ECE 5654 - Digital Communications Spring 2005

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Lecture #8 - Probability of Error Calculations for Optimal Receivers

Problem Statement

- We transmit a signal $s(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\}$, where $s(t)$ is nonzero only on $t \in [0,T]$.
- Let the various signals be transmitted with probability: $p_1 = \Pr[s_1(t)], \ldots, p_M = \Pr[s_M(t)]$
- The received signal is corrupted by noise:

$$r(t) = s(t) + n(t)$$

- Given $r(t)$, the receiver forms an estimate $\hat{s}(t)$ of the signal $s(t)$ with the goal of minimizing symbol error probability $P_s = \Pr[\hat{s}(t) \neq s(t)]$

Final Form of MAP Receiver

- Multiplying through by the constant $N_0/2$:

$$\hat{\mathbf{s}} = \arg\max_{\{\mathbf{s}_1,\ldots,\mathbf{s}_M\}} \frac{N_0}{2}\ln[p_m] + \sum_{k=1}^{K} r_k s_{m,k} - \frac{1}{2}\sum_{k=1}^{K} s_{m,k}{}^2$$

Symbol Error Probability

- $P_S(e) = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}]$ is the average probability of symbol error

$$P_S(e) = \sum_{i=1}^{M} \Pr[\mathbf{s}_i] \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i]$$

  - where $\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i]$ is the conditional probability of the receiver not deciding on $\mathbf{s}_i$ given $\mathbf{s}_i$ was transmitted

$$\Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i] = 1 - \int_{R_i} p(\mathbf{r} | \mathbf{s}_i) d\mathbf{r}$$

Symbol Error Probability

$$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right] = 1 - \int_{R_i} p\left(\mathbf{r} \middle| \mathbf{s}_i\right) d\mathbf{r}$$

- This is a multidimensional integral over the decision region $\mathfrak{R}_i$ where

$$p\left(\mathbf{r} \middle| \mathbf{s}_i\right) = \left(\pi N_0\right)^{-K/2} \exp\left(-\sum_{k=1}^{K} \left(r_k - s_{i,k}\right)^2 \Big/ N_0 \right)$$

Symbol Error Probability Calculation for BPSK

- Two antipodal signals $(M = 2)$:

$$s_1(t) = \sqrt{2P}\cos(2\pi f_c t)\Big|_0^T \qquad s_2(t) = -\sqrt{2P}\cos(2\pi f_c t)\Big|_0^T$$

- One basis function:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T$$

- Signal Space Representation:

$$s_1(t) = \sqrt{P \cdot T}\, f_1(t) = \sqrt{E_b}\, f_1(t), s_2(t) = -\sqrt{E_b}\, f_1(t)$$

$$\mathbf{s}_1 = \sqrt{E_b}, \mathbf{s}_2 = -\sqrt{E_b}$$

$$\Pr[\mathbf{s}_1] = \Pr[\mathbf{s}_2] = \frac{1}{2}$$

Note: One dimensional signal space

Boundaries of Decision Regions for BPSK

- Decision rule says choose $\mathbf{s}_1$ if:

$$p(\mathbf{r}|\mathbf{s}_2)\Pr(\mathbf{s}_2) \le p(\mathbf{r}|\mathbf{s}_1)\Pr(\mathbf{s}_1)$$

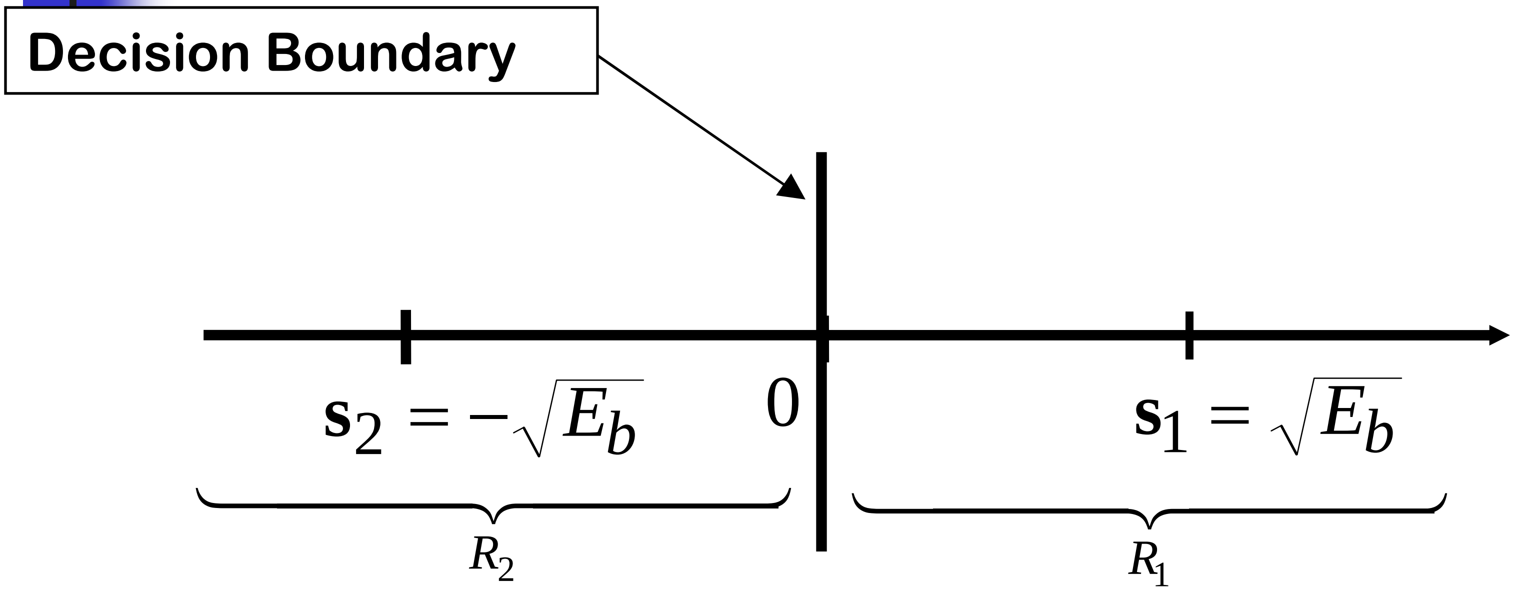
$$\Leftrightarrow \frac{1}{2}\cdot\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{\left(r-\sqrt{E_b}\right)^2}{N_0}\right) \ge \frac{1}{2}\cdot\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{\left(r+\sqrt{E_b}\right)^2}{N_0}\right)$$

$$\Leftrightarrow \exp\left(-\frac{\left(r-\sqrt{E_b}\right)^2}{N_0}\right) \ge \exp\left(-\frac{\left(r+\sqrt{E_b}\right)^2}{N_0}\right)$$

$$\Leftrightarrow \left(r-\sqrt{E_b}\right)^2 \le \left(r+\sqrt{E_b}\right)^2$$

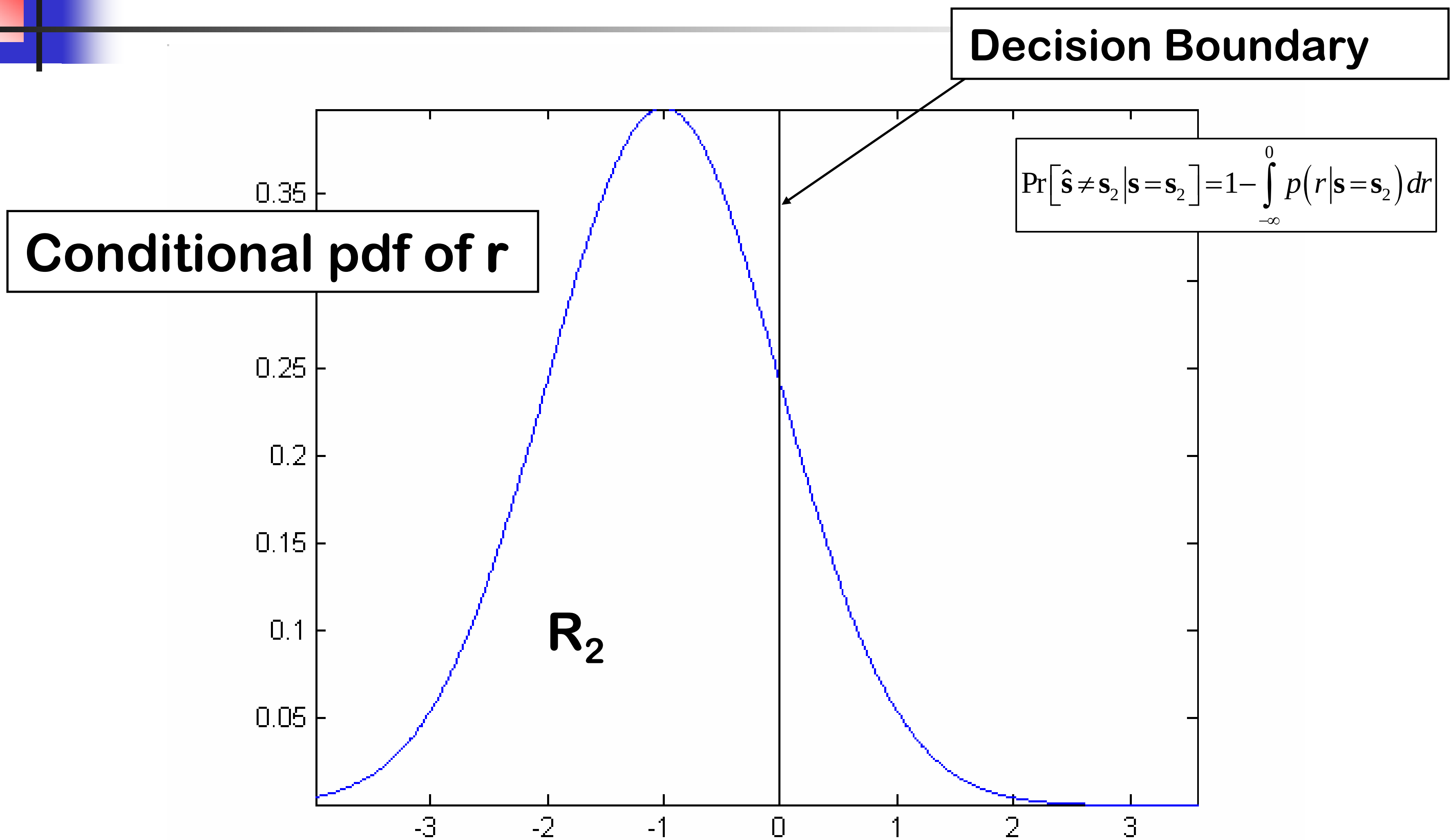
$$\Leftrightarrow r \ge 0$$

Decision Region for BPSK



Note: This is a *one-dimensional* decision region

Decision Region for BPSK



Calculation of Error Probability for BPSK

$$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_2\right] = 1 - \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(r + \sqrt{E_b}\right)^2}{N_0} \right) dr$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(r + \sqrt{E_b}\right)^2}{N_0} \right) dr$$

$$= \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-y^2 / N_0 \right) dy, \qquad y = r + \sqrt{E_b}$$

$$= \int_{\sqrt{2E_b / N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-x^2 / 2 \right) dx, \qquad x = y \Big/ \sqrt{\frac{N_0}{2}}$$

Error Probability for BPSK

- $$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_2\right] = Q\left(\sqrt{2E_b / N_o}\right),$$
  where $\quad Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) dx$
- By symmetry:
  $$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_1 \middle| \mathbf{s} = \mathbf{s}_1\right] = \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_2\right] = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$
- Although we have derived the result with BPSK in mind, the result holds for any signal set with this constellation.

Probability of Error - BPSK

- Recall that

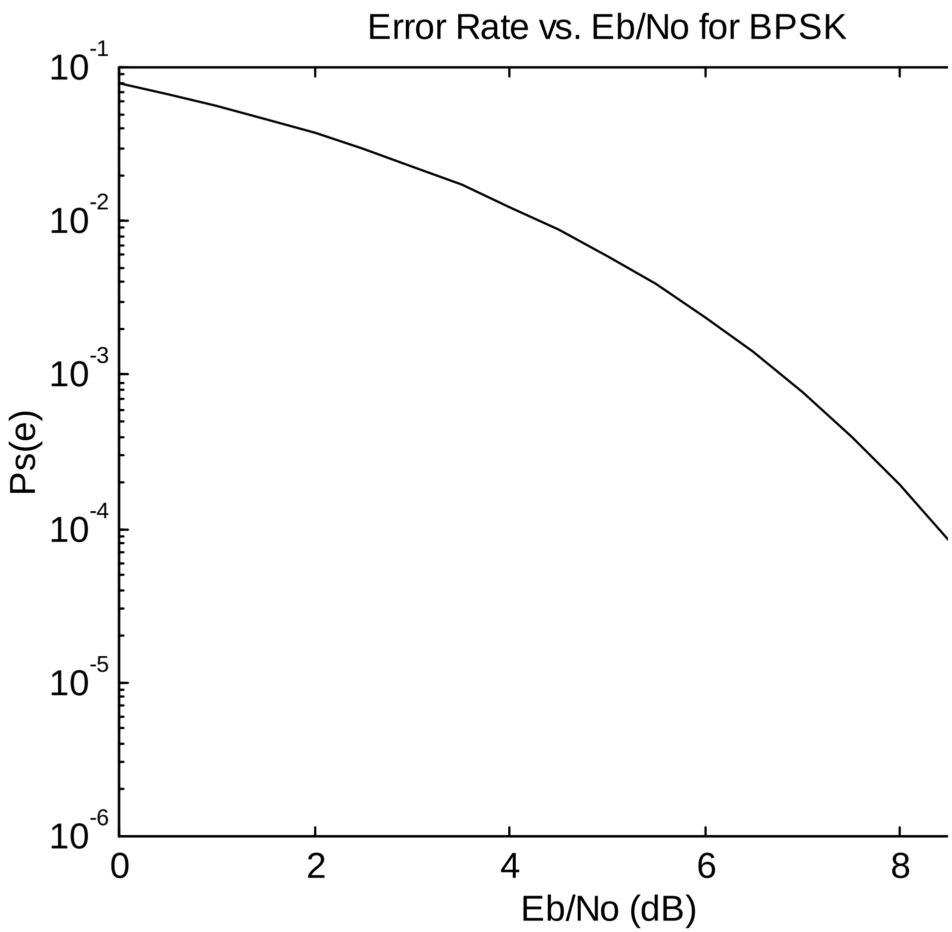
$$P_s(e) = \sum_{i=1}^{M} \Pr[\mathbf{s}_i] \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right]$$

- Thus,

$$\begin{aligned} P_s(e) &= \Pr[\mathbf{s}_1] \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_1 \middle| \mathbf{s} = \mathbf{s}_1\right] + \Pr[\mathbf{s}_2] \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_2\right] \\ &= \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_o}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \\ &= Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \end{aligned}$$

Error Probability Curve for BPSK

$$P_s(e) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$



Symbol Error Probability Calculation for Binary Coherent FSK

- Two orthogonal signals $(M = 2)$:

$$s_1(t) = \sqrt{2P}\cos(2\pi f_1 t)\Big|_0^T \qquad s_2(t) = \sqrt{2P}\cos(2\pi f_2 t)\Big|_0^T$$

- Two basis functions:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_1 t)\Big|_0^T \qquad f_2(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_2 t)\Big|_0^T$$

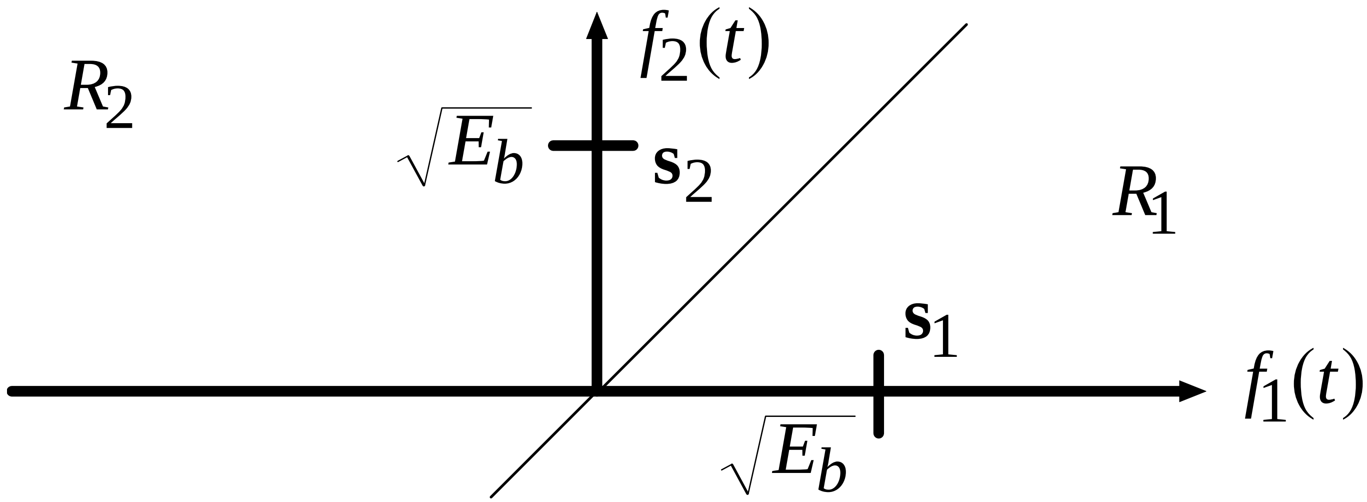
- Signal space representation:

$$s_1(t) = \sqrt{E_b}\, f_1(t), \qquad s_2(t) = \sqrt{E_b}\, f_2(t)$$

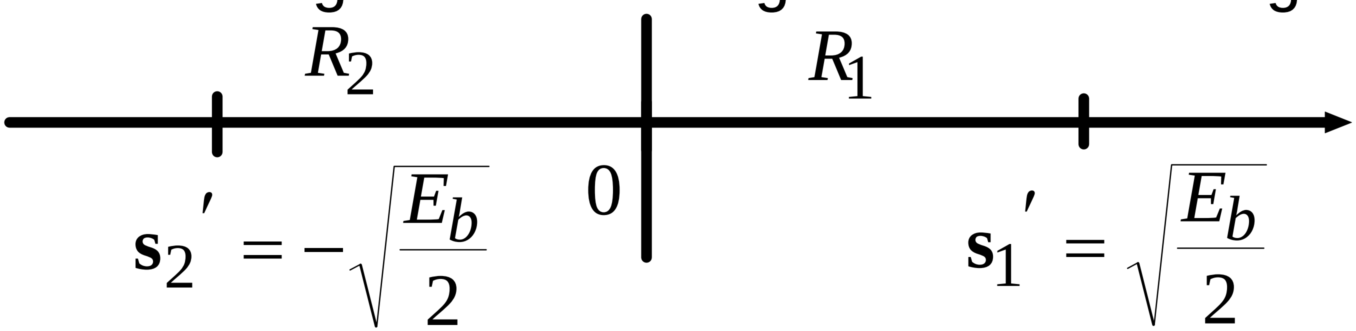
$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

Two-dimensional signal set

Decision Regions for Binary Coherent FSK



- Rotating and translating the coordinates gives:



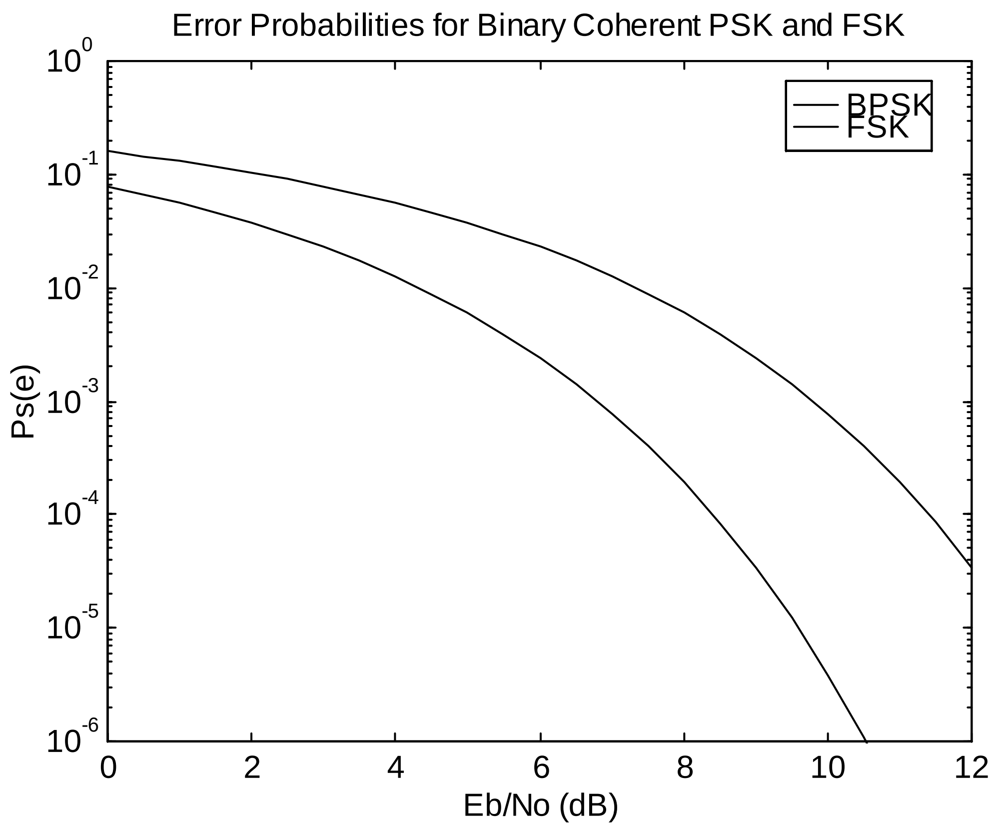
Error Calculation for Binary Coherent FSK

- Any translation, rotation, or reflection operation on the coordinates which does not change the distance between signals will not effect the error probability.
- Note that we must keep the average energy per symbol the same
  - If we define the points in terms of $E_{s,}$ this will happen automatically.
- By repeating the calculation for BPSK (with $\sqrt{E_b/2}$ substituted for $\sqrt{E_b}$), we find that:

$$P_s(e) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

BER Curves for BPSK and FSK

- FSK is approximately 3dB worse than BPSK



Symbol Error Probability Calculation for Binary Coherent ASK

- Two signals $(M = 2)$:

$$s_1(t) = \sqrt{2P}\cos(2\pi f_c t)\Big|_0^T$$

$$s_2(t) = 0$$

- One basis function:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T$$

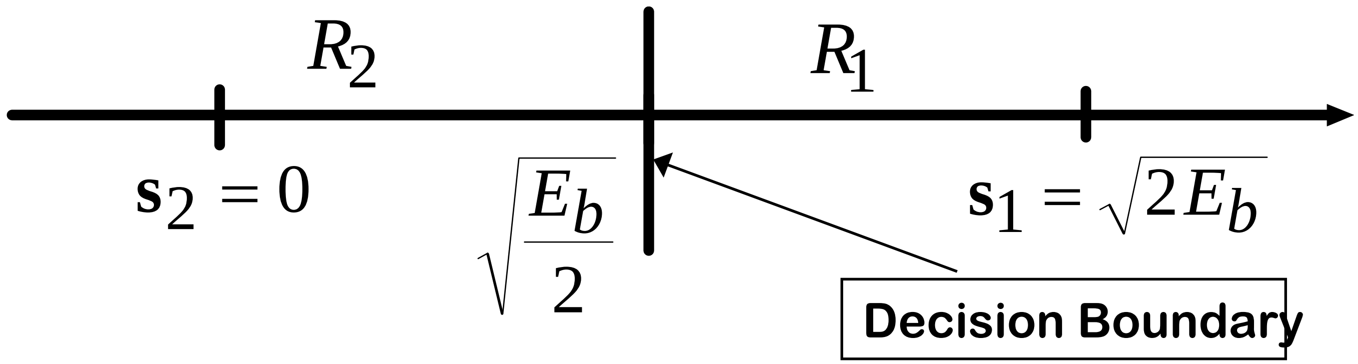
- Signal space representation:

$$s_1(t) = \sqrt{2E_b}\, f_1(t), \qquad s_2(t) = 0$$

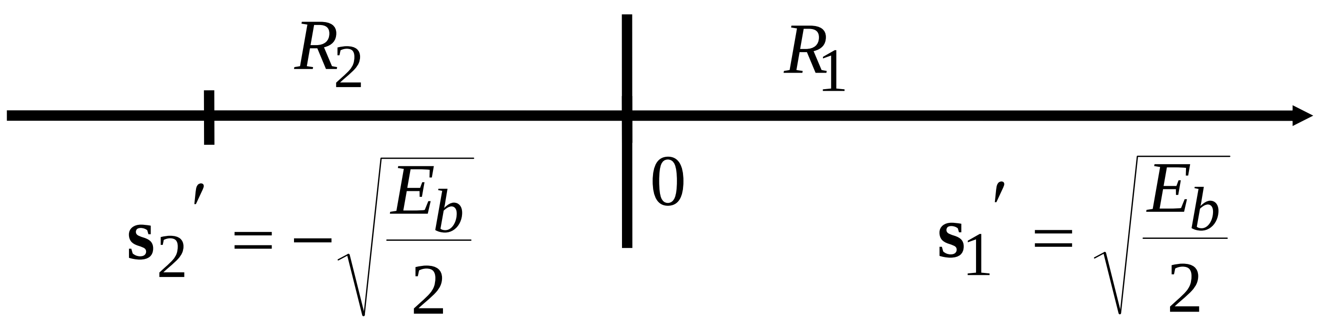
$$\mathbf{s}_1 = \sqrt{2E_b}, \mathbf{s}_2 = 0$$

Note that in this case $E_b = PT/2$

Decision Regions for Binary Coherent ASK



- Translating the coordinates gives:



Error Calculation for Binary Coherent ASK

- Translated signal constellation is identical to that for FSK
- Error calculation is identical to that of FSK, so:

$$P_S(e) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

General Binary Modulation

- For any arbitrary binary modulation scheme

$$\mathbf{s}_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

- Arbitrarily choosing

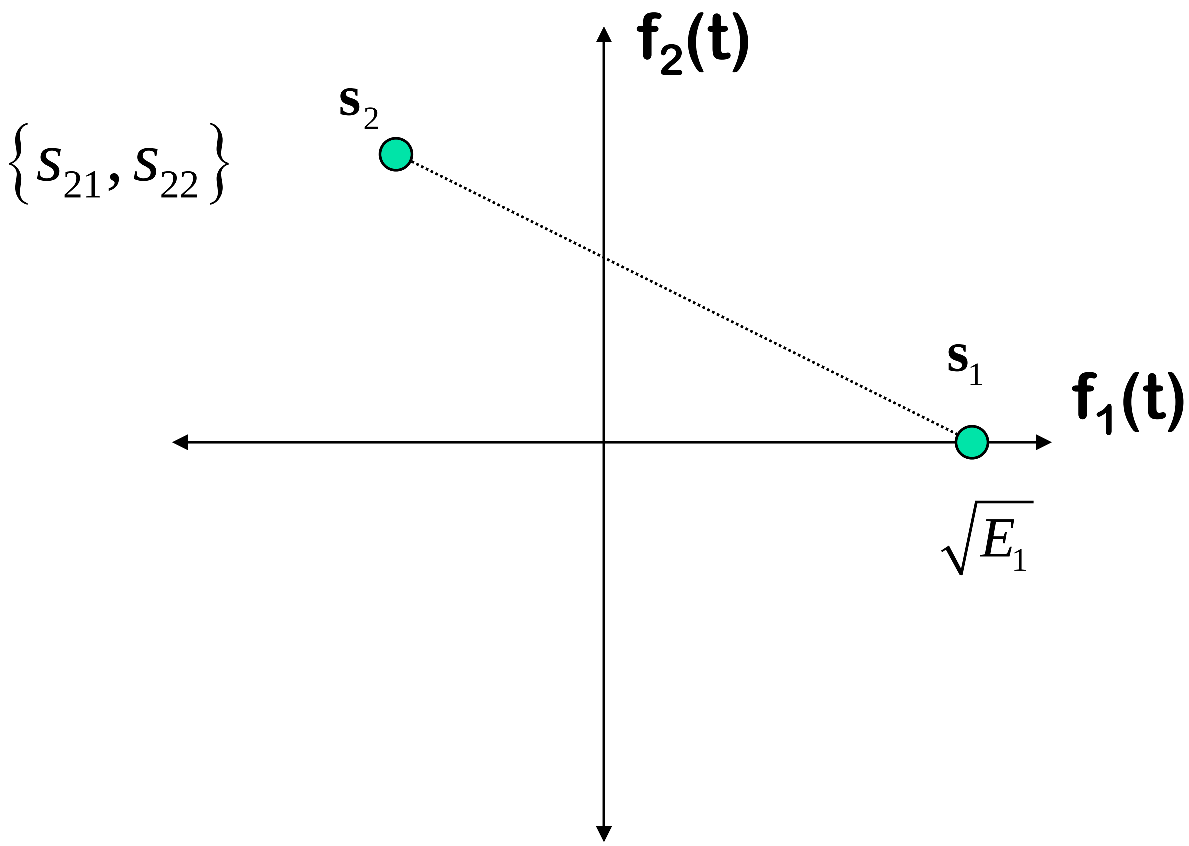
$$f_1(t) = \frac{s_1(t)}{\sqrt{E_b}} \quad 0 \le t \le T$$

- We have

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_1} \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

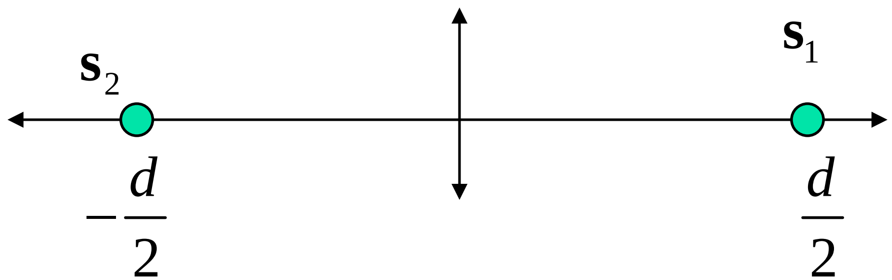
Constellation Diagram

- The resulting signal space diagram



Constellation Diagram

- Rotating to a single axis and assuming equal energy symbols:



$$d = \sqrt{\left(\sqrt{E_1} - s_{21}\right)^2 + \left(0 - s_{22}\right)^2}$$

$$= \sqrt{E_1 - 2\sqrt{E_1} s_{21} + s_{21}{}^2 + s_{22}{}^2}$$

$$= \sqrt{E_1 - 2\sqrt{E_1} s_{21} + E_2}$$

Probability of Error

- The resulting probability of error is then

$$P_s(e) = Q\left(\sqrt{\frac{2(d/2)^2}{N_o}}\right)$$

$$= Q\left(\sqrt{\frac{\left(E_1 - 2\sqrt{E_1}s_{21} + E_2\right)}{2N_o}}\right)$$

- If the symbols have equal energy

$$P_s(e) = Q\left(\sqrt{\frac{\left(E_b - \sqrt{E_b}s_{21}\right)}{N_o}}\right)$$

$$= Q\left(\sqrt{\frac{E_b(1-\rho_{21})}{N_o}}\right)$$

$$\rho_{12} = \frac{1}{\sqrt{E_1E_2}}\int_0^T s_1(t)s_2(t)dt$$

Examples

- BPSK → $\rho_{12} = -1$

$$P_s(e) = Q\left(\sqrt{\frac{E_b\left(1-(-1)\right)}{N_o}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

- BFSK, BASK → $\rho_{12} = 0$

$$P_s(e) = Q\left(\sqrt{\frac{E_b\left(1-0\right)}{N_o}}\right)$$

$$= Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

2-D Symbol Error Probability Example: QPSK

- Consider a QPSK signal set:

$$s_1(t) = \sqrt{2P}\cos(2\pi f_c t)\Big|_0^T, s_2(t) = \sqrt{2P}\sin(2\pi f_c t)\Big|_0^T,$$

$$s_3(t) = -\sqrt{2P}\cos(2\pi f_c t)\Big|_0^T, s_4(t) = -\sqrt{2P}\sin(2\pi f_c t)\Big|_0^T$$

- These can be represented with the basis functions:

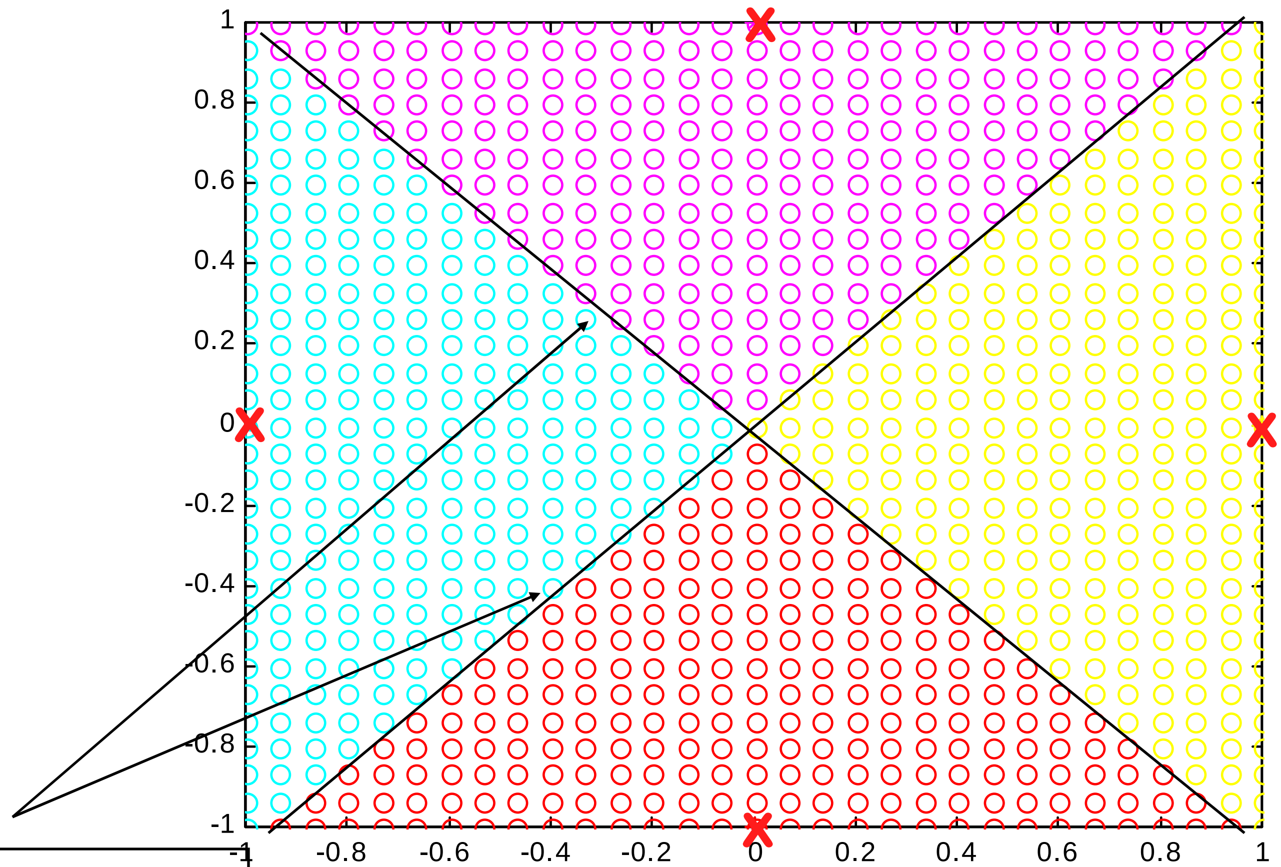
$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T, f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Big|_0^T$$

- This representation results in the signal vectors:

$$\mathbf{s}_1 = \left[\sqrt{P\cdot T} = \sqrt{E_s} \quad 0\right], \mathbf{s}_2 = \left[0 \quad \sqrt{E_s}\right],$$

$$\mathbf{s}_3 = \left[-\sqrt{E_s} \quad 0\right], \mathbf{s}_4 = \left[0 \quad -\sqrt{E_s}\right]$$

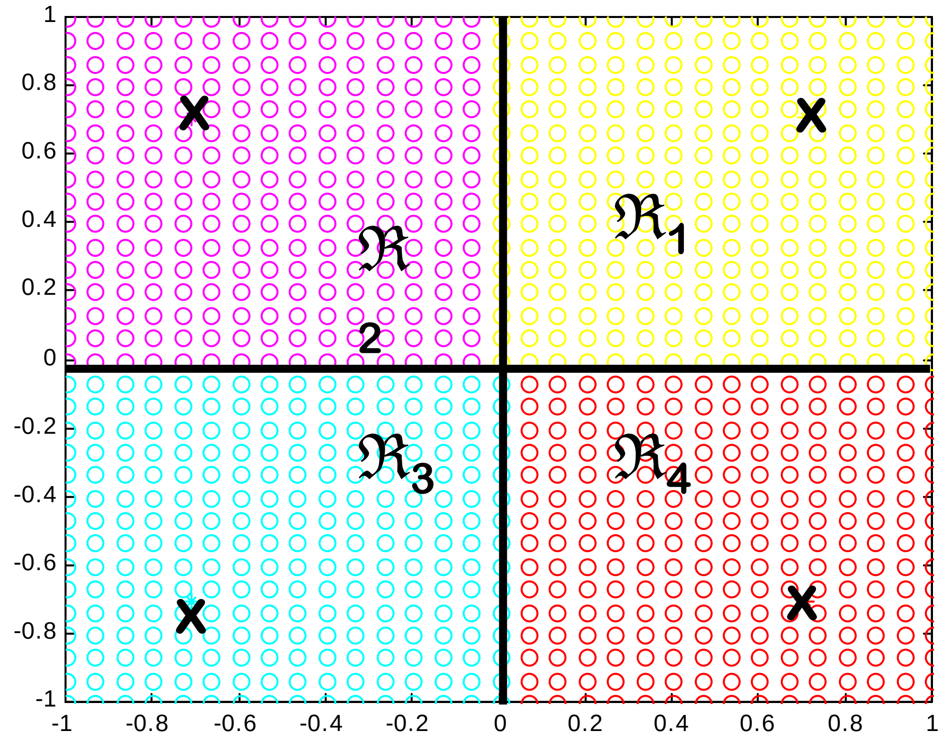
QPSK Signal Constellation and Decision Regions



Decision boundaries

QPSK Signal Constellation after 45 Degree Rotation

$$\mathbf{s}_1 = \left[\sqrt{E_s/2} \quad \sqrt{E_s/2}\right]$$
$$\mathbf{s}_2 = \left[-\sqrt{E_s/2} \quad \sqrt{E_s/2}\right]$$
$$\mathbf{s}_3 = \left[-\sqrt{E_s/2} \quad -\sqrt{E_s/2}\right]$$
$$\mathbf{s}_4 = \left[\sqrt{E_s/2} \quad -\sqrt{E_s/2}\right]$$



Symbol Error Probability Calculation for QPSK

$$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_1 \middle| \mathbf{s} = \mathbf{s}_1\right] = 1 - \int_{R_1} p\left(\mathbf{r} \middle| \mathbf{s}_1\right) d\mathbf{r}$$

$$= 1 - \int_0^\infty \int_0^\infty \frac{1}{\pi N_0} e^{-\left(x - \sqrt{\frac{E_s}{2}}\right)^2 / N_0} \cdot e^{-\left(y - \sqrt{\frac{E_s}{2}}\right)^2 / N_0} dxdy$$

$$= 1 - \int_{-\sqrt{\frac{E_s}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_{-\sqrt{\frac{E_s}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Symbol Error Probability Calculation for QPSK (cont.)

$$= 1 - \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right] \cdot \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]$$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2 \quad \text{(Because } E_s = 2E_b\text{)}$$

$$\approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{(Because } Q^2(\cdot) << Q(\cdot)\text{)}$$

Symbol Error Probability of QPSK

- The conditional error probability given any of the four signals is identical:

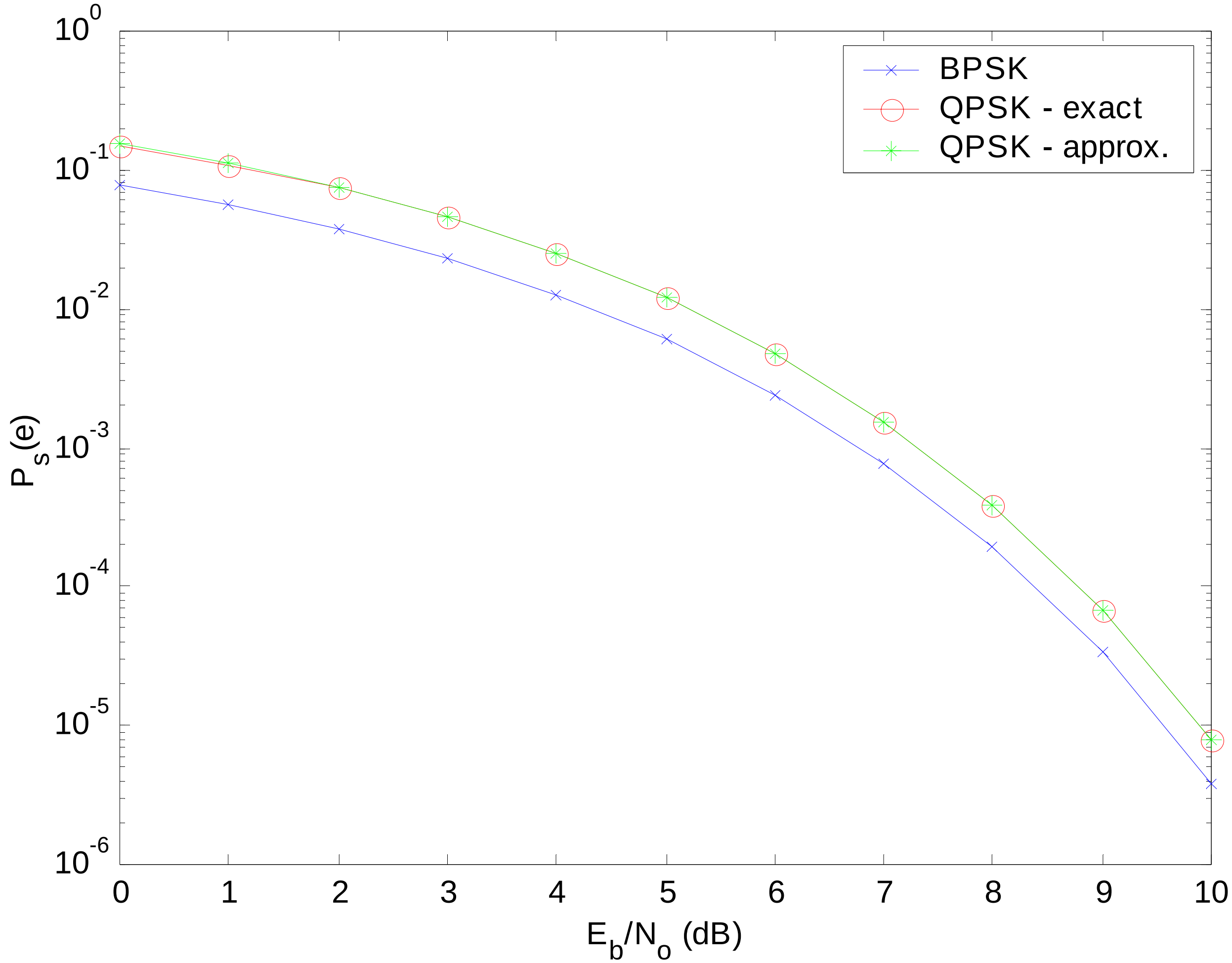
$$P_s(e) = \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_1 \middle| \mathbf{s} = \mathbf{s}_1\right] = \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_2 \middle| \mathbf{s} = \mathbf{s}_2\right] = \cdots$$

- The symbol error probability of QPSK is approximately twice that of BPSK:

$$P_s(e) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- When we discuss Bit Error Rate, we will find that the BER of QPSK and BPSK are identical

Symbol Error Probabilities for BPSK and QPSK



Notes on Error Probability Calculations

- Error probability is found by integrating conditional probability of error over the decision region
  - This becomes difficult for a large number of dimensions
  - Difficult multidimensional integrations can be simplified by appropriate rotation, translation or reflection of coordinates
- Error performance depends only on the distance properties of the signal constellation.
- Calculations for nonbinary signal constellations may be reduced to a set of binary calculations via the “Union Bound”