Digital Communications
Midterm Exam
March 3, 2005

SOLUTION

I pledge that I have neither given nor received any assistance on this exam.

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(signed)

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Name (print) Student Number

1. (10 points) Short answer. Please answer the following questions.

   a. (5 points) A source has a bit rate of 100kbps which must be transmitted over a wireless link. The system currently uses the following symbols to transmit the data:

$$s_1(t) = \cos(\omega_1 t)$$
$$s_2(t) = \cos(\omega_1 t + \pi)$$
$$s_3(t) = \cos(\omega_2 t)$$
$$s_4(t) = \cos(\omega_2 t + \pi)$$

where $\omega_1 - \omega_2 = 2\pi*(100\text{kHz})$

Name three distinct ways to improve the bandwidth efficiency of the link.

1. *Pulse shaping*
2. *Change Δf to 50 kHz (remains orthogonal)*
3. *Use QAM, M-PSK or other bandwidth efficient modulation scheme.*

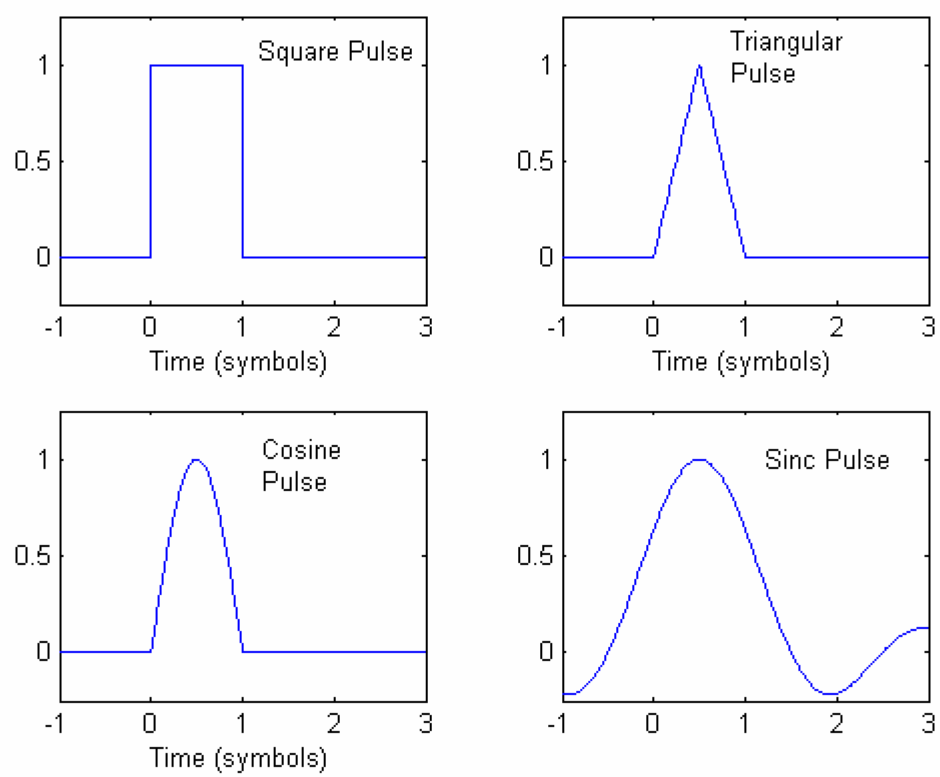
   b. (5 points) Explain how bandwidth efficiency and energy efficiency change as the number of symbols *M* increases.

For PSK, QAM, ASK or any modulation scheme where increasing M does not increase the number of basis functions (dimensions), bandwidth efficiency improves while energy efficiency degrades as M increases.

For FSK, M-ary orthogonal modulation or any modulation scheme where increasing M increases the number of basis functions, bandwidth efficiency degrades while energy efficiency improves as M increases.

2. (10 points) Pulse Shaping

Rate the following pulse shapes in terms of their bandwidth efficiency (1 being the most efficient and 4 being the least efficient).



1. Sinc Pulse
2. Cosine pulse
3. Triangular pulse
4. Square pulse

3. (30 points) MAP Decision Rule

Assume that a receiver processes the received signal (which uses a binary modulation scheme) such that the received decision metric **r** has the following conditional probability distribution functions:

$$p(\mathbf{r}|\mathbf{s}_1) = \frac{b/\pi}{b^2 + \left(r - \sqrt{E_s}\right)^2}$$

$$p(\mathbf{r}|\mathbf{s}_2) = \frac{b/\pi}{b^2 + \left(r + \sqrt{E_s}\right)^2}$$

where b is a positive constant.

(a) [10 points] Determine the MAP decision rule. Simplify as much as possible.

The general MAP rule can be written as

$$p(\mathbf{s}_1)\, p(\mathbf{r}|\mathbf{s}_1) \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} p(\mathbf{s}_2)\, p(\mathbf{r}|\mathbf{s}_2)$$

substituting for the conditional error probabilities:

$$p(\mathbf{s}_1)\frac{b/\pi}{b^2 + \left(r - \sqrt{E_s}\right)^2} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} p(\mathbf{s}_2)\frac{b/\pi}{b^2 + \left(r + \sqrt{E_s}\right)^2}$$

$$p(\mathbf{s}_1)\left\{b^2 + \left(r + \sqrt{E_s}\right)^2\right\} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} p(\mathbf{s}_2)\left\{b^2 + \left(r - \sqrt{E_s}\right)^2\right\}$$

(b) [10 points] Determine the ML decision rule. Simplify as much as possible.

Starting with the MAP rule

$$p(\mathbf{s}_1)\left\{b^2+\left(r+\sqrt{E_s}\right)^2\right\} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} p(\mathbf{s}_2)\left\{b^2+\left(r-\sqrt{E_s}\right)^2\right\}$$

$$\left\{b^2+\left(r+\sqrt{E_s}\right)^2\right\} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} \left\{b^2+\left(r-\sqrt{E_s}\right)^2\right\}$$

$$b^2+r^2+2r\sqrt{E_s}+E_s \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} b^2+r^2-2r\sqrt{E_s}+E_s$$

$$2r\sqrt{E_s} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} -2r\sqrt{E_s}$$

$$4r\sqrt{E_s} \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} 0$$

$$r \underset{\mathbf{s}_2}{\overset{\mathbf{s}_1}{\gtrless}} 0$$

(c) [10 points] For the ML case, determine the probability of error. You may find the following integral useful:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

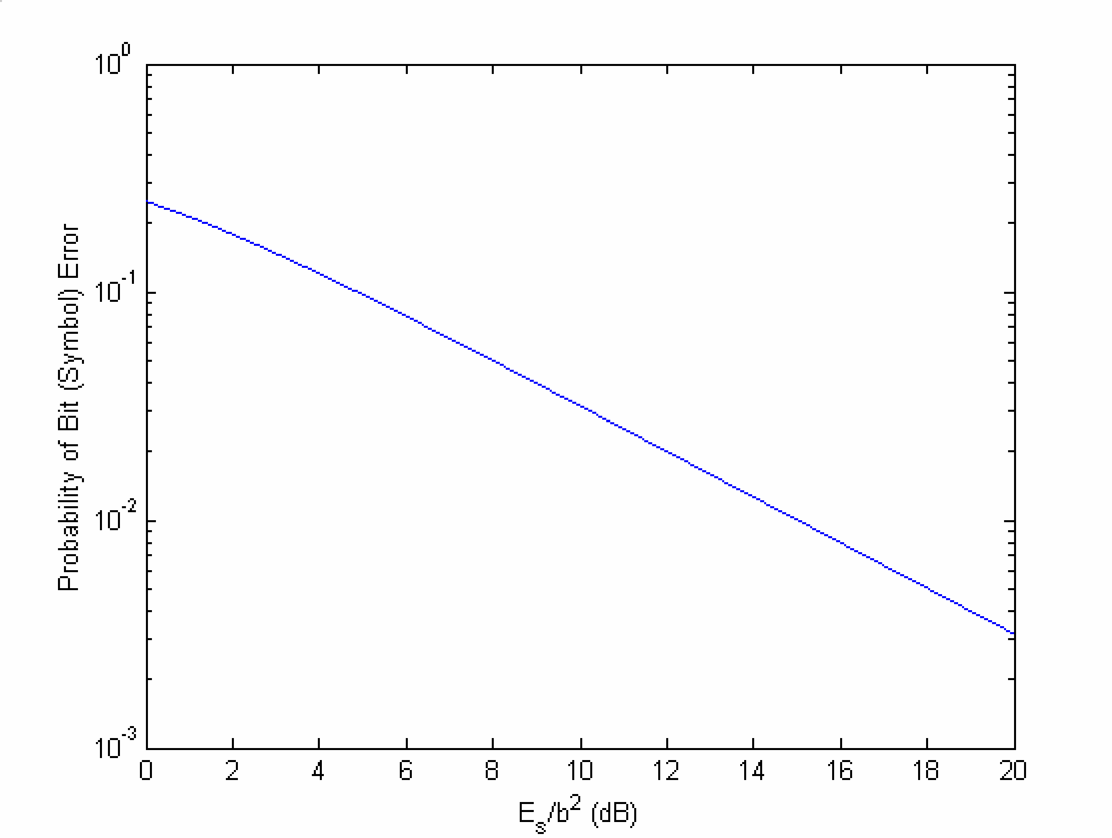
The probability of symbol error can be written as

$$\begin{aligned} P_s(e) &= \frac{1}{2}\Pr\{r < 0 | \mathbf{s}_1\} + \frac{1}{2}\Pr\{r > 0 | \mathbf{s}_2\} \\ &= \Pr\{r < 0 | \mathbf{s}_1\} \\ &= \int_{-\infty}^{0} \frac{b/\pi}{b^2 + \left(r - \sqrt{E_s}\right)^2} dr \end{aligned}$$

Now, making a substitution of variables:

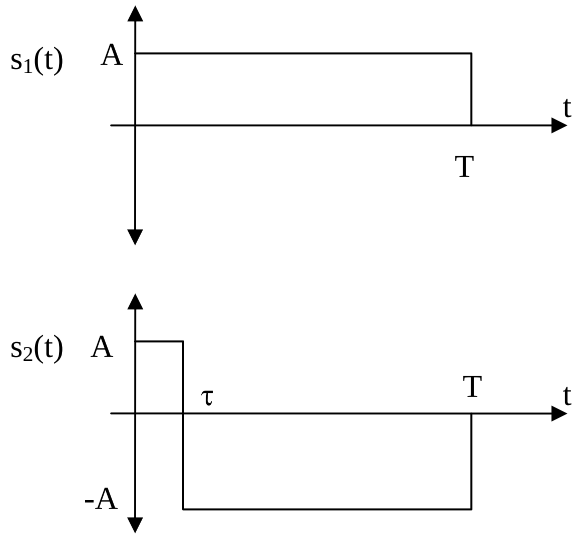
$$\begin{aligned} P_s(e) &= \frac{b}{\pi} \int_{-\infty}^{-\sqrt{E_s}} \frac{dx}{b^2 + x^2} \\ &= \frac{b}{\pi}\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\Bigg|_{-\infty}^{-\sqrt{E_s}} \\ &= \frac{1}{\pi}\left\{\tan^{-1}\left(\frac{-\sqrt{E_s}}{b}\right) - \tan^{-1}(-\infty)\right\} \\ &= \frac{1}{\pi}\left\{\tan^{-1}\left(\frac{-\sqrt{E_s}}{b}\right) + \frac{\pi}{2}\right\} \\ &= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{-\sqrt{E_s}}{b}\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{E_s}{b^2}}\right) \end{aligned}$$

The resulting probability of error is shown below.


10^0
10^-1
10^-2
10^-3
Probability of Bit (Symbol) Error
0
2
4
6
8
10
12
14
16
18
20
E_s/b^2 (dB)

4. (30 points) Signal Space

Consider the following signal set used for binary modulation:



(a) [5 points] Determine a set of basis functions for this set in terms of A, T, and τ.

By inspection we can choose the basis functions as

$$f_1(t) = \begin{cases} \dfrac{1}{\sqrt{\tau}} & 0 \le t \le \tau \\ 0 & else \end{cases}$$

$$f_2(t) = \begin{cases} \dfrac{1}{\sqrt{T-\tau}} & \tau \le t \le T \\ 0 & else \end{cases}$$

Note that the Gram-Schmidt procedure could also be used if desired.

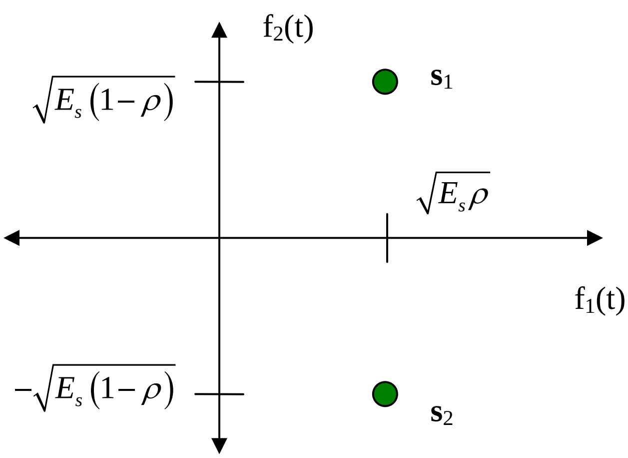
(b) [5 points] Plot the signal-space representation of this set for τ = T/4.

$$\mathbf{s}_1 = \begin{bmatrix} A\sqrt{\tau} \\ A\sqrt{T-\tau} \end{bmatrix}$$
$$\mathbf{s}_2 = \begin{bmatrix} A\sqrt{\tau} \\ -A\sqrt{T-\tau} \end{bmatrix}$$

Since the average energy per symbol is A^2T, we can rewrite this as

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_s\rho} \\ \sqrt{E_s(1-\rho)} \end{bmatrix}$$
$$\mathbf{s}_2 = \begin{bmatrix} \sqrt{E_s\rho} \\ -\sqrt{E_s(1-\rho)} \end{bmatrix}$$

where $\rho = \tau/T$. The signal space diagram is then:



(c) [10 points] Determine the probability of error in an AWGN channel for τ = T/2 and τ = T/4. (Assume matched filtering and the maximum likelihood decision rule.)

From the diagram, we can find the probability of error as

$$P_s(e) = Q\left(\frac{d}{\sqrt{2N_o}}\right) = Q\left(\frac{2\sqrt{E_s(1-\rho)}}{\sqrt{2N_o}}\right) = Q\left(\sqrt{\frac{2E_b(1-\rho)}{N_o}}\right)$$

For τ = T/2, ρ = ½:

$$P_s(e) = Q\left(\sqrt{\frac{2E_b\left(1-\frac{1}{2}\right)}{N_o}}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

which is the same as BFSK or orthogonal modulation. This makes sense since for T=1/2, the two symbols are orthogonal. For τ = T/4, ρ = 1/4:

$$P_s(e) = Q\left(\sqrt{\frac{2E_b\left(1-\frac{1}{4}\right)}{N_o}}\right) = Q\left(\sqrt{\frac{3E_b}{2N_o}}\right)$$

which is better than orthogonal modulation.

(d) [10 points] What value of τ will minimize the probability of error?

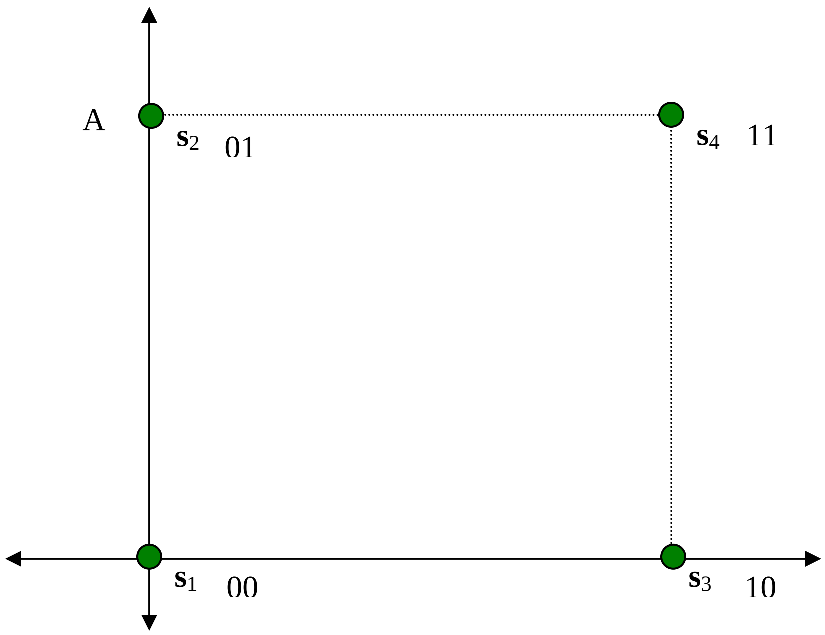
From the equation above we can see that $\rho = 0$ ($\tau = 0$) minimizes probability of error. The minimum probability of error is then

$$P_s(e) = Q\left(\sqrt{\frac{2E_b(1-0)}{N_o}}\right)$$
$$= Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

This result can also be ascertained from the figure, since the best performance will occur when the two symbols are antipodal. This occurs when $\tau = 0$.

5. (20 points) The Union Bound

Consider the following signal space representation for a modulation scheme used in additive white Gaussian noise.



(a) [10 points] Using the standard Union Bound, determine an upper bound on the average probability of *bit* error in terms of the average energy per bit and the noise power spectral density N_o. Assume that a matched filter is used with the ML decision rule.

Since the distance properties of all the symbols are equal, we can find the probability of error for symbol one and it will be the same for all symbols:

$$P_s(e) \le \sum_{i=2}^{4} \Pr\left\{\hat{\mathbf{s}} = \mathbf{s}_i \middle| \mathbf{s}_1\right\}$$
$$= \sum_{i=2}^{4} Q\left(\frac{d_{1i}}{\sqrt{2N_o}}\right)$$

From the diagram we can see that $d_{12}=d_{13}=A$ *and* $d_{14} = \sqrt{2}A$. *Thus,*

$$
\begin{aligned}
P_s(e) &\le \sum_{i=2}^{4} Q\left(\frac{d_{1i}}{\sqrt{2N_o}}\right) \\
&= 2Q\left(\frac{A}{\sqrt{2N_o}}\right) + Q\left(\frac{\sqrt{2}A}{\sqrt{2N_o}}\right) \\
&= 2Q\left(\sqrt{\frac{A^2}{2N_o}}\right) + Q\left(\sqrt{\frac{A^2}{N_o}}\right)
\end{aligned}
$$

Now, the average energy per symbol is:

$$
\begin{aligned}
E_s &= \frac{1}{4}\left\{0 + A^2 + A^2 + 2A^2\right\} \\
&= A^2
\end{aligned}
$$

Thus,

$$
P_s(e) \le 2Q\left(\sqrt{\frac{E_s}{2N_o}}\right) + Q\left(\sqrt{\frac{E_s}{N_o}}\right)
$$

Since there are two bits per symbol we have

$$
P_s(e) \le 2Q\left(\sqrt{\frac{E_b}{N_o}}\right) + Q\left(\sqrt{\frac{2E_b}{N_o}}\right)
$$

Now the bit error rate is determined by the number of bits in error per symbol error. Two errors will result in one bit error, while the other will result in two:

$$
\begin{aligned}
P_b(e) &\approx \frac{1}{2} * 2Q\left(\sqrt{\frac{E_b}{N_o}}\right) + \frac{2}{2} * Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \\
&\approx Q\left(\sqrt{\frac{E_b}{N_o}}\right) + Q\left(\sqrt{\frac{2E_b}{N_o}}\right)
\end{aligned}
$$

(b) Using the *improved Union Bound*, determine the average probability of *bit* error in terms of the average energy per bit and the noise power spectral density N_o. Assume that a matched filter is used with the ML decision rule. *How does the result compare to QPSK?*

The improved Union Bound only considers nearest neighbors. From the equation above, this can be easily found by eliminating the second term which is due to the furthest point:

$$P_b(e) \approx Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

Note that this is 3dB worse than QPSK. This is because the constellation is not centered at zero.