Simulation of Gaussian Noise

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Complex Envelope

- For simulation and analysis of communication systems we rely on the complex baseband (complex envelope) of bandpass signals.
- Desired signal:

$$s_{bp}(t) = \mathrm{Re}\left\{ \underbrace{s_{ce}(t)}_{\substack{complex \\ envelope}} e^{j2\pi f_c t} \right\}$$

$$= s_I(t)\cos\left(2\pi f_c t\right) - s_Q(t)\sin\left(2\pi f_c t\right)$$

$$= A(t)\cos\left(2\pi f_c t + \varphi(t)\right)$$

Complex Envelope (cont.)

- The complex envelope can be represented as:

$$s_{ce}(t) = A(t)e^{j\varphi(t)}$$
$$= s_I(t) + js_Q(t)$$

- In our simulation we have use a sampled version of the complex envelope:

$$s_k = s_{ce}(kt_s)$$
$$= A(kt_s)e^{j\varphi(kt_s)}$$
$$= s_I(kt_s) + js_Q(kt_s)$$
$$= s_{Ik} + js_{Qk}$$

Power

- The power of the different representations can be found as:

$$P_{bp} = \underbrace{\lim}_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} s_{bp}^2(t)dt$$

$$= \underbrace{\lim}_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} A^2(t) \left(\frac{1 + \cos\left(4\pi f_c t + 2\varphi(t)\right)}{2} \right) dt$$

$$= \frac{A^2}{2}$$

$$P_{ce} = \underbrace{\lim}_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} s_{ce}(t) s_{ce}^*(t)dt$$

$$= \underbrace{\lim}_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} A^2(t) e^{j\varphi(t)} e^{-j\varphi(t)} dt$$

$$= A^2$$

Note factor of 2!

Assuming constant envelope transmission

Power (cont.)

- Additionally the sampled complex envelope has power:

$$
\begin{aligned}
P_{sim} &= \underbrace{\lim}_{L\to\infty} \frac{1}{L+1} \sum_{-L/2}^{L/2} s_k s_k^* \\
&= \underbrace{\lim}_{L\to\infty} \frac{1}{L+1} \sum_{-L/2}^{L/2} A_k^2 e^{j\varphi_k} e^{-j\varphi_k} \\
&= A^2
\end{aligned}
$$

Assuming constant envelope transmission

Same as complex envelope power

Symbol Energy

- We can then determine the energy per symbol as:

$$E_s^{bp} = P_{bp}T = \frac{A^2T}{2}$$

$$E_s^{bp} = P_{ce}T = A^2T$$

$$E_s^{sim} = P_{sim}m = A^2mt_s$$

m is the number of samples per symbol
and t_s is the time between samples.

Noise

- Bandpass noise:

$$n_{bp}(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$N_{bp}(f) = \begin{cases} \frac{N_o}{2} & |f \pm f_c| \le \frac{B}{2} \\ 0 & else \end{cases}$$

$$\begin{aligned} R_{nn}^{bp}(\tau) &= E\left\{n_{bp}(t)n_{bp}(t+\tau)\right\} \\ &= R_{II}(\tau)\cos(2\pi f_c \tau) - R_{IQ}(\tau)\sin(2\pi f_c \tau) \\ &= R_{II}(\tau)\cos(2\pi f_c \tau) \end{aligned}$$

Noise (cont.)

- Baseband noise

$$n_{ce}(t) = n_I(t) + jn_Q(t)$$

- The expression for the autocorrelation function of the bandpass noise tells us that the complex envelope PSD is

$$N_I(f) = \begin{cases} N_o & |f| \le \frac{B}{2} \\ 0 & else \end{cases}$$

$$N_Q(f) = \begin{cases} N_o & |f| \le \frac{B}{2} \\ 0 & else \end{cases}$$

Noise (cont.)

- The auto-correlation function is then:

$$R_{II}(\tau) = R_{QQ}(\tau) = N_o B \operatorname{sinc}\left(B\tau\right)$$

- The autocorrelation function of the bandpass process is:

$$R_{nn}^{bp}(\tau) = N_o B \operatorname{sinc}\left(B\tau\right)\cos(2\pi f_c \tau)$$

- The noise powers are then:

$$\sigma_{nn}^2 = \sigma_{II}^2 = \sigma_{QQ}^2 = N_o B$$

$$\sigma_{ce}^2 = \sigma_{II}^2 + \sigma_{QQ}^2 = 2N_o B$$

Sampled Noise

- If we sample the complex envelope at $t_s = 1/B$:

$$\begin{aligned} n_k &= n_I(kt_s) + jn_Q(kt_s) \\ &= x_k + jy_k \end{aligned}$$

- The autocorrelation function for *x* or *y* is:

$$\begin{aligned} R_{xx}(k) = R_{yy}(k) &= R_{II}(kt_s) \\ &= N_o B \operatorname{sinc}(Bkt_s) \\ &= N_o B \operatorname{sinc}\left(Bk\frac{1}{B}\right) \\ &= \begin{cases} N_o B & k = 0 \\ 0 & else \end{cases} \end{aligned}$$

Sampled Noise

- Thus, the noise samples are created as independent I and Q Gaussian samples:

$$n_k = n_I(kt_o) + jn_Q(kt_o)$$
$$= x_k + jy_k$$

- Where x_k and y_k are independent GRV with zero mean and variance:

$$\sigma_x^2 = \sigma_y^2 = \sigma_{II}^2 = N_o B$$

$$\sigma_{sim}^2 = \sigma_x^2 + \sigma_y^2 = 2N_o B$$

Signal-to-Noise Ratio

- The SNR for the bandpass system is:

$$\left(\frac{S}{N}\right)_{bp} = \frac{A^2}{2\sigma_{nn}^2} = \frac{A^2}{2N_o B}$$

- The SNR for the complex envelope is:

$$\left(\frac{S}{N}\right)_{ce} = \frac{A^2}{\sigma_{ce}^2} = \frac{A^2}{2N_o B}$$

- The SNR in the simulated system is:

$$\left(\frac{S}{N}\right)_{sim} = \frac{A^2}{\sigma_{sim}^2} = \frac{A^2}{2N_o B}$$

Note: All achieve the same SNR which we have defined in terms of the *bandpass* bandwidth B and the *bandpass* noise power spectral density N_o

Matched Filter

- Typically, when we simulate digital communication systems, we assume the use of a matched filter (or correlator version of the matched filter).
- When matched filtering is assumed, we need not actually simulate the pulse shape and matched filter.
- This is because the output of the matched filter has the same SNR regardless of pulse shape.
  - $SNR_{out} = 2E_s/N_o$
- Thus, we normally simulate the sampled output of the matched filter with an SNR equal to $2E_s/N_o$

E_s/N_o

- The noise power spectral density for the bandpass system is defined as N_o and is equal to

$$N_o = \frac{\sigma_{nn}^2}{B}$$

- The noise power spectral density for the complex envelope system is defined as N_o^{ce} and is equal to

$$N_o^{ce} = \frac{\sigma_{ce}^2}{B} = \frac{2\sigma_{nn}^2}{B}$$

- The noise power spectral density for the complex envelope system is defined as N_o^{sim} and is equal to

$$N_o^{ce} = \frac{\sigma_{sim}^2}{B} = \frac{\sigma_{sim}^2}{1/t_s}$$

E_s/N_o (cont)

- The E_b/N_o for the bandpass system is then

$$\left(\frac{E_b}{N_o}\right)_{bp} = \frac{A^2T}{2\frac{\sigma_{nn}^2}{B}} = \frac{A^2TB}{2\sigma_{nn}^2} = TB\left(\frac{S}{N}\right)_{bp}$$

- The E_b/N_o for the complex envelope system is

$$\left(\frac{E_b}{N_o}\right)_{ce} = \frac{A^2T}{\frac{\sigma_{ce}^2}{B}} = \frac{A^2TB}{\sigma_{ce}^2} = TB\left(\frac{S}{N}\right)_{ce} = TB\left(\frac{S}{N}\right)_{bp}$$

- The E_b/N_o for the simulated system is

$$\left(\frac{E_b}{N_o}\right)_{ce} = \frac{A^2mt_s}{\frac{\sigma_{sim}^2}{B}} = \frac{A^2mt_s}{\sigma_{sim}^2 t_s} = m\left(\frac{S}{N}\right)_{sim}$$

Determining σ^2_{sim}

- In a simulation we often wish to create the noise variance to provide a desired E_s/N_o given a signal level *A*.

$$\left(\frac{E_b}{N_o}\right)_{ce} = \frac{A^2 m t_s}{\sigma^2_{sim} t_s}$$

$$\sigma^2_{sim} = 2\sigma^2_x = 2\sigma^2_y \frac{A^2 m}{\left(E_b / N_o \right)}$$

- Thus, the random variables *x* and *y* must have variance

$$\sigma^2_x = \sigma^2_y \frac{A^2 m}{2\left(E_b / N_o \right)}$$

Specifying E_b/N_o

- Often we are interested in specifying E_b/N_o rather than E_s/N_o. In that case the variance is:

$$\sigma_x^2 = \sigma_y^2 = \frac{A^2 m}{2\left(E_s/N_o\right)} = \frac{A^2 m}{2n\left(E_b/N_o\right)} = \frac{A^2 m/n}{2\left(E_b/N_o\right)}$$

- Where *n* is the number of bits per symbol.
- If coding with rate *r* is envolved,

$$\sigma_x^2 = \sigma_y^2 = \frac{A^2 m/(nr)}{2\left(E_b/N_o\right)}$$

Specifying E_b/N_o

- Finally, if there are multiple users in the system, we may want to fix the noise variance and vary the signal amplitudes. In such a case, we can set

$$\sigma_x^2 = \sigma_y^2 = \frac{1}{2}$$

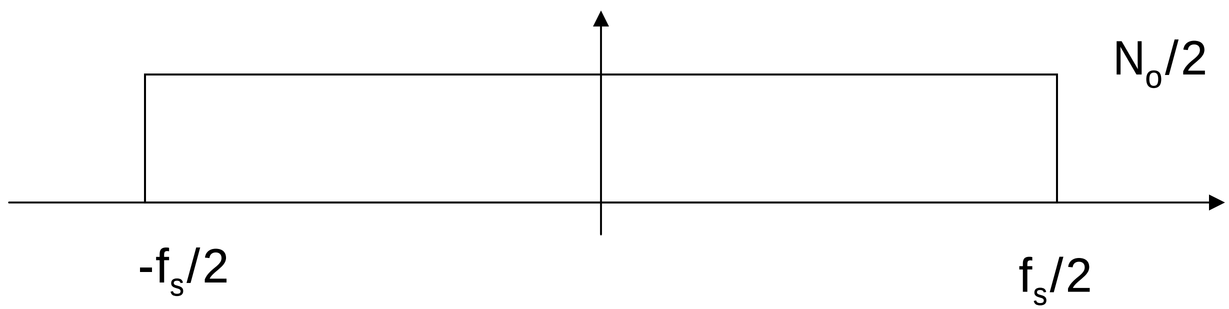
- And the amplitude is determined as

$$A^2 = \frac{2\left(E_b / N_o\right)\sigma_x^2}{m/(nr)} = \frac{\left(E_b / N_o\right)}{m/(nr)}$$

$$A = \sqrt{\frac{\left(E_b / N_o\right)}{m/(nr)}}$$

Simulating AWGN

- We would like to simulate White Noise with a Gaussian distribution. The noise PSD is then



- The noise power is then

$$\sigma_n^2 = \frac{f_s N_o}{2}$$

Specifying SNR

- If I want to simulate a specific SNR we know the SNR is stated as

$$\frac{S}{N} = \frac{P}{\sigma_n^2}$$

  - Where $P=E\{A^2\}$ and A is the amplitude of a sample of the desired signal.
- The simplest method is then to simply define $E\{A^2\}=1$ and the noise variance is set to

$$\sigma_n^2 = \frac{1}{\left(S/N\right)_{desired}}$$

Specifying E_s/N_o

- Typically, however, we are interested in specifying a specific value of E_s/N_o rather than SNR.
- The energy per symbol is simply the equal to the sum of the square amplitudes over the *N* samples per symbol.

$$E_s = \sum_{i=1}^{N} A_i^2 T_s = \frac{\sum_{i=1}^{N} A_i^2}{f_s}$$

- Thus, from our definition previous relation we find

$$\frac{E_s}{N_o} = \frac{\sum_{i=1}^{N} A_i^2}{f_s} \frac{f_s}{2\sigma_n^2} = \frac{\sum_{i=1}^{N} A_i^2}{2\sigma_n^2}$$

Specifying E_s/N_o

- Thus, we simply define the variance per sample as

$$\sigma_n^2 = \frac{\sum_{i=1}^{N} A_i^2}{2\left(\frac{E_s}{N_o}\right)_{des}}$$

- If we have square pulses and force $A_i = 1$, this reduces to

$$\sigma_n^2 = \frac{N}{2\left(\frac{E_s}{N_o}\right)_{des}}$$

- Where N is the number of samples per symbol