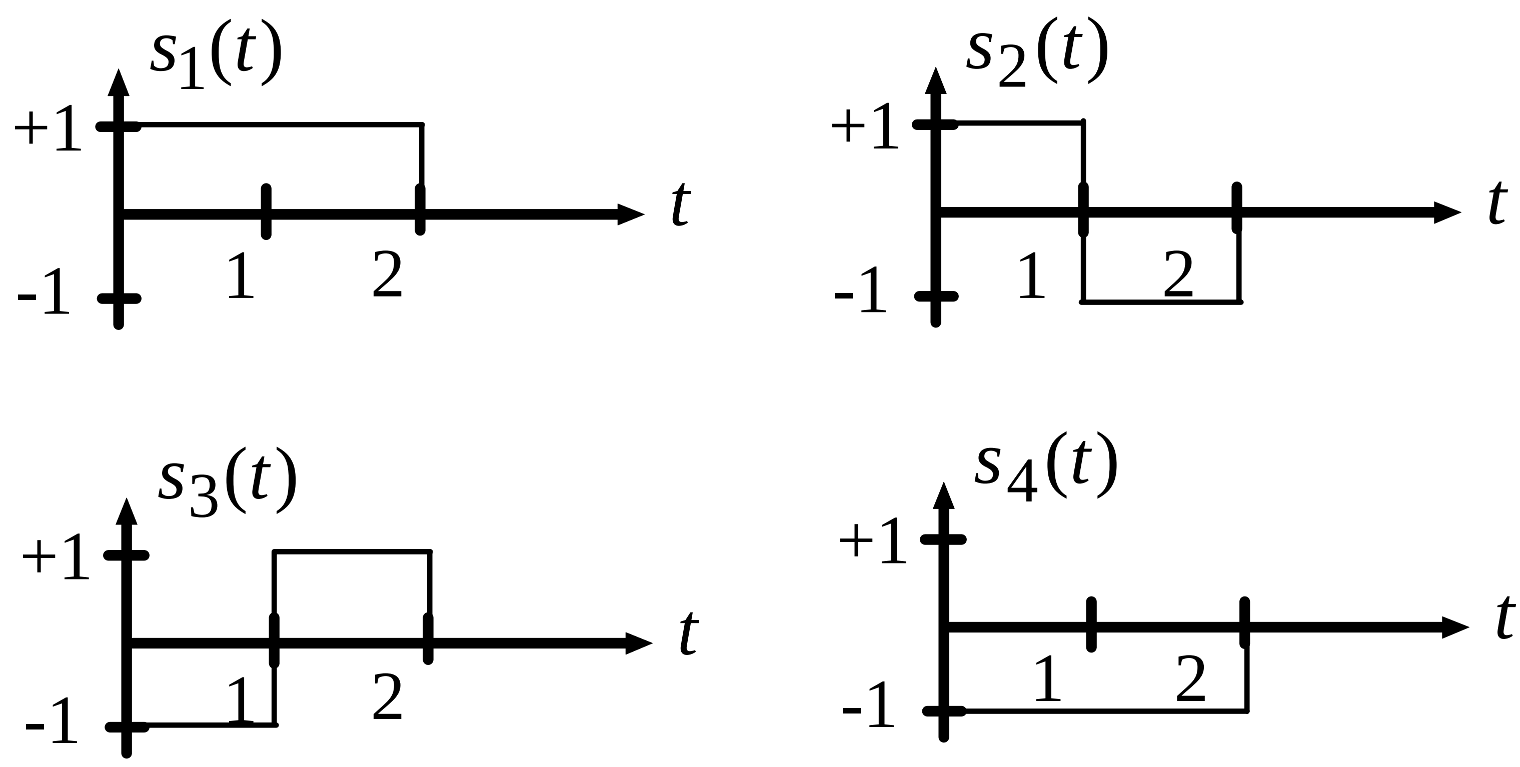
- We can implement each correlation by passing $r(t)$ through a filter with impulse response $h_k(t)$

Matched Filter Implementation of Correlation



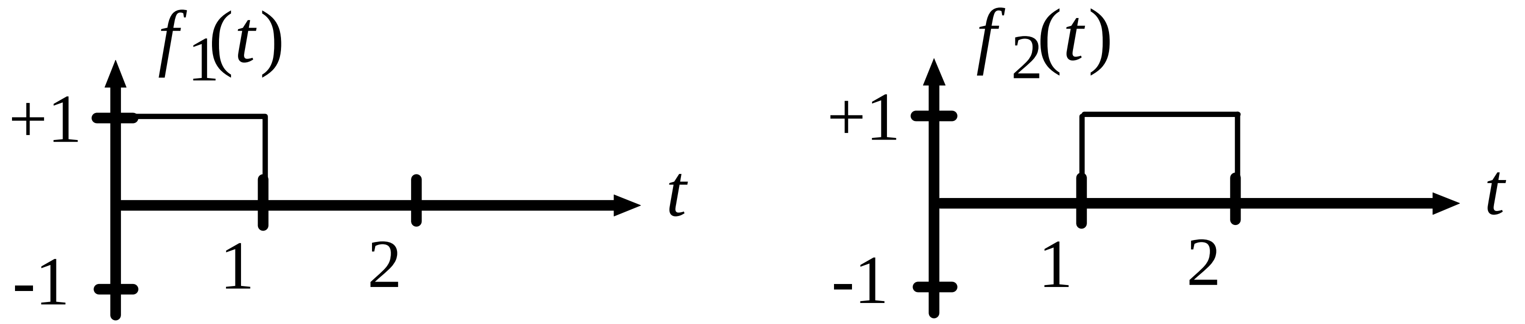
Example of Optimal Receiver Design

- Consider the signal set:



Example of Optimal Receiver Design (continued)

- Suppose we use the basis functions:



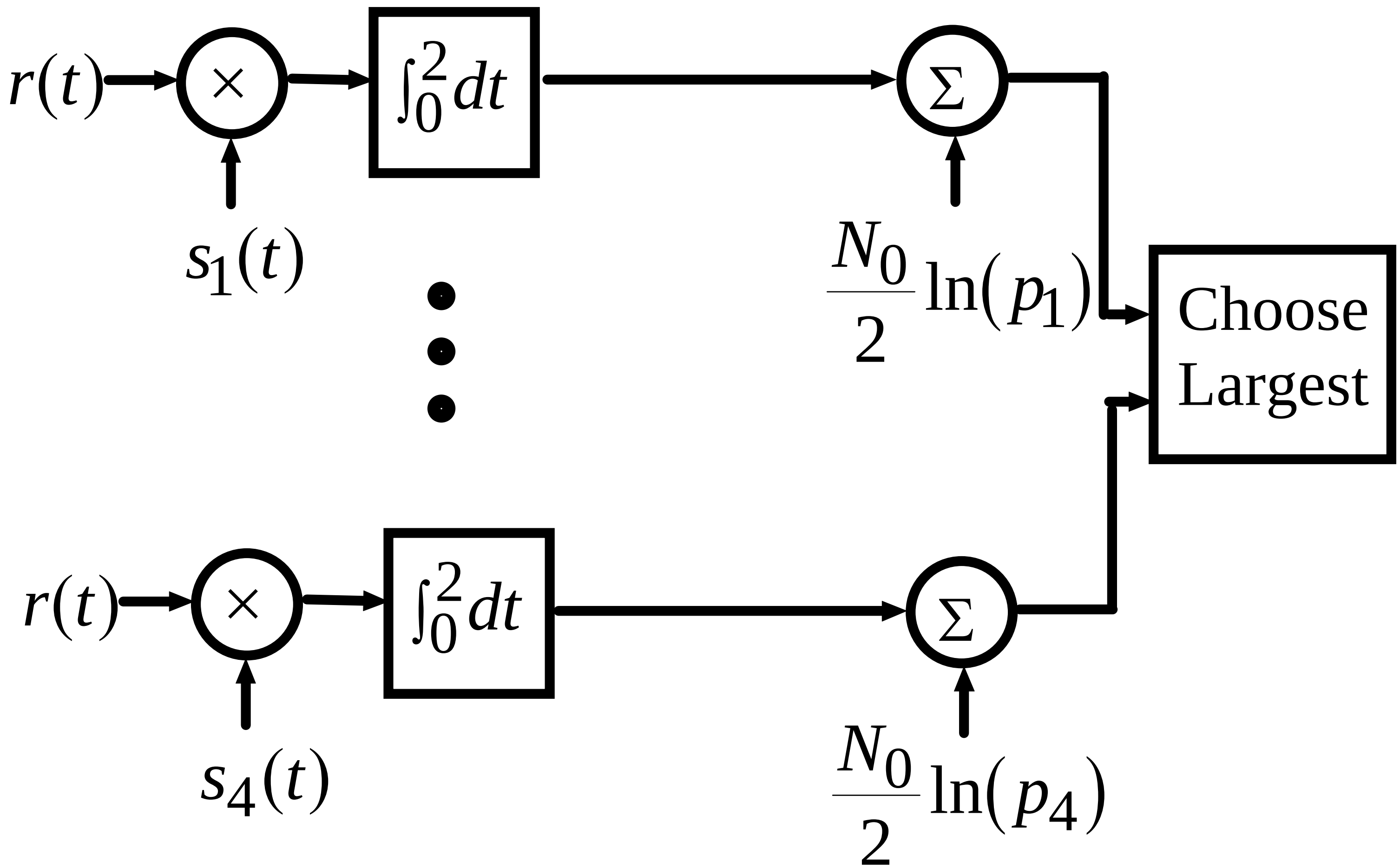
$$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$$

$$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t) \qquad s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$$

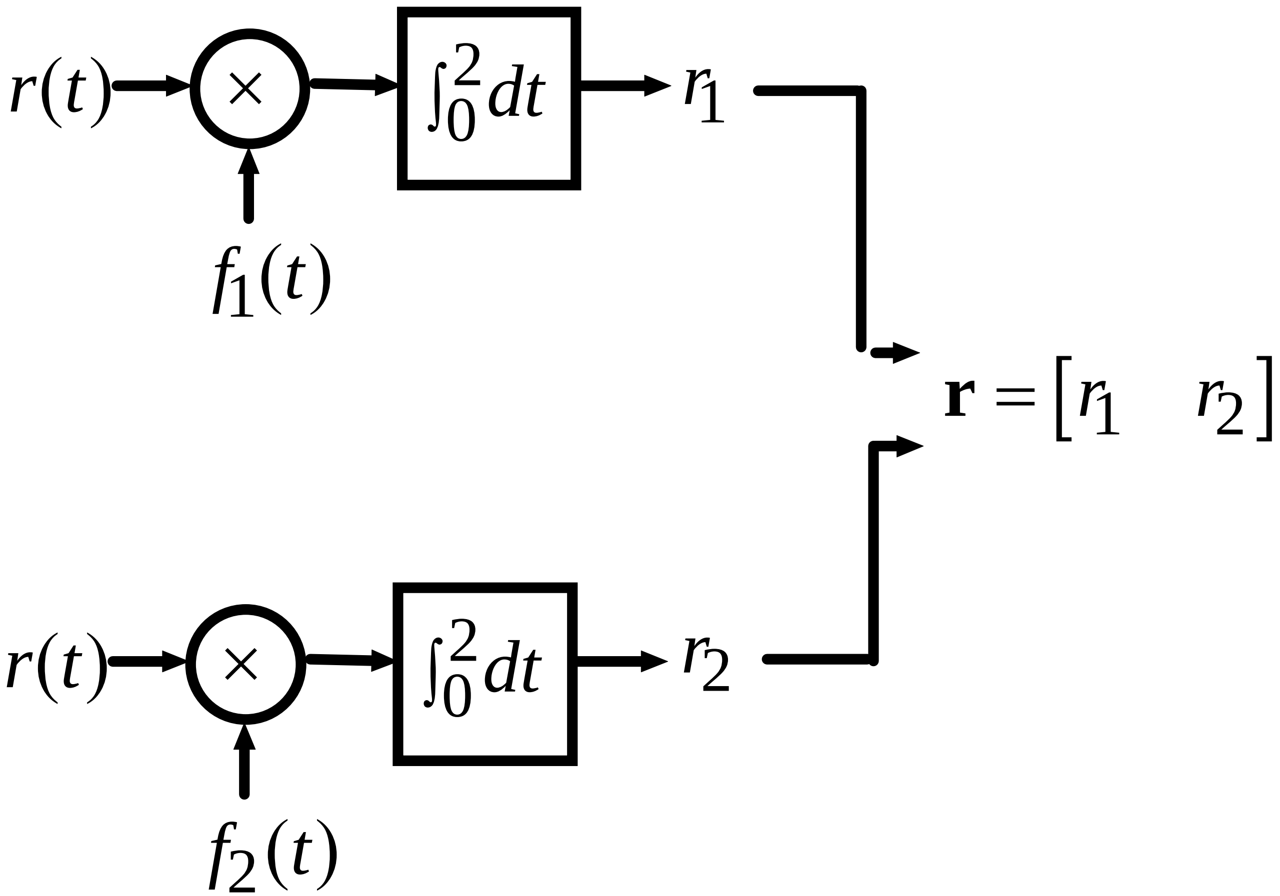
$$T = 2$$

$$E_1 = E_2 = E_3 = E_4 = 2$$

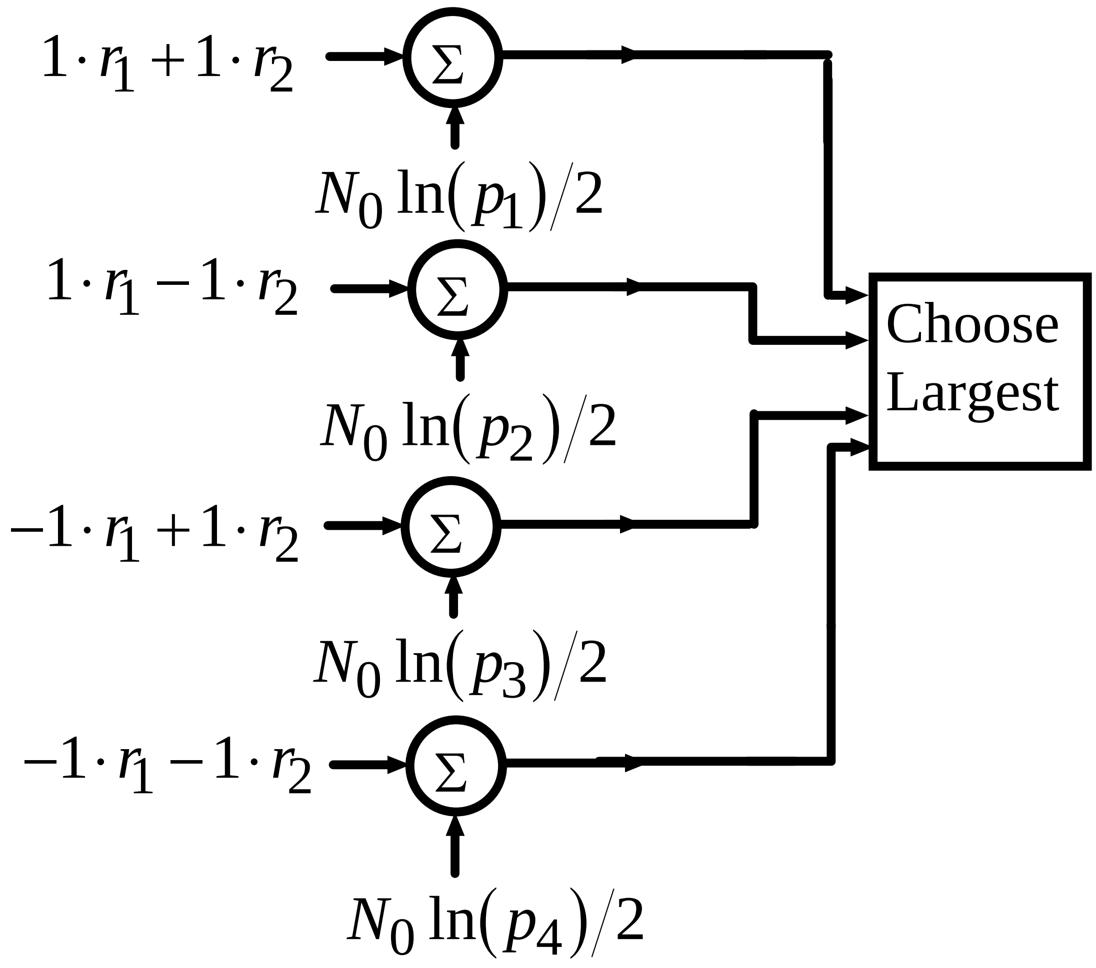
1st Implementation of Correlation Receiver



Reduced Complexity Correlation Receiver - Correlation Stage

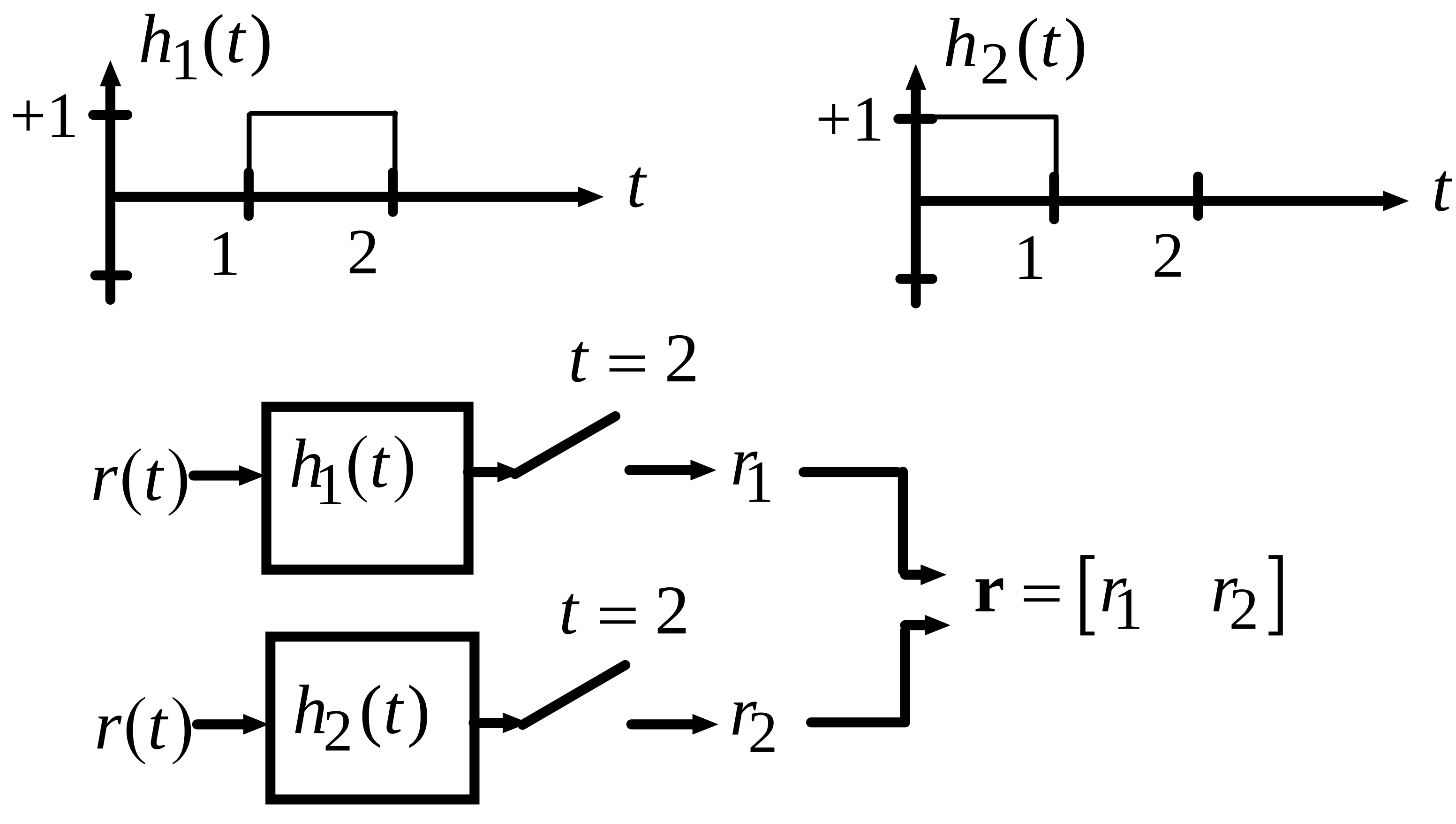


Reduced Complexity Correlation Receiver - Processing Stage



Matched Filter Implementation of Correlations

$$h_k(t) = f_k(2 - t)$$



Summary of Optimal Receiver Design

- Optimal coherent receiver for AWGN has three parts:
  - Correlates the received signal with each possible transmitted signal signal *(We will show later that this is optimal demodulator)*
  - Normalizes the correlation to account for energy
  - Weights the a priori probabilities according to noise power
- This receiver is completely general for any signal set
- Simplifications are possible under many circumstances