EE 5654 - Digital Communications Spring 2005

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Lecture #7 - Optimum Receiver Structures for Digital Modulation

Modulation

- Modulation is used to transmit digital data over a channel
- Gramm-Schmidt procedure allows vector representation of any signal constellation
- Optimal receiver consists of a correlator, weighted to adjust for signal energy and a priori probabilities
- Receiver implementation can be simplified with reduced correlation and matched filter
- We now turn to calculating performance

Problem Statement

- We transmit a signal $s(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where $s(t)$ is nonzero only on $t \in [0, T]$.
- Let the various signals be transmitted with probability: $p_1 = \Pr[s_1(t)], \ldots, p_M = \Pr[s_M(t)]$
- The received signal is corrupted by noise:
  $r(t) = s(t) + n(t)$
- Given $r(t)$, the receiver forms an estimate $\hat{s}(t)$ of the signal $s(t)$ with the goal of minimizing symbol error probability $P_s = \Pr[\hat{s}(t) \neq s(t)]$

Final Form of MAP Receiver

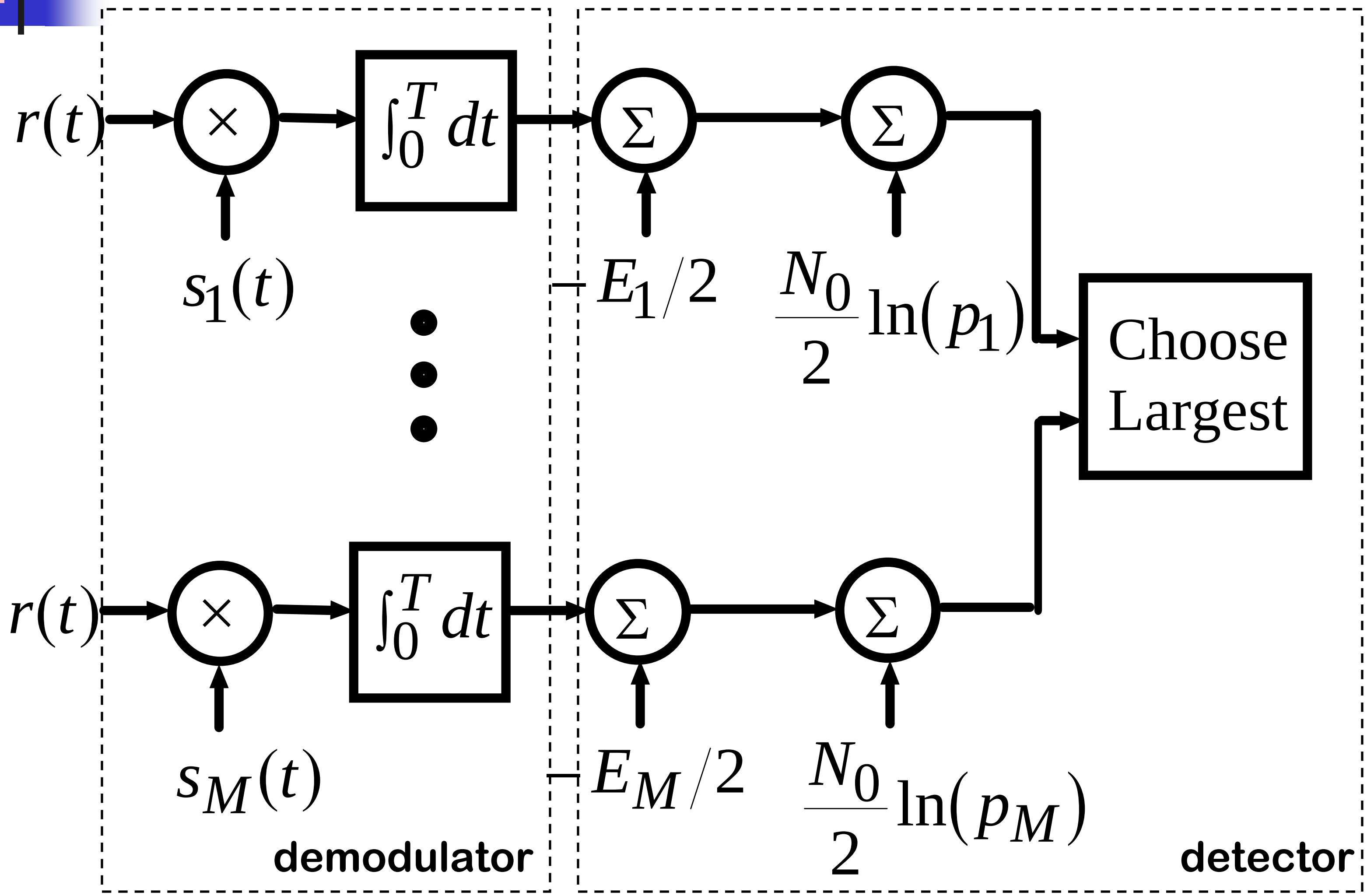
$$\hat{\mathbf{s}} = \arg\max_{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}} \left\{ p\left\{\mathbf{s}_m\right\} p\left\{\mathbf{r} \middle| \mathbf{s}_m\right\} \right\}$$

$$= \arg\max_{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}} \left\{ \frac{N_0}{2} \ln\left[p_m\right] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^{K} {s_{m,k}}^2 \right\}$$

Interpreting This Result

- $\frac{N_0}{2}\ln\left[p_m\right]$ weights the a priori probabilities
  - If the noise is large, p_m counts a lot
  - If the noise is small, our received signal will be an accurate estimate and p_m counts less
- $\sum_{k=1}^{K} r_k s_{m,k} = \int_0^T s_m(t) r(t) dt$ represents the correlation with the received signal
- $\frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2 = \frac{1}{2}\int_0^T s_m{}^2(t) dt = \frac{E_m}{2}$ represents signal energy

An Implementation of the Optimal Receiver -Correlation Receiver



Simplifications for Special Cases

- ML case: All signals are equally likely ($p_1 = \cdots = p_M$). A priori probabilities can be ignored.
- All signals have equal energy ($E_1 = \cdots = E_M$). Energy terms can be ignored.
- We can reduce the number of correlations by directly implementing:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2$$

Summary of Optimal Receiver Design

- Optimal coherent receiver for AWGN has three parts:
  - Correlates the received signal with each possible transmitted signal
  - Normalizes the correlation to account for energy
  - Weights the a priori probabilities according to noise power
- This receiver is completely general for any signal set
- Simplifications are possible under many circumstances

Example

- **Problem:** Consider the case of binary PAM signals in which the two possible signal points are

$$s_1 = \sqrt{E_b}\, p(t)$$
$$s_2 = -\sqrt{E_b}\, p(t)$$

where E_b is the energy per bit and $p(t)$ is the transmitted pulse shape normalized to unit energy. The *a priori* probabilities are $P(s_1) = p$ and $P(s_2) = 1-p$. Let's determine the decision rule for the optimum MAP detector when the transmitted signal is corrupted with AWGN.

Example (cont.)

Solution: The received signal is one-dimensional and with basis function *p(t)*

The received vector is then represented as

$$\mathbf{s}_1 = \sqrt{E_b}$$

$$\mathbf{s}_2 = -\sqrt{E_b}$$

$$\mathbf{r} = \pm\sqrt{E_b} + n$$

Example (cont.)

- n is a Gaussian random variable with variance $\sigma_n^2 = \frac{N_o}{2}$
- Thus the conditional pdf's of the two signals are

$$p(r|s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{\left(r - \sqrt{E_b}\right)^2}{2\sigma_n^2} \right]$$

$$p(r|s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{\left(r + \sqrt{E_b}\right)^2}{2\sigma_n^2} \right]$$

Example (cont.)

- If we define the metrics upon which we make a decision as *pm(r,s$_i$)*

$$\hat{\mathbf{s}} = \arg\max_{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}} \left\{ p\{\mathbf{s}_m\}\, p\{\mathbf{r}|\mathbf{s}_m\} \right\}$$

$$= \arg\max_{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}} \left\{ pm(\mathbf{r},\mathbf{s}_m) \right\}$$

$$pm(\mathbf{r},\mathbf{s}_m) = p\{\mathbf{s}_m\}\, p\{\mathbf{r}|\mathbf{s}_m\}$$

Note that these are scalars in this case since there is only one basis function.

Example (cont.)

- The metrics for the two symbols are then

$$pm(r,s_1)=\frac{p}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]$$

$$pm(r,s_2)=\frac{(1-p)}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r+\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]$$

- If $pm(r,s_1) > pm(r,s_2)$, then we select s_1 as the transmitted signal. Otherwise we select s_2.

Example (cont.)

- Since there are only two symbols, We may express the decision rule as

$$pm(r, s_1) \underset{s_1}{\overset{s_2}{\lessgtr}} pm(r, s_2)$$

$$\frac{pm(r, s_1)}{pm(r, s_2)} \underset{s_1}{\overset{s_2}{\lessgtr}} 1$$

Example (cont.)

- Substituting:

$$\frac{pm(r,s_1)}{pm(r,s_2)} \underset{s_1}{\overset{s_2}{\lessgtr}} 1$$

$$\frac{\frac{p}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]}{\frac{(1-p)}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r+\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]} \underset{s_1}{\overset{s_2}{\lessgtr}} 1$$

$$\frac{p}{1-p}\exp\left[\frac{\left(r+\sqrt{E_b}\right)^2-\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\right] \underset{s_1}{\overset{s_2}{\lessgtr}} 1$$

Example (cont.)

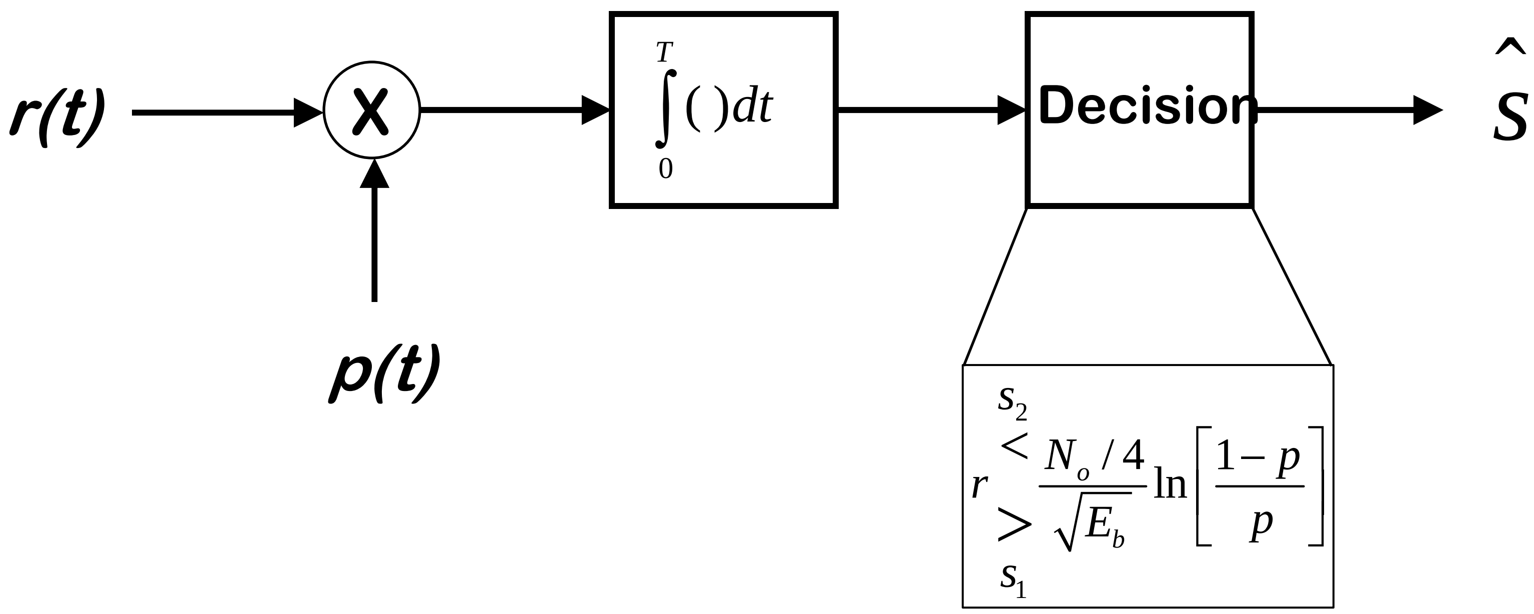
$$\frac{p}{1-p}\exp\left[\frac{\left(r+\sqrt{E_b}\right)^2-\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]\underset{s_1}{\overset{s_2}{\lessgtr}} 1$$

$$\frac{\left(r+\sqrt{E_b}\right)^2-\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\underset{s_1}{\overset{s_2}{\lessgtr}}\ln\left[\frac{1-p}{p}\right]$$

$$\frac{2r\sqrt{E_b}}{\sigma_n^2}\underset{s_1}{\overset{s_2}{\lessgtr}}\ln\left[\frac{1-p}{p}\right]$$

$$r\underset{s_1}{\overset{s_2}{\lessgtr}}\frac{N_o/4}{\sqrt{E_b}}\ln\left[\frac{1-p}{p}\right]\xrightarrow{\text{If } p=1/2} r\underset{s_1}{\overset{s_2}{\lessgtr}} 0$$

Example (cont.) – Optimal Receiver



Decision Regions

- Optimal Decision Rule:

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\dots,\mathbf{s}_M\}}{\arg\max} \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}^2$$

- Let $R_i \subset \Re^K$ be the region in which

$$\frac{N_0}{2}\ln[p_i] + \sum_{k=1}^{K} r_k s_{i,k} - \frac{1}{2}\sum_{k=1}^{K} s_{i,k}^2$$

$$\geq \frac{N_0}{2}\ln[p_j] + \sum_{k=1}^{K} r_k s_{j,k} - \frac{1}{2}\sum_{k=1}^{K} s_{j,k}^2, \forall i \neq j$$

- Then R_i is the *i* th "Decision Region"

Decision Regions

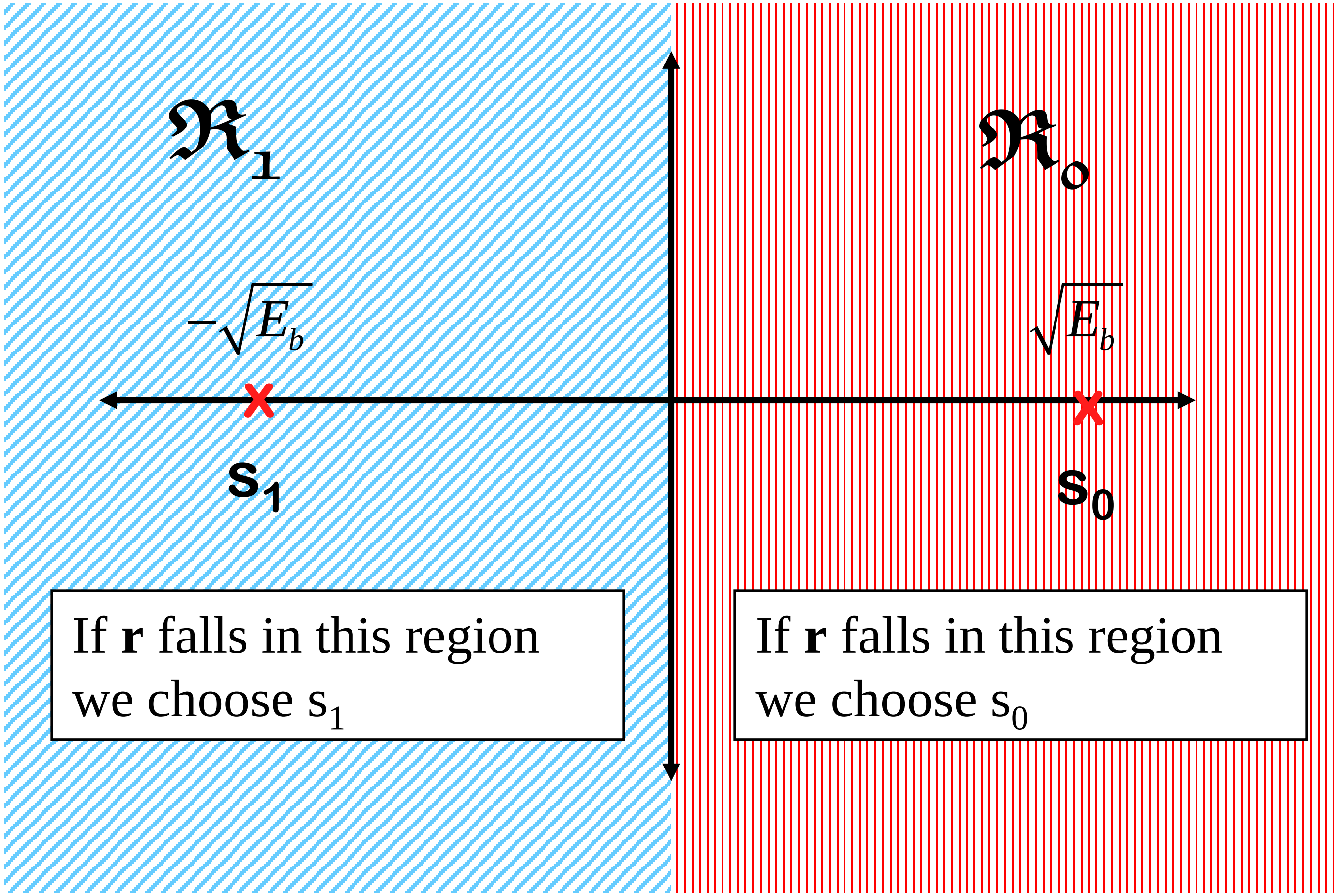
- Maximum Likelihood Case ($p_1 = p_2 = .. p_M$)

$$\hat{\mathbf{s}} = \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} \ln[p_m] - \frac{K}{2}\ln[\pi N_0] - \frac{1}{N_0}\sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

$$= \underset{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}}{\arg\max} -\frac{1}{N_0}\sum_{k=1}^{K}\left(r_k - s_{m,k}\right)^2$$

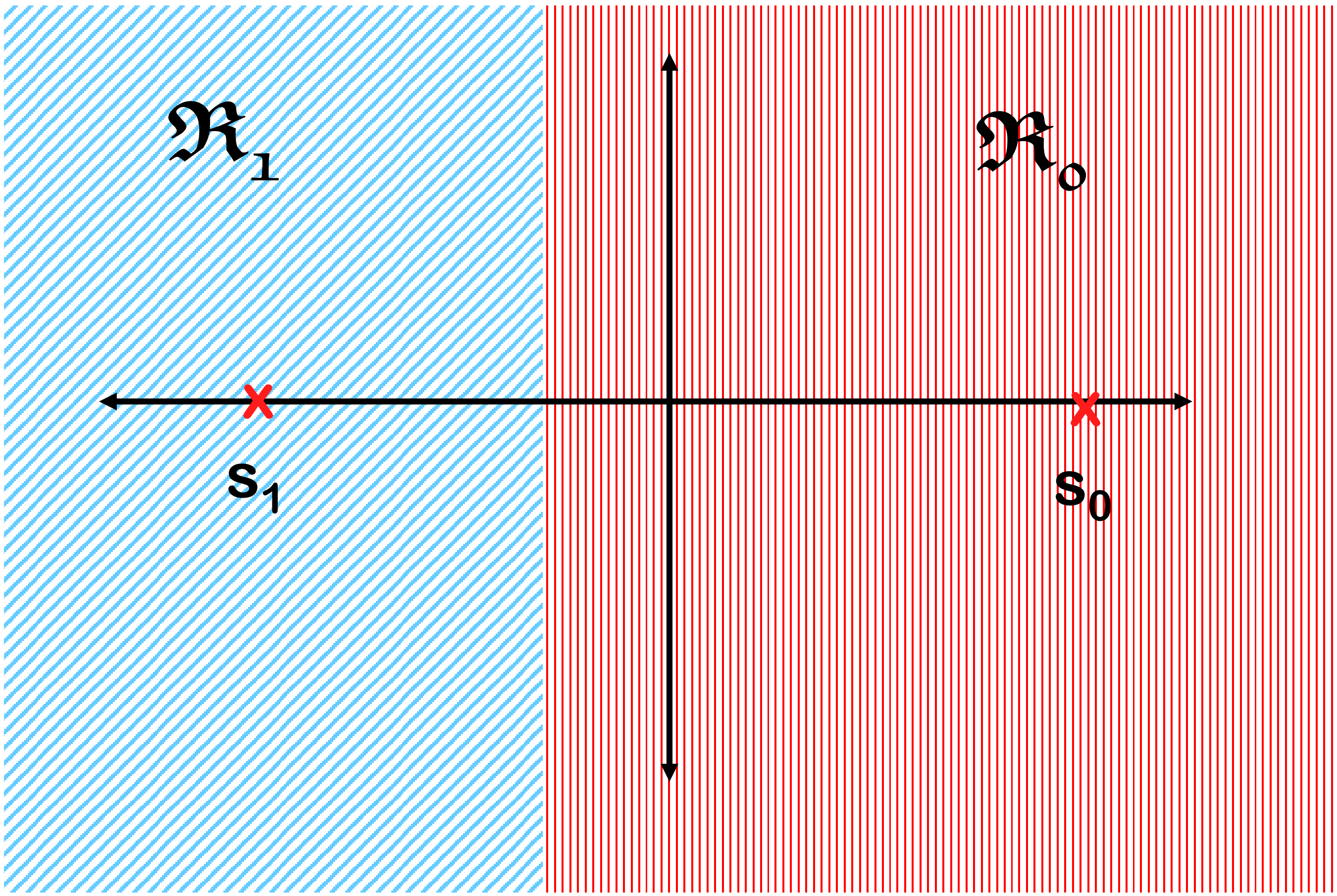
- Thus, we for maximum likelihood we choose the symbol that is closest to the received signal in *K*-dimensional space

Example: BPSK



$p_1=p_0$

Example: BPSK



Decision region for s_0 is larger since it is more likely

$p_1 < p_0$

A Matlab Function for Visualizing Decision Regions

- The Matlab Script File “sigspace.m” (on course web page) can be used to visualize two dimensional signal spaces and decision regions
- The function is called with the following syntax:

  sigspace($[x_1\ y_1\ p_1; x_2\ y_2\ p_2; \cdots; x_M\ y_M\ p_M]$, E_b/N_0)
  - x_i and y_i are the coordinates of the *i*th signal point
  - p_i is the probability of the *i*th signal (omitting gives ML)
  - E_b/N_0 is the signal to noise ratio of digital system in dB

Average Energy Per Bit: E_b

- $E_i = \sum_{k=1}^{K} s_{i,k}^2$ is the energy of the *i* th signal
- $E_s = \frac{1}{M} \sum_{i=1}^{M} E_i$ is the average energy per symbol
- $\log_2 M$ is the number of bits transmitted per symbol
- $E_b = \frac{E_s}{\log_2 M}$ is the average energy per bit
  - allows fair comparisons of the energy requirements of different sized signal constellations

Signal to Noise Ratio for Digital Systems

- $N_0/2$ is the (two-sided) power spectral density of the background noise
- The ratio E_b/N_0 measures the relative strength of signal and noise at the receiver
- E_b has units of Joules = Watts *sec
- N_0 has units of Watts/Hz = Watts*sec
- The unitless quantity E_b/N_0 is frequently expressed in dB

Example: QPSK

- symbols

$$s_i(t) = \cos\left(2\pi ft + (i-1)\frac{\pi}{4}\right) \quad 0 \le t \le T_s$$

- basis functions

$$f_1(t) = \cos\left(2\pi ft\right) \quad 0 \le t \le T_s$$

$$f_2(t) = \sin\left(2\pi ft\right) \quad 0 \le t \le T_s$$

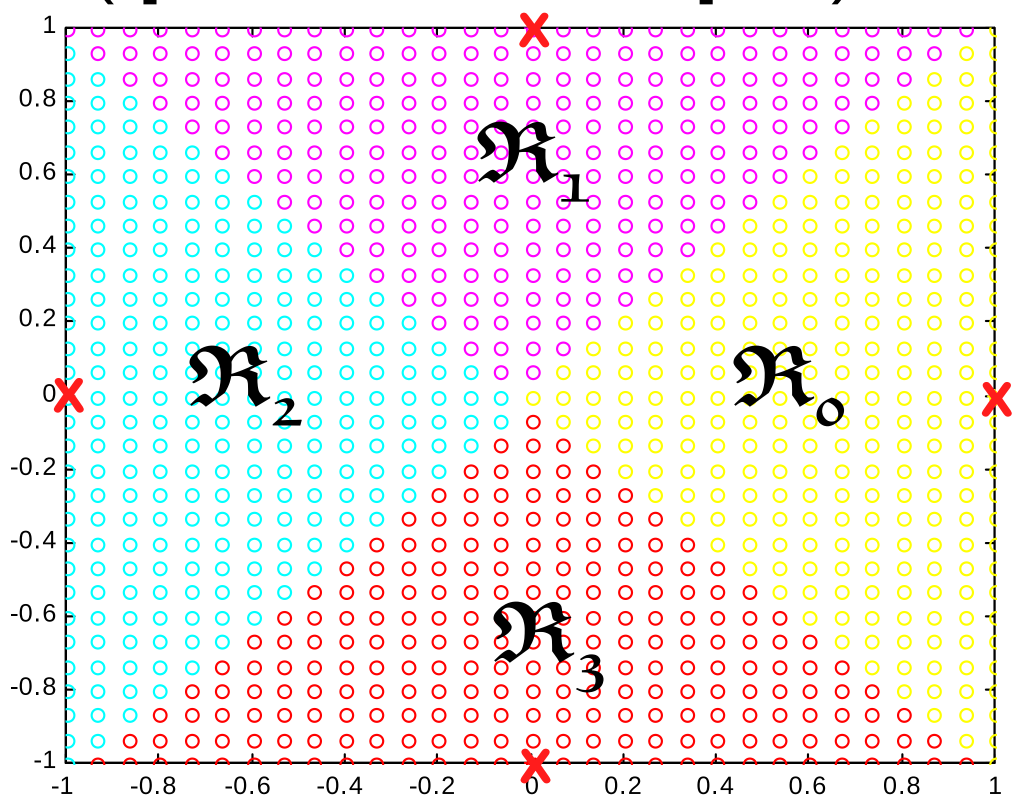
- symbol vectors

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_s} \end{bmatrix}$$

$$\mathbf{s}_3 = \begin{bmatrix} -\sqrt{E_s} \\ 0 \end{bmatrix} \quad \mathbf{s}_4 = \begin{bmatrix} 0 \\ -\sqrt{E_s} \end{bmatrix}$$

Examples of Decision Regions - QPSK

- sigspace([1 0; 0 1; -1 0; 0 -1], 20)



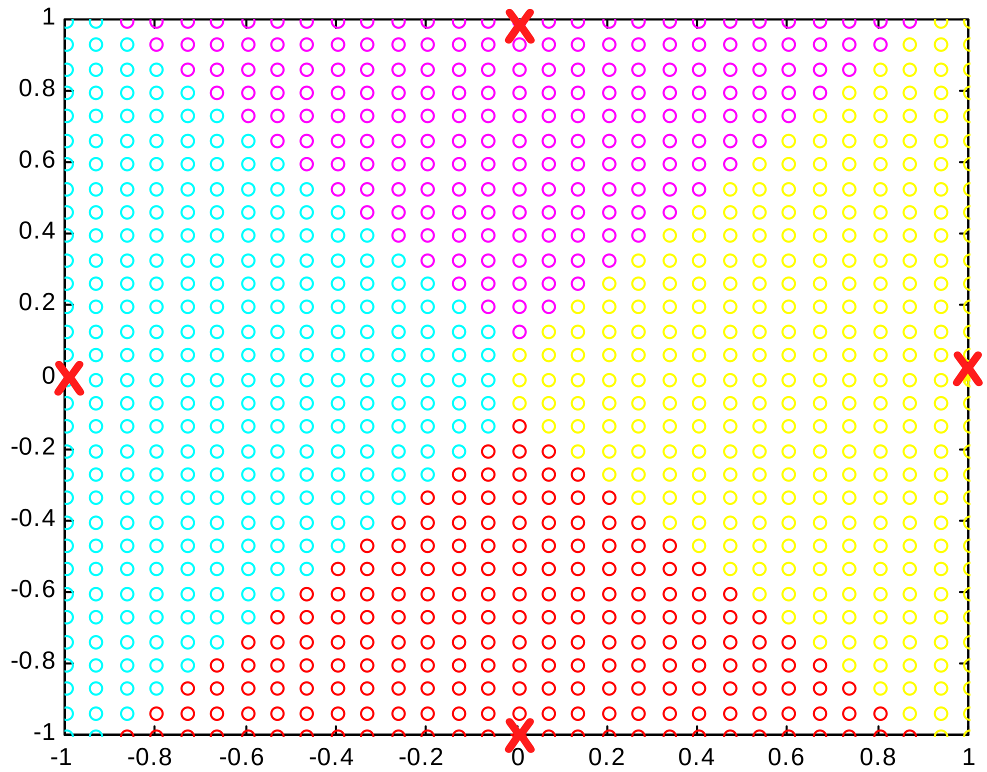
E_b/N_o = 20dB
But is irrelevant in this case

$$\hat{\mathbf{s}} = \arg\max_{\{\mathbf{s}_1,\dots,\mathbf{s}_M\}} \ln[p_m] - \frac{1}{N_0}\sum_{k=1}^{K}(r_k - s_{m,k})^2$$

$p_1 = p_2 = p_3 = p_4$

QPSK with Unequal Signal Probabilities

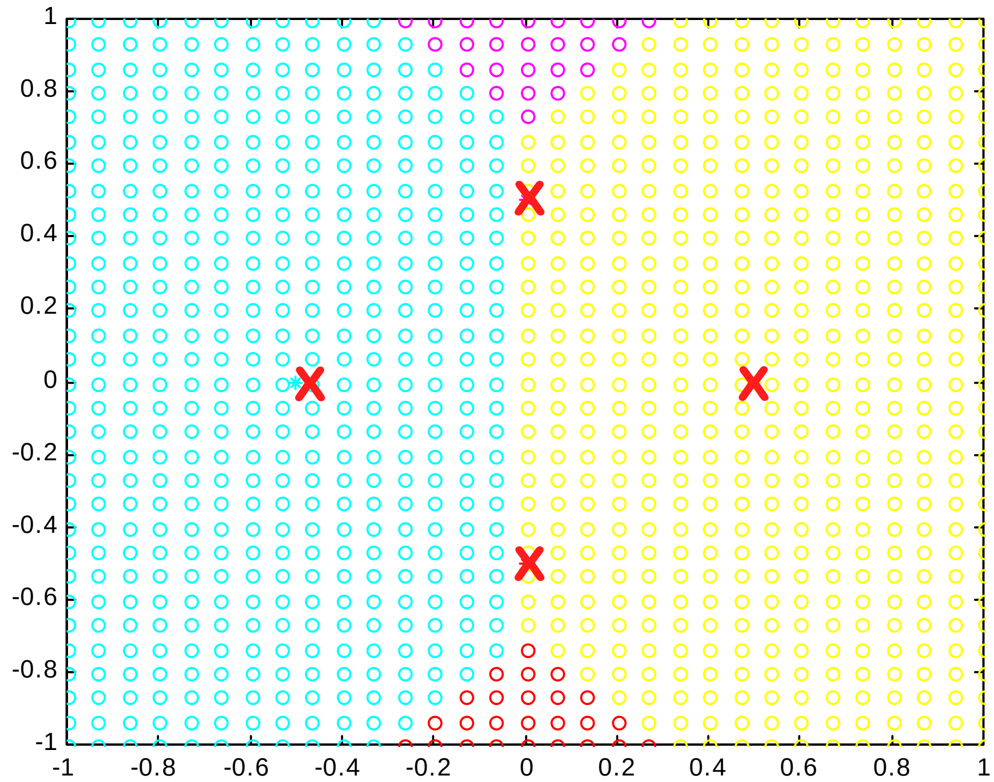
- sigspace([1 0 0.4; 0 1 0.1; -1 0 0.4; 0 -1 0.1], 5)



$p_1 = 0.4$
$p_2 = 0.1$
$p_3 = 0.4$
$P_4 = 0.1$
$E_b/N_o = 5dB$

QPSK with Unequal Signal Probabilities - Extreme Case

- sigspace([0.5 0 0.4; 0 0.5 0.1; -0.5 0 0.4; 0 -0.5 0.1],-6)

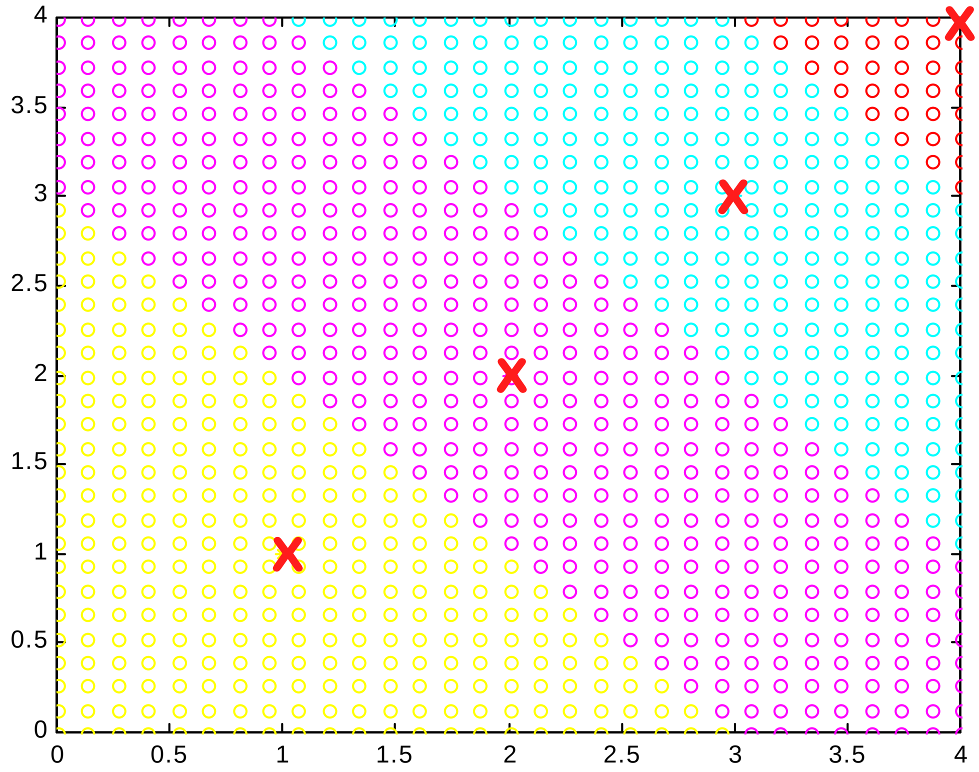


$p_1 = 0.4$
$p_2 = 0.1$
$p_3 = 0.4$
$P_4 = 0.1$
$E_b/N_o = -6dB$

Low SNR gives heavy weighting to *a priori probabilities*

Unequal Signal Powers

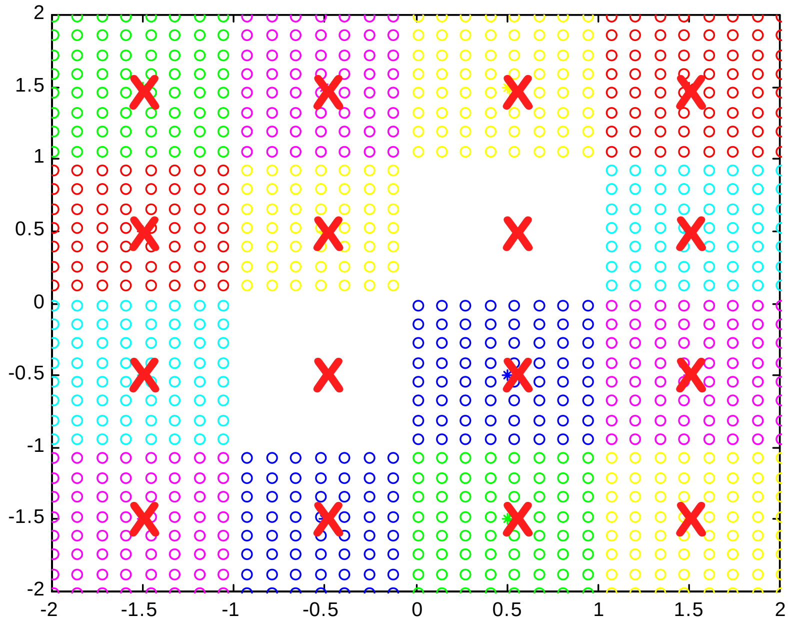
- sigspace([1 1 ; 2 2; 3 3; 4 4], 10)



ASK scheme
Could use single dimension

Signal Constellation for 16-ary QAM

- sigspace([1.5 -1.5; 1.5 -0.5; 1.5 0.5; 1.5 1.5; 0.5 -1.5; 0.5 -0.5; 0.5 0.5; 0.5 1.5;-1.5 -1.5; -1.5 -0.5; -1.5 0.5; -1.5 1.5; -0.5 -1.5; -0.5 -0.5; -0.5 0.5; -0.5 1.5],10)



Notes on Decision Regions

- Boundaries are perpendicular to a line drawn between two signal points
- If signal probabilities are equal, decision boundaries lie exactly halfway in between signal points
- If signal probabilities are unequal, the region of the less probable signal will shrink.
- Signal points need not lie within their decision regions for case of low E_b/N_0 and unequal probabilities.

Receiver Operations

- What we have been discussing so far is the *optimum detector*. That is given the observed vector

$$\mathbf{r} = \mathbf{s} + \mathbf{n}$$

  we have determined the optimal detection or decision rule.
- However, we haven't shown whether the creation of the vector $\mathbf{r}$ is from the received signal is optimal. That is, we haven't shown that the *demodulator* is optimal.

Optimal Demodulation

- We have described the *correlation demodulator* and the equivalent *matched filter demodulator.*
- Consider a general correlator demodulator where the received signal is correlated with a function *h(t)*

$$
\begin{aligned}
r_k &= \int_0^T r(t)h(t)dt \\
&= \int_0^T \big(s(t)+n(t)\big)h(t)dt \\
&= \int_0^T s(t)h(t)dt + \int_0^T n(t)h(t)dt
\end{aligned}
$$

Optimal Demodulation (cont.)

- Consider a general correlator demodulator where the received signal is correlated with a function *h(t)*

$$r_k = \int_0^T s(t)h(t)dt + \int_0^T n(t)h(t)dt$$

Desired Component

Noise Component

$$SNR = \frac{\left[\int_0^T s(t)h(t)dt\right]^2}{E\left[\left(\int_0^T n(t)h(t)dt\right)^2\right]}$$

Optimal Demodulator (cont.)

- The noise power can be determined as

$$E\left\{\left[\int_0^T n(t)h(t)dt\right]^2\right\} = \int_0^T\int_0^T E\left\{n(t)n(\tau)\right\}h(t)h(\tau)dtd\tau$$

$$= \frac{N_o}{2}\int_0^T\int_0^T \delta(t-\tau)h(t)h(\tau)dtd\tau$$

$$= \frac{N_o}{2}\int_0^T h^2(t)dt$$

Optimal Demodulator - SNR

- From the Cauchy-Schwarz Inequality we know

$$SNR = \frac{\left[\int_0^T s(t)h(t)dt\right]^2}{\frac{N_o}{2}\int_0^T h^2(t)dt} \leq \frac{\int_0^T s^2(t)dt \int_0^T h^2(t)dt}{\frac{N_o}{2}\int_0^T h^2(t)dt}$$

where the equality is obtained when *h(t) = s(t).* Thus, the demodulator that maximizes SNR at the output is the demodulator correlator where *h(t) = s(t)* or equivalently a matched filter with an impulse response equal to *s(T-t).*

Optimal Demodulator

- Thus, the demodulator which correlates with the symbol waveform or which uses a matched filter matched to the symbol waveform *maximizes SNR.*
- Thus, the demodulator is optimal in that it provides the max SNR. The detector is optimal in that given r it minimizes probability of error.
- Further the SNR obtained is

$$SNR = \frac{\int_0^T s^2(t)dt \int_0^T s^2(t)dt}{\frac{N_o}{2}\int_0^T s^2(t)dt}$$

$$= \frac{\int_0^T s^2(t)dt}{\frac{N_o}{2}}$$

$$= \frac{2E}{N_o}$$