ECE 5654 - Digital Communications Spring 2005

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Lecture #9 - The Union Bound

Announcements

- Exam #1 Thursday March 3
  - Covers material through Lecture 12
  - Bring: calculator, one page of notes
  - Sample exam on web page
- New Homework posted today
  - Due Feb 22

Modulation

- Modulation is used to transmit digital data over a channel
- Gramm-Schmidt procedure allows vector representation of any signal constellation
- Optimal receiver consists of a correlator, weighted to adjust for signal energy and a priori probabilities
- Receiver implementation can be simplified with reduced correlations and matched filter
- Receiver decision rule corresponds to decision regions in signal space

Performance Evaluation

- Probability of error can be calculated by integrating probability density function of received signal vector over decision regions.
- This integration can become extremely complicated
- Today we explore a simpler method for error probability calculations called the Union Bound
- This approximation is based on computing the *pair-wise* error probability (*i.e.,* the probability of error assuming only two of the M symbols were used) for all pairs

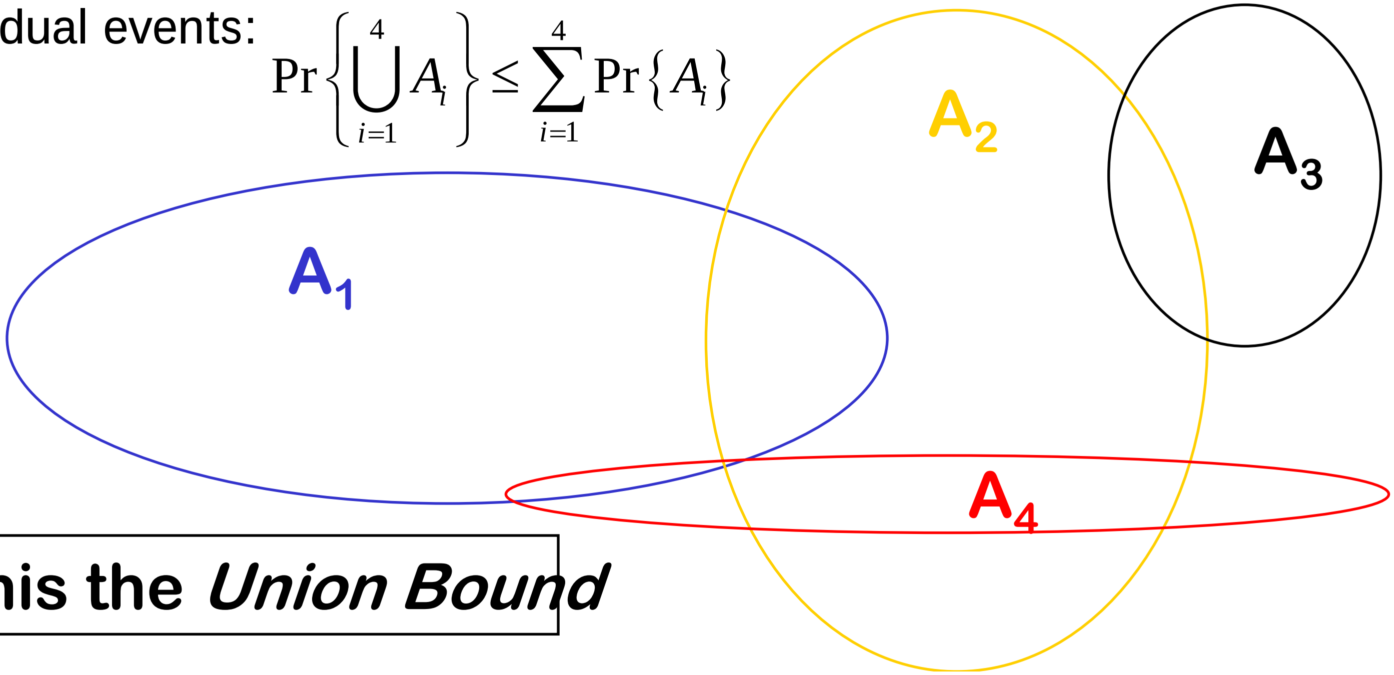
The Union Bound

- In theory, we can compute any error probability by the integration of the K-dimensional conditional probability density function over the appropriate K-dimensional regions.
- In practice, even with appropriate rotations and translations of the signal set, integration will become tedious for large signal constellations
- The Union Bound allows us to reduce error probability calculations to a series of binary error calculations.

The Union Bound

- Consider the four events A_1, A_2, A_3, and A_4. The probability of the union of the events $\Pr\left\{\bigcup_{i=1}^{4} A_i\right\}$ is analogous to the combined area of the four events.
- Clearly the total area covered by the four events is less than the sum of the four individual areas. Similarly we can say that the probability of the union of the events is less than the sum of the probability of the individual events:

$$\Pr\left\{\bigcup_{i=1}^{4} A_i\right\} \leq \sum_{i=1}^{4} \Pr\left\{A_i\right\}$$



We call this the *Union Bound*

Derivation of Union Bound

$$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right] = \int_{R_i{}^C} p\left(\mathbf{r} \middle| \mathbf{s}_i\right) d\mathbf{r}$$

R_i^C **is compliment of** R_i

$$= \sum_{j=1, j\neq i}^{M} \int_{R_j} p\left(\mathbf{r} \middle| \mathbf{s}_i\right) d\mathbf{r}$$

$$\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right] = \underbrace{\Pr\left[\bigcup_{\substack{j=1\\ j\neq i}}^{M} \Pr\left[Z_j \geq Z_i\right]\right]}_{\text{true error probability}} \leq \underbrace{\sum_{j=1, j\neq i}^{M} \Pr\left[Z_j \geq Z_i\right]}_{\text{union bound}}$$

- where

$$Z_i = \frac{N_0}{2} \ln[p_i] + \sum_{k=1}^{K} r_k s_{i,k} - \frac{1}{2} \sum_{k=1}^{K} s_{i,k}{}^2$$

Decision
Metrics

Derivation of Union Bound (continued)

- Assume ML case:

$$\Pr\left[Z_j \geq Z_i\right] = \Pr\left[\sum_{k=1}^{K} r_k s_{j,k} - \frac{1}{2} s_{j,k}{}^2 \geq \sum_{k=1}^{K} r_k s_{i,k} - \frac{1}{2} s_{i,k}{}^2\right]$$

$$= \Pr\left[\sum_{k=1}^{K} s_{j,k}{}^2 - 2 r_k s_{j,k} + r_k^2 \leq \sum_{k=1}^{K} s_{i,k}{}^2 - 2 r_k s_{i,k} + r_k^2\right]$$

$$= \Pr\left[\sum_{k=1}^{K} \left(s_{j,k} - r_k\right)^2 \leq \sum_{k=1}^{K} \left(s_{i,k} - r_k\right)^2\right]$$

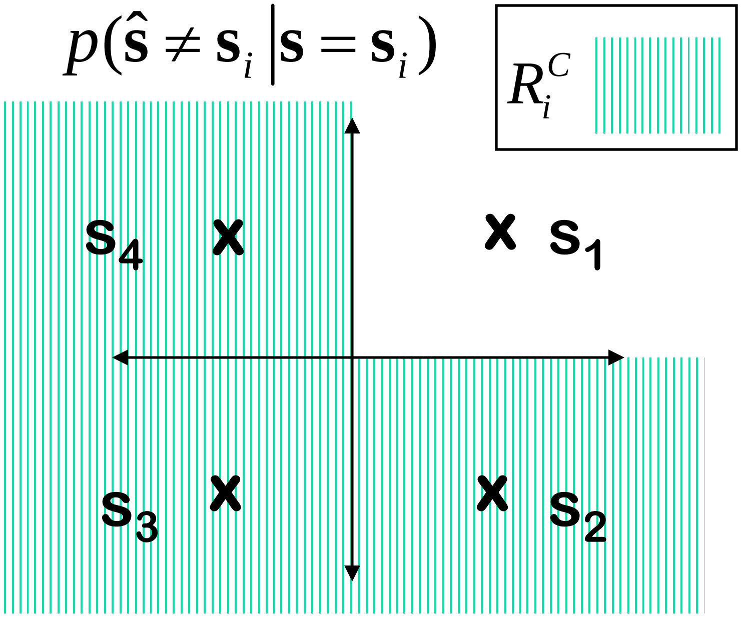
Expression of Decision Statistic in terms of Distance Between Signals

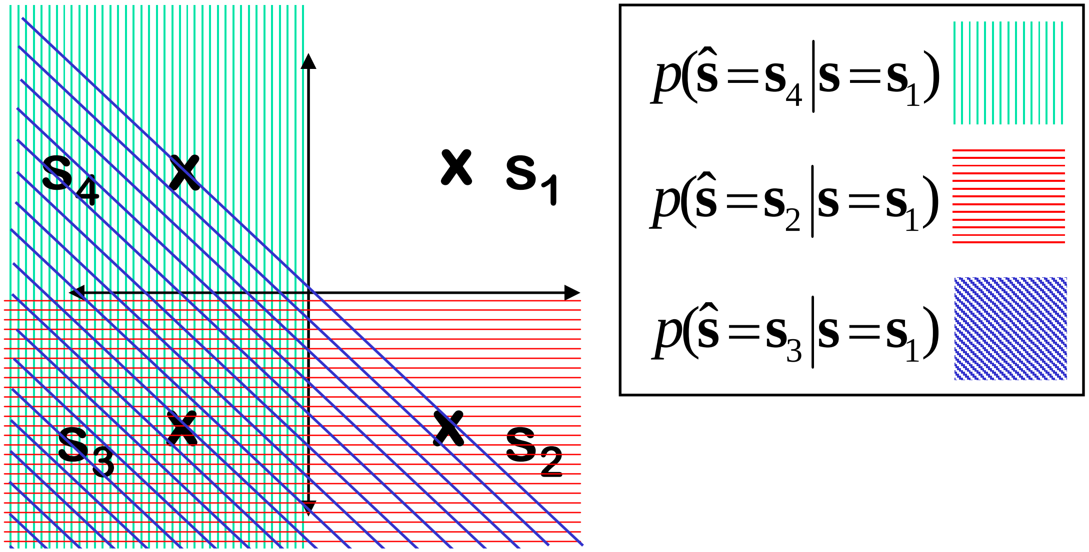
- $$\Pr\left[Z_j \geq Z_i\right] = \Pr\left[\sum_{k=1}^{K}\left(s_{j,k} - r_k\right)^2 \leq \sum_{k=1}^{K}\left(s_{i,k} - r_k\right)^2\right]$$
  $$= \Pr\left[d\left(\mathbf{r},\mathbf{s}_j\right) \leq d\left(\mathbf{r},\mathbf{s}_i\right)\right]$$
- where $d\left(\mathbf{r},\mathbf{s}_j\right) = \sqrt{\sum_{k=1}^{K}\left(s_{j,k} - r_k\right)^2}$ is the Euclidean distance between the points $\mathbf{s}_j$ and $\mathbf{r}$.
- This means the pair-wise error probability between $\mathbf{s}_i$ and $\mathbf{s}_j$ is equal to the probability that the received signal vector $\mathbf{r}$ is closer to $\mathbf{s}_j$ than to $\mathbf{s}_i$.

Pictorial Description of Union Bound

The Union Bound can be interpreted as a way of approximating the integration over one large region $\Re_i^C$ by integrating over several smaller regions. Since these regions have some overlap, the Union Bound is an *upper bound.*

Example: QPSK (equal *apriori* probabilities)


$p(\hat{\mathbf{s}} \neq \mathbf{s}_i | \mathbf{s} = \mathbf{s}_i)$



$p(\hat{\mathbf{s}} = \mathbf{s}_4 | \mathbf{s} = \mathbf{s}_1)$ 🟩

$p(\hat{\mathbf{s}} = \mathbf{s}_2 | \mathbf{s} = \mathbf{s}_1)$ 🟥

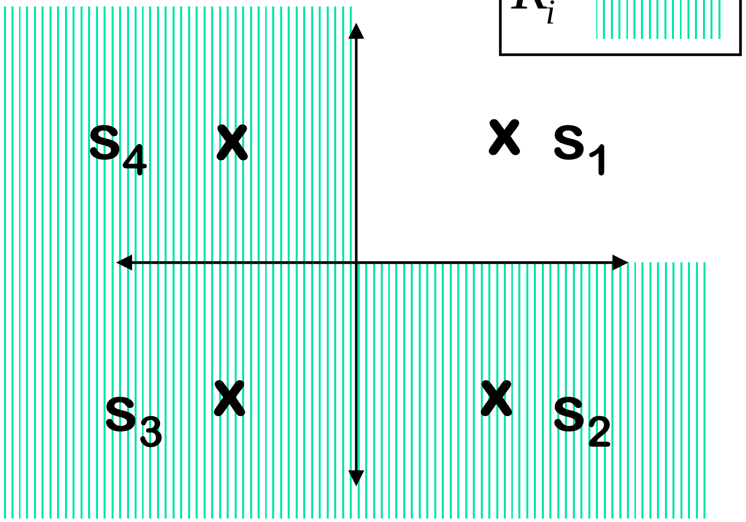
$p(\hat{\mathbf{s}} = \mathbf{s}_3 | \mathbf{s} = \mathbf{s}_1)$ ▨

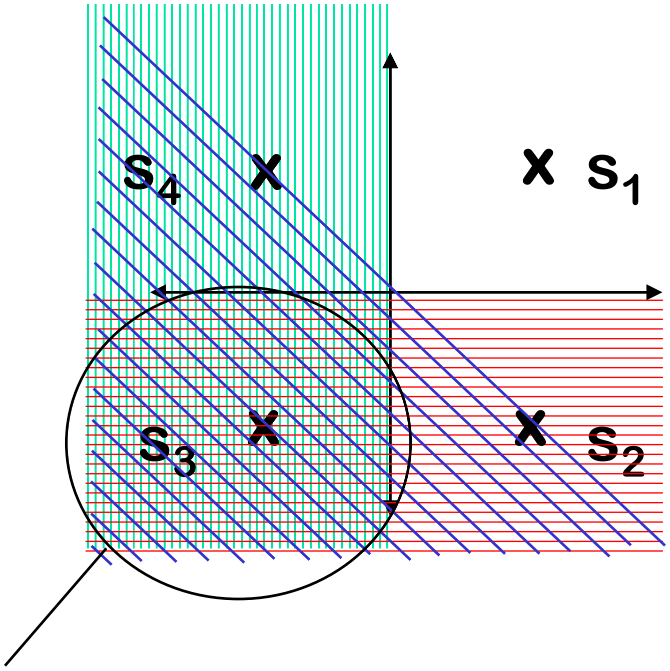
Pictorial Description of Union Bound

Example: QPSK (equal *apriori* probabilities)

$p(\hat{s} \neq s_i | s = s_i)$

R_i^C ▥





$p(\hat{s} = s_4 | s = s_1)$ ▥

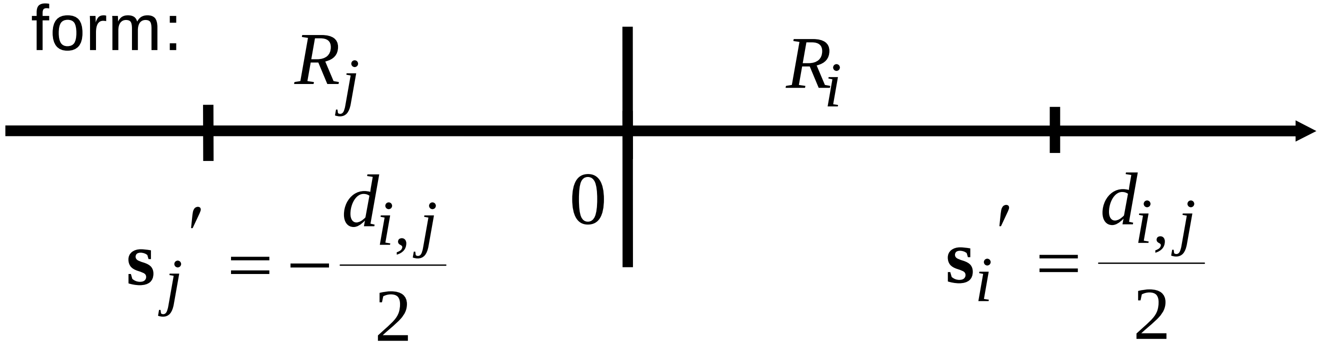
$p(\hat{s} = s_2 | s = s_1)$ ▤

$p(\hat{s} = s_3 | s = s_1)$ ▨

We integrate over this region three times, but fortunately, it is the least likely quadrant given s_1 was sent.

Computation of Pair-wise Error Probability

- By appropriate rotation and translation, we can always express the pair-wise decision problem in the form:



- where $d_{i,j} = d(\mathbf{s}_i, \mathbf{s}_j) = \sqrt{\sum_{k=1}^{K} (s_{i,k} - s_{j,k})^2}$ is the Euclidean distance between the points $\mathbf{s}_i$ and $\mathbf{s}_j$.
- This implies that

$$\Pr\left[Z_j \geq Z_i\right] = Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

The Union Bound (final form)

- $\Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right] \leq \sum_{j=1, j \neq i}^{M} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$
- $P_s(e) \leq \sum_{i=1}^{M} \frac{1}{M} \sum_{j=1, j \neq i}^{M} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$ (ML Assumption)
- An improved form of the Union Bound:

$$P_s(e) \leq \sum_{i=1}^{M} \frac{1}{M} \sum_{j \in A_i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

where A_i is the set of signals with decision regions *directly* adjacent to R_i.

Application of Union Bound to QPSK

- Distances between signals:

$$d_{1,2} = d_{1,4} = d_{2,3} = d_{3,4} = \sqrt{2E_s} = \sqrt{4E_b}$$

$$d_{1,3} = d_{2,4} = 2\sqrt{E_s} = \sqrt{8E_b}$$

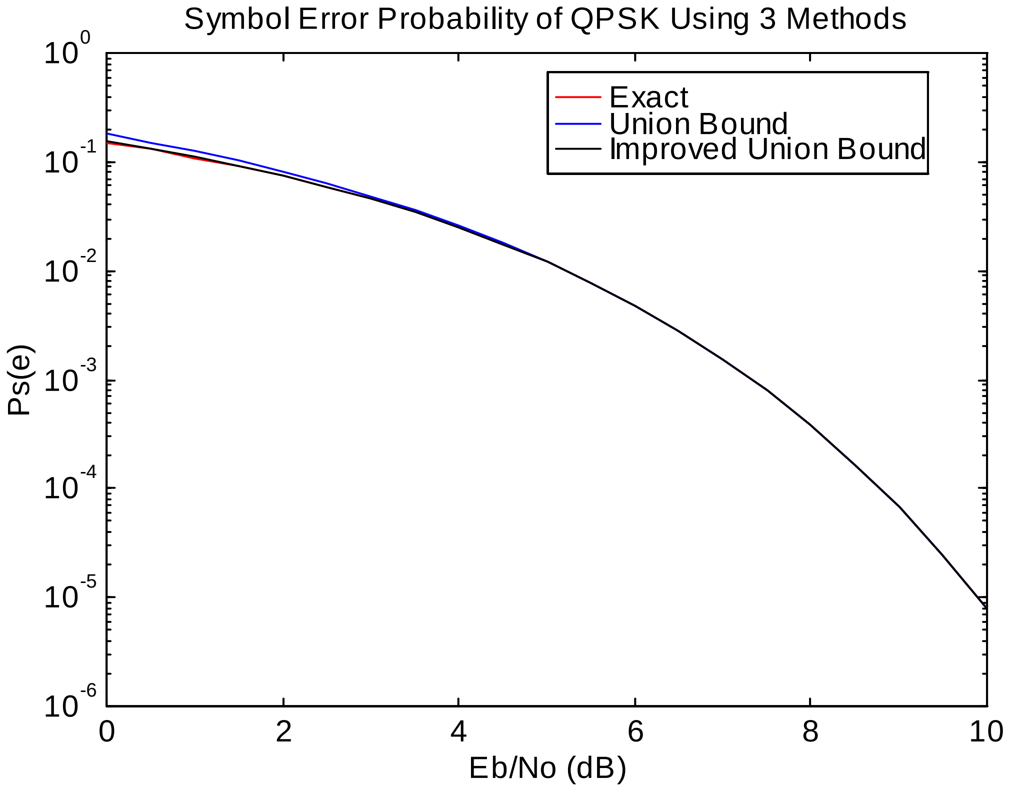
- Exact Calculation: $P_s(e) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]^2$

- Union Bound: $P_s(e) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \left[Q\left(\sqrt{\frac{4E_b}{N_0}}\right)\right]$

- Improved Union Bound:

$$P_s(e) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The Accuracy of the Union Bound



- The Union Bound is an upper bound because we may integrate over some regions twice.
- Union Bound is very accurate, especially for high E_b/N_o

Union Bound for *M*-ary PSK

- Signal Set:

$$s_m(t) = \sqrt{2E_s}\cos\left(2\pi f_c t + \frac{2\pi}{M}(m-1)\right)\Bigg|_0^T, m = 1, \ldots, M$$

- Basis Functions:

$$f_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\Big|_0^T,\ f_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\Big|_0^T$$

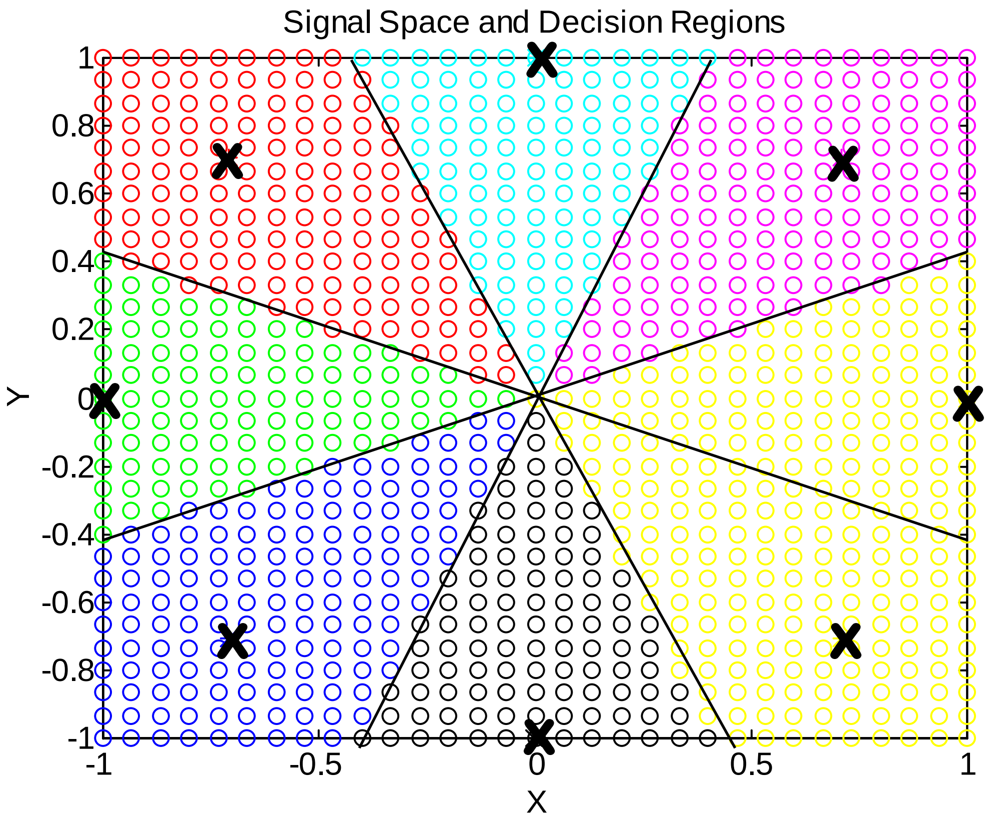
- Signal Vectors:

$$s_m = \left[\sqrt{E_s}\cos\left(\frac{2\pi(m-1)}{M}\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi(m-1)}{M}\right)\right],$$

$$m = 1, \ldots, M$$

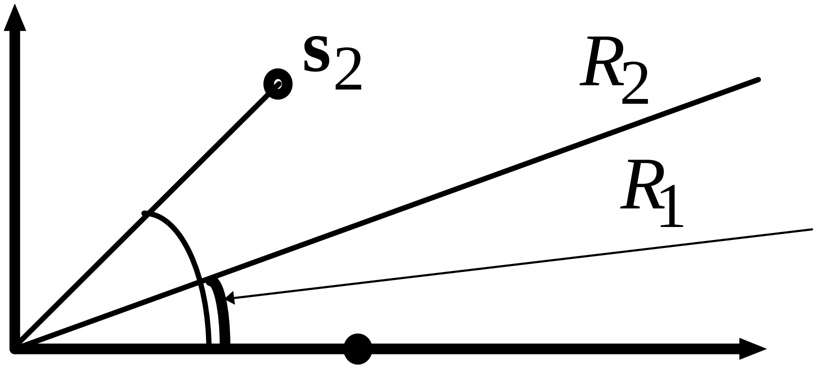
M-PSK is a 2-dimensional signal set

8-ary PSK Signal Space



Error Calculation for *M*-ary PSK

- Each signal has two adjacent decision regions.
- By symmetry, the distance $d_{i,j}$ between all adjacent signals is identical.



$\text{Total Angle} = 2\pi/M$

$\text{Half Angle} = \pi/M$

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_s} & 0 \end{bmatrix}$$

$$d_{i,j}/2 = \sqrt{E_s}\sin(\pi/M) = \sqrt{E_b \log_2 M}\,\sin(\pi/M)$$

Symbol Error Probability for *M*-ary PSK

- Union Bound (*M*>2):

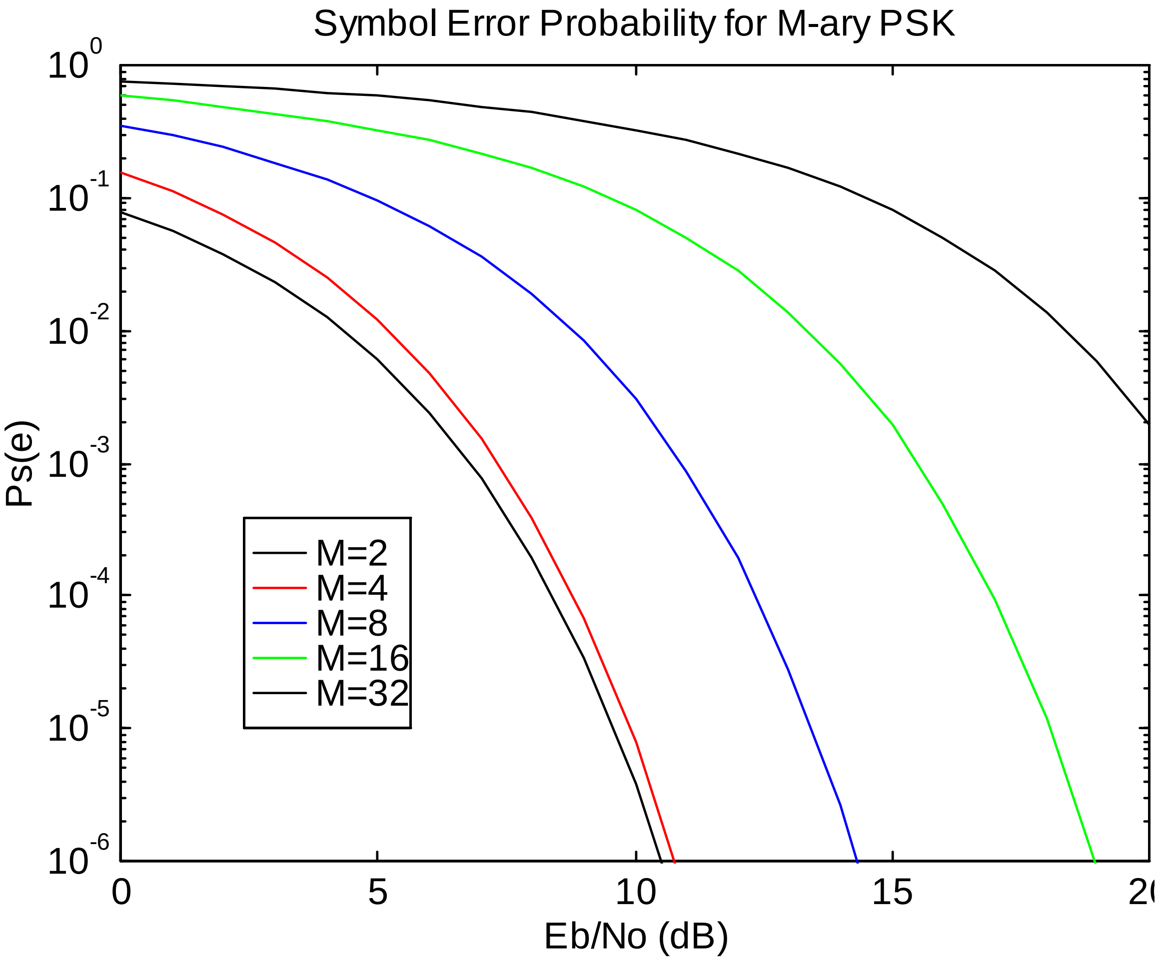
$$P_s(e) \le \sum_{i=1}^{M} \frac{1}{M} \sum_{j \in A_i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}} \right)$$

$$\le Q\left(\frac{2\sqrt{E_b \log_2 M} \sin(\pi/M)}{\sqrt{2N_0}} \right) + Q\left(\frac{2\sqrt{E_b \log_2 M} \sin(\pi/M)}{\sqrt{2N_0}} \right)$$

Note the above simplification was made since (a) all symbols have the same distance properties and (b) there are two adjacent regions.

$$P_s(e) \le 2Q\left(\sqrt{2E_b \log_2 M / N_0} \sin(\pi/M) \right)$$

Performance Comparison for *M*-ary PSK



Orthogonal Signal Sets

- *M* Basis Functions: $\{f_1(t),\ldots,f_M(t)\}$
- *M* Signals:

$$s_m(t)=\sqrt{E_s}\,f_m(t)=\sqrt{E_b\log_2 M}\,f_m(t),\quad m=1,\ldots,M$$

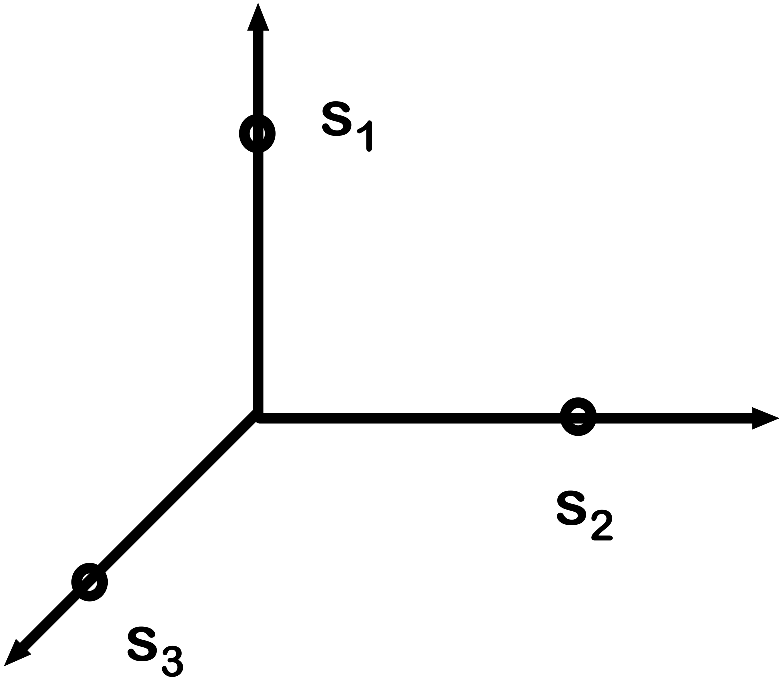
- Signal Space Representation:

$$\mathbf{s}_m=\begin{bmatrix}s_{m,1} & \cdots & s_{m,M}\end{bmatrix}, s_{m,i}=\begin{cases}\sqrt{E_s}, & m=i\\ 0, & m\neq i\end{cases}$$

- Example: *M*-ary FSK

$$s_m(t)=\sqrt{2P}\cos(2\pi f_m t)\Big|_0^T$$

3-ary Orthogonal Signal Space



Probability of Symbol Error

- It is more convenient to find the probability of a correct symbol P_c and then $P_s(e) = 1-P_c$
- Assume that symbol 1 was sent. The correlator outputs are

$$\begin{aligned}\mathbf{r} &= \mathbf{s}_1 + \mathbf{n} \\ &= \begin{bmatrix} \sqrt{E_s} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix}\end{aligned}$$

- As shown previously, the conditional probabilities of the correlator outputs are uncorrelated (and thus independent) Gaussian random variables

$$p(r_1|s_1) = \frac{1}{\sqrt{\pi N_o}} \exp\left[-\frac{\left(r_1 - \sqrt{E_s}\right)^2}{N_o}\right] \qquad p(r_i|s_1) = \frac{1}{\sqrt{\pi N_o}} \exp\left[-\frac{r_i^2}{N_o}\right] \quad i \neq 1$$

Probability of Error (cont.)

- The probability of a correct decision is the probability that r_1 is the largest correlator output

$$P_C = \int_{-\infty}^{\infty} P\left(n_2 < r_1, n_3 < r_1, \ldots n_M < r_1, \middle| r_1\right) p\left(r_1\right) dr_1$$

- Since the correlator outputs are independent

$$P\left(n_m < r_1 \middle| r_1\right) = \int_{-\infty}^{r_1} p_{r_m}\left(x_m\right) dx_m$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1} e^{-\frac{x^2}{2}} dx$$

Probability of Error (cont.)

- Thus,

$$P_C = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1} e^{-\frac{x^2}{2}} dx \right)^{M-1} p(r_1)\, dr_1$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1} e^{-\frac{x^2}{2}} dx \right)^{M-1} \frac{1}{\sqrt{\pi N_o}} \exp\left[-\frac{\left(r_1 - \sqrt{E_s}\right)^2}{N_o} \right] dr_1$$

$$P_s(e) = 1 - P_C$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1} e^{-\frac{x^2}{2}} dx \right)^{M-1} \right] \exp\left[-\frac{1}{2} \left(y - \sqrt{\frac{2E_s}{N_o}} \right) \right] dy$$

Union Bound

- The preceding expression provides an exact probability of symbol error, but is somewhat unwieldy.
- We can obtain a simpler expression by using the union bound.

$$P_s\left(e\middle|\mathbf{s}_1\right)=P\left(\bigcup_{i\neq 1} n_i>r_1\right)\leq\sum_{i=2}^{M}P\left(n_i>r_1\right)$$

- Or another way of viewing it is:

$$P_s\left(e\middle|\mathbf{s}=\mathbf{s}_1\right)=P\left(\bigcup_{i\neq 1}\hat{\mathbf{s}}=\mathbf{s}_i\middle|\mathbf{s}=\mathbf{s}_1\right)\leq\sum_{i=2}^{M}P\left(\hat{\mathbf{s}}=\mathbf{s}_i\middle|\mathbf{s}=\mathbf{s}_1\right)$$

Union Bound Calculation for *M* -ary Orthogonal Signaling

Since all of the symbols are equidistant from $\mathbf{s}_1$ the pairwise error probability between $\mathbf{s}_1$ and all other symbols is

$$P\left(\hat{\mathbf{s}} = \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_1\right) = Q\left(\sqrt{\frac{E_s}{N_o}}\right)$$

Note all symbols have equal energy E_s

The union bound is then

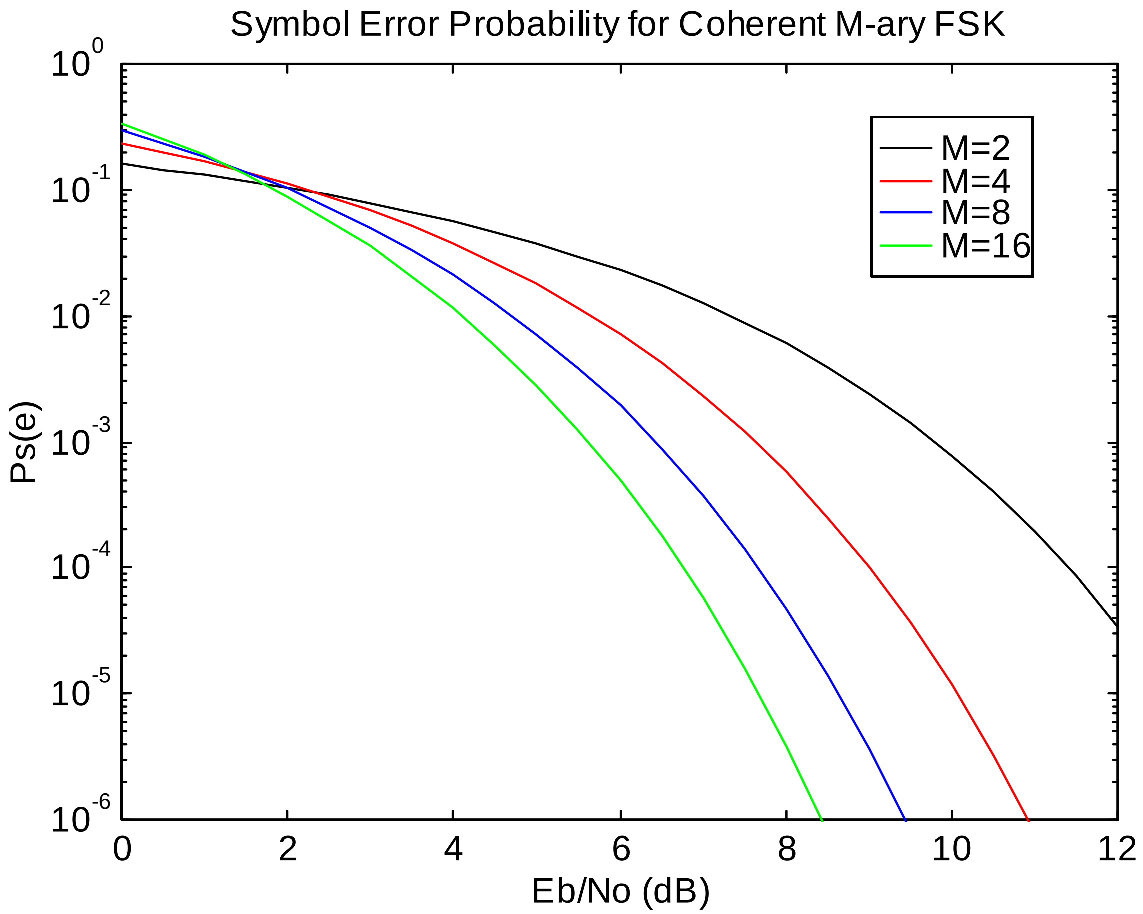
$$P_s\left(e \middle| \mathbf{s} = \mathbf{s}_1\right) \le (M-1)\, Q\left(\sqrt{\frac{E_s}{N_o}}\right)$$

Again, due to symmetry and if symbols are equally likely, we can use the probability of error for one symbol as the $P_s(e)$ if all

$$P_S(e) = (M-1)Q\left(\sqrt{E_S / N_0}\right)$$

$$= (M-1)Q\left(\sqrt{E_b \log_2 M \; / N_0}\right)$$

Performance Comparison for M-ary Orthogonal Signaling



Comments on Error Probability for Orthogonal Signals

- Performance improves as M increases
- In the limit (as $M \to \infty$), error probability can be made arbitrarily small as long as $E_b / N_0 > -1.6\text{dB}$
- Most practical systems use noncoherent FSK rather than coherent FSK. We will discuss noncoherent FSK soon.

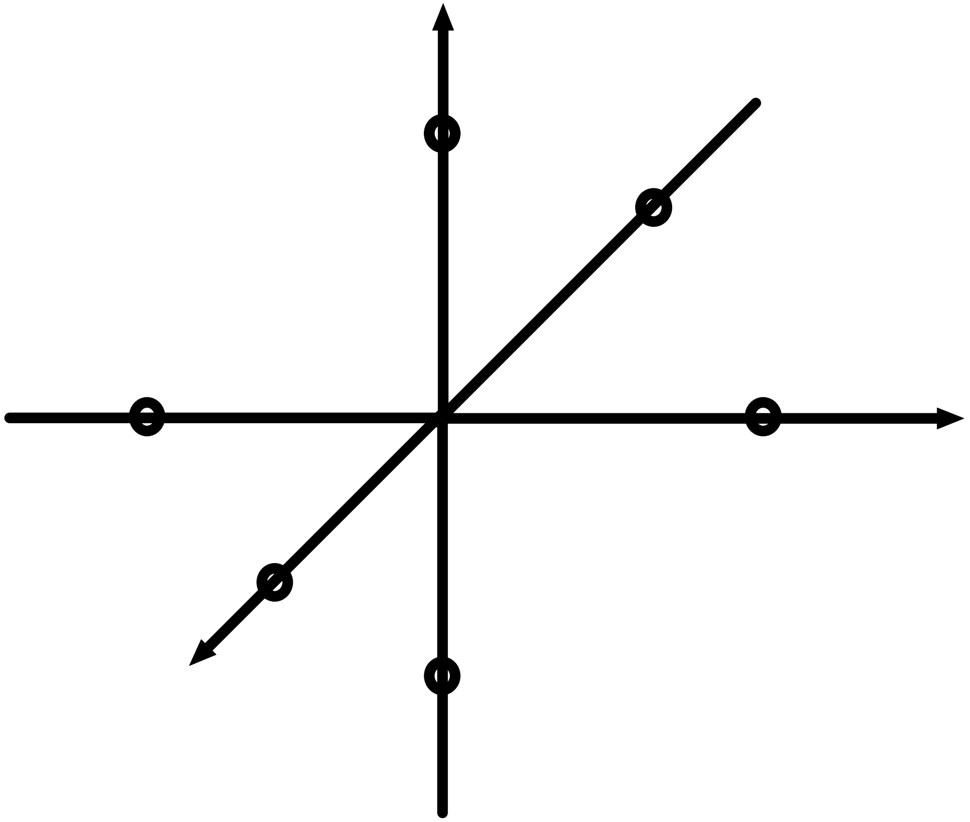
Improvements on Orthogonal Signaling

- Orthogonal signaling achieves optimum performance as $M \to \infty$, but for finite M improvements are possible.
- Biorthogonal Signaling:
  - $s_m(t)$ are orthogonal for $m = 1, \ldots, M/2$
  - $s_m(t) = -s_{m-M/2}(t)$ for $m = 1 + M/2, \ldots, M$
- Union Bound:

$$P_s(e) = (M-2)Q\left(\sqrt{E_b \log_2 M \quad / N_0}\right)$$

- Error probability about the same, but bandwidth efficiency better (only $M/2$ dimensions needed)

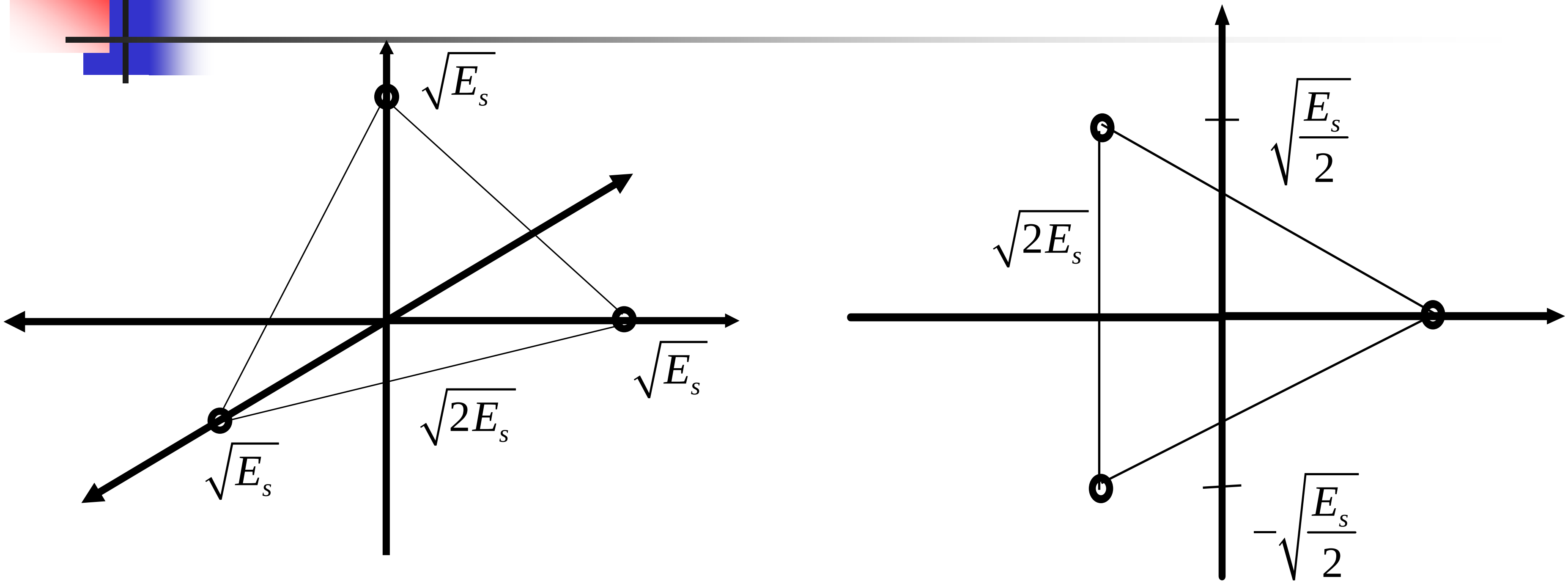
6-ary Biorthogonal Signal Constellation



Simplex Signal Constellation:

- Note that the centroid of an orthogonal constellation is located at: $\mathbf{c} = \left[\sqrt{E_s}/M \quad \cdots \quad \sqrt{E_s}/M \right]$
- A more energy efficient signal constellation can be achieved by moving the centroid to the origin.
- This can be accomplished by letting:
  - $\mathbf{s}_m{}' = \mathbf{s}_m - \mathbf{c}$ for $m = 1, \ldots, M$
- **Simplex Conjecture:** The Simplex signal constellation is believed to be the most energy efficient signal constellation for any value of *M*

3-ary Simplex Signal Constellation



•Note that simplex signal constellation reduces dimension by one from orthogonal signaling

•Distance properties unchanged with a reduction in necessary signal energy.

Summary of Union Bound

- Union Bound allows evaluation of error probability for arbitrary signal constellations
- High dimensional signal constellations (such as FSK) can become very energy efficient as M becomes large
- Fixed dimensional signal constellations (such as PSK and QAM) use energy inefficiently as M becomes large