The Impact of Angular Energy Distribution on Spatial Correlation

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Abstract— In this paper we derive simple generalized formula for spatial correlation and show that it provides a good approximation for spatial correlation for three very different angular energy distributions: a Gaussian angle distribution, a Laplacian angle distribution, and a uniform angular distribution. We also derive an equation for spatial correlation when the angular energy distribution is bi-modal. These generalized equations are parameterized by $\frac{d\sigma}{\lambda}$ where d is the distance between antennas, λ is the carrier wavelength and σ is the standard deviation of the angular energy distribution. This allows for a single curve to be used for any value of d/λ and σ. It is found that the standard deviation of the angular energy distribution or angle spread is the dominating factor for determining correlation as opposed to the specific angular energy distribution.

I. Introduction

Signal fading due to multipath is the dominant impairment in mobile radio systems. To overcome this problem multiple antennas are typically employed to provide diversity. However, the performnace of antenna arrays is dependent on the fading correlation between the antennas. This is usually termed spatial correlation. Previous work on spatial correlation has relied on numerical integration or infinite series to evaluate the correlation between two points in space based on the angular energy distribution [1], [2], [3]. As a result, separate curves must be generated for each distribution and parameter of interest (e.g., each variance of a Gaussian distribution). In this paper we show that an approximate equation can be derived which is very general and can be used for a wide range of distributions with good approximation. Further, this equation is parameterized by $\frac{d\sigma}{\lambda}$ which allows a single curve to not only be used for a variety of distributions, but also for a wide range of values for σ. The approximation error increases as σ increases. However, as σ increases, the correlation decreases as does its impact on performance. Thus, we will show that in effect the approximation can be used to accurately predict bit-error rate performance over all practical values of σ. Further, we show that with slight modification, the equation can be used to predict correlation when the angular energy distribution is bi-modal.

II. Spatial Correlation

Consider a plane wave signal arriving at an array from angle θ with respect to the normal bisecting two points of interest separated by d meters. The signals seen at the two points can be represented as $s_1(t) = m(t)$ and $s_2(t) = m(t)e^{-j2\pi d/\lambda \sin(\theta)}$. If the power of the message signal $m(t)$ is unity, then $\mathrm{E}\left\{s_1(t)s_2^*(t)\right\} = e^{j2\pi d/\lambda \sin(\theta)}$. Thus, if a signal of interest arriving at an array can be described by the summation of plane waves arriving from angles with distribution $p_\Theta(\theta)$, then the spatial correlation between two points a distance d apart can be determined as [4]

$$\rho(d) = \int_{-\pi}^{\pi} \exp\left\{ j2\pi \frac{d}{\lambda} \sin(\theta) \right\} p_\Theta(\theta) d\theta \quad (1)$$

where θ is defined relative to the normal.

First, let us assume a truncated Gaussian distribution for angular energy which is common for spatial channel modeling. Thus, the angular distribution function can be represented as

$$p_\Theta(\theta) = \frac{C_g}{\sqrt{2\pi}\sigma_g} \exp\left\{ -\frac{(\theta-\phi)^2}{2\sigma_g^2} \right\} \quad (2)$$

where σ_g is the standard deviation of the distribution in radians, $C_g = \frac{1}{\mathrm{erf}\left(\frac{\pi}{\sqrt{2}\sigma_g}\right)}$ is a constant to guarantee that $p_\Theta(\theta)$ is a density function, and ϕ is the central angle of arrival in radians. It is shown in Appendix A that in the case of $p_\Theta(\theta)$ defined by (2) the correlation can be approximated by

$$\rho(d) \approx \exp\left\{ j\frac{2\pi d}{\lambda} \sin(\phi) \right\} \exp\left\{ -\frac{\left(\frac{2\pi d}{\lambda} \cdot \sigma \cos(\phi)\right)^2}{2} \right\} \quad (3)$$

where we now see that we can parameterize the spatial correlation by $\frac{2\pi d}{\lambda}\sigma \cdot \cos(\phi)$.

Another common assumption for angular energy distribution is a uniform distribution [2]. A uniform distribution of angular energy is defined as

$$p_\Theta(\theta) = \frac{1}{2\Delta} \qquad \phi - \Delta \leq \theta \leq \phi + \Delta \tag{4}$$

where 2Δ is the range of angles about a central angle-of-arrival ϕ. It is shown in Appendix B that the spatial correlation in this case can be approximated by

$$\rho(d) \approx \exp\left(j\frac{2\pi d}{\lambda}\sin(\phi)\right)\operatorname{sinc}\left(\frac{2\pi d}{\lambda}\cos(\phi)\,\Delta\right) \tag{5}$$

where for the purpose of this paper we define $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$. If we substitute $\sigma = \frac{\Delta}{\sqrt{3}}$ (the standard deviation of a uniform distribution) we can compare the Gaussian and uniform distributions as shown in Figure 1 for $\phi = 0$ and $\sigma = 2°$. Also plotted in Figure 1 is the approximation from (3). As expected, the Gaussian distribution decreases more slowly in the main lobe, but lacks the secondary correlation peaks. Otherwise the approximate correlation functions are similar. It would thus appear that we could use equation (3) to approximate both a uniform and a Gaussian AOA distribution. Let us examine this more carefully. It is well known that e^x can be approximated by the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ Further, we know that $\sin(x)$ can be approximated by the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Examining the first three terms of the two approximations we find for the Gaussian distribution

$$\rho(d) \approx 1 - \frac{1}{2}\left(2\pi\frac{d}{\lambda}\sigma\cos(\phi)\right)^2 + \frac{1}{8}\left(2\pi\frac{d}{\lambda}\sigma\cos(\phi)\right)^4$$

while for the uniform distribution

$$\rho(d) \approx 1 - \frac{1}{2}\left(2\pi\frac{d}{\lambda}\sigma\cos(\phi)\right)^2 + \frac{3}{40}\left(2\pi\frac{d}{\lambda}\sigma\cos(\phi)\right)^4$$

Thus, since the two correlation equations are similar, we propose using equation (3) to approximate the spatial correlation for both Gaussian and Uniform distributions. In fact we also maintain that the approximation will hold well for other distributions such as a Laplacian distribution [5] where we define the Laplacian distribution as

$$p_\Theta(\theta) = \frac{2C_l}{b}e^{-b|\theta-\phi|} \tag{6}$$

b is a positive constant and $C_l = \frac{1}{1-e^{-a\pi}}$ is a constant to ensure that $p_\Theta(\theta)$ is a probability distribution. We examine this claim in more detail in the next section.

III. Analytical Results

Figures 1 and 2 plot the spatial correlation between two antennas versus $\frac{2\pi d\sigma}{\lambda}$ for the three distributions for low and high angle spreads, $\sigma = \{2^o, 20^o\}$ using numerical integration as well as our approximation. We have assumed that $\phi = 0^o$. We note that the more narrow the distribution the higher the correlation for the same distance and angle spread. One of our claims with the above approximation, is that we can use a single equation for all three distributions and for any value of σ. Comparing the two figures, we see that they are nearly identical despite the large difference in angle spread, which supports the latter claim. We can see that the approximation works very well for each of the cases considered. The approximation is slightly optimistic (lower correlation) for a Laplcian distribution and slightly pessimistic (higher correlation) for a uniform distribution. As expected, it is very accurate for a Gaussian distribution since it was originally derived from the Gaussian distribution. We would expect that the approximation would be worse for higher values of σ since our approximation in getting to (3) assumed that σ was small. From Figure 2 we can see that this is indeed the case. The other condition where we might suspect the approximation to be poor is for values of ϕ much larger than zero. Figure 3 plots the estimated correlation for all three distributions as well as the approximation for $\sigma = 10^o$ and $\phi = 60^o$. We note that indeed the approximation is worse, but still good. This represents a worst case practically speaking. The accuracy of the approximation will degrade as *phi* increases beyond 60, but since most outdoor wireless systems use sectorication (120° sectors), it is not typically necessary to consider angles beyond ±60.

The appproximation is best at high correlation values and worst at low correlation values. This is fortunate since it is the higher correlation values that impact the performance of diversity arrays more profoundly than low correlation values. That is, while there is a large performance difference between a diversity array with correlations of 1 and 0.5, there is very little difference between the performance of diversity arrays with correlations of 0.5 and 0. To show this we plot the performance of BPSK in Rayleigh fading with two correlated receive antennas [6]. The perfomance is plotted assuming each of the three distributions along with the approximated correlation for threee values of SNR per branch. Specifically we plot the BER performance that would be achieved given each of the three distributions with $\sigma = 2°$, as well as the performance predicted by using the approximation in Figure 4. The SNR per branch was assumed to be 5dB, 8dB and 15dB. The performance based on the acutal distributions is extremely close to the predicted performance using the approximation. The dominating factor on the performance (from a diversity perspective) is the angle spread, not the exact distribution.

IV. Bi-Modal Distributions

It has been suggested that the AOA distribution may be generated by multiple dominant reflectors. In such a case, the AOA distribution may be bi-modal. In this section we wish to examine such a case and determine a simplified equation which can cover this case. To do this let us assume that the angular energy (AOA) distribution is equal to the sum of two Guassian distributions:

$$\begin{aligned} p_\Theta(\theta) &= \frac{C_g}{2\sqrt{2\pi}\sigma_a}\exp\left\{-\frac{(\theta-\phi_a)^2}{2\sigma_b^2}\right\}\cdots \\ &\quad+\frac{C_g}{2\sqrt{2\pi}\sigma_b}\exp\left\{-\frac{(\theta-\phi_b)^2}{2\sigma_b^2}\right\} \end{aligned} \quad (7)$$

where ϕ_a and ϕ_b are the centers of the two individual distributions. Then, following the previous work we can find the correlation function to be

$$\begin{aligned} \rho(d) &\approx \exp\left\{j\frac{2\pi d}{\lambda}\sin(\phi_a)\right\} \\ &\quad \exp\left\{-\frac{\left(\frac{2\pi d}{\lambda}\cdot\sigma_a\cos(\phi_a)\right)^2}{2}\right\}+ \\ &\quad \exp\left\{j\frac{2\pi d}{\lambda}\sin(\phi_b)\right\} \\ &\quad \exp\left\{-\frac{\left(\frac{2\pi d}{\lambda}\cdot\sigma_b\cos(\phi_b)\right)^2}{2}\right\} \end{aligned}$$

The correlation function is very similar to the case of a single-mode distribution with an oscillation term which depends on the difference between the AOA's of the two modes. If the angles are close, the oscillation will be slow (versus d), while if the difference is large, the oscillation will be fast (versus d). Further, the single distribution correlation function is then an approximation of the envelope of the correlation oscillation. This can be seen in Figure 5. This figure plots the spatial correlation versus $2\pi d/\lambda$ for each of the three dsitributions when $\sigma = 2°$ and $\phi_a = 30°$ $\phi_b = -30°$. It can be shown that if we measure the variance of the total distribution, using (3) with this measured variance will provide us an approximation of the first lobe in the correlation function. This is also plotted in Figure 5 where the measured variance for each of the distributions is approximately 10°. As in the case of single-mode distributions, we find that bi-modal distributions can be approximated using a simple formula for a wide range of angle-of-arrival distributions and values for angle spread (σ). Further, if we use the variance of a single mode of the distribution ($\sigma = 2°$) in equation (3) we obtain

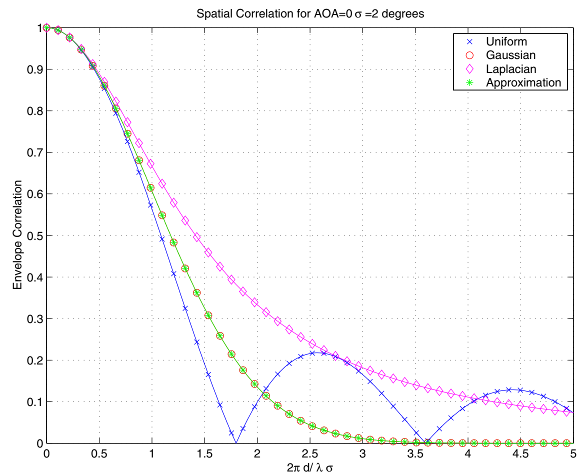

Fig. 1. Spatial Correlation versus $2\pi\frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions and Proposed Approximation ($\sigma = 2°$, $\phi = 0$)

the envelope of the correlation function also shown in Figure 5. However, the preceding example assumed symetric central AOA's ($\phi_a = -\phi_b$) and $\sigma_a = \sigma_b$. We would like to determine if the equations still hold when these two conditions are violated. Figure 6 (right) plots the correlation for a bi-modal distribution with $\sigma_a = 5°$,$\sigma_b = 10°$, $\phi_a = -30°$ and $\phi_b = 0°$ along with the distributions assumed. Figure 6 (left) plots the correlation for a bi-modal distribution with $\sigma_a = 2°$,$\sigma_b = 20°$, $\phi_a = -30°$ and $\phi_b = 30°$ and the distributions. These two examples show that the correlation function for bi-modal distributions provides a good approximation for a variety of conditions.

V. Conclusions

In this paper we have investigated a generalized (i.e.,for multiple values of σ and for a wide range of AOA distributions) approximation for spatial correlation for single mode and bi-modal distributions. The generalized equations allow the correlation to be found for any practical standard deviation and distance between antetnnas. We have examined the error in the approximation and have shown that it is negligible if the ultimate concern is diversity performance. This arises because the approximation is worst at low values of correlation where the diversity performance is less sensitive to the exact correlation value. It is found that it is the standard deviation of the angular energy distribution, not the specific distribution, that matters in spatial correlation.

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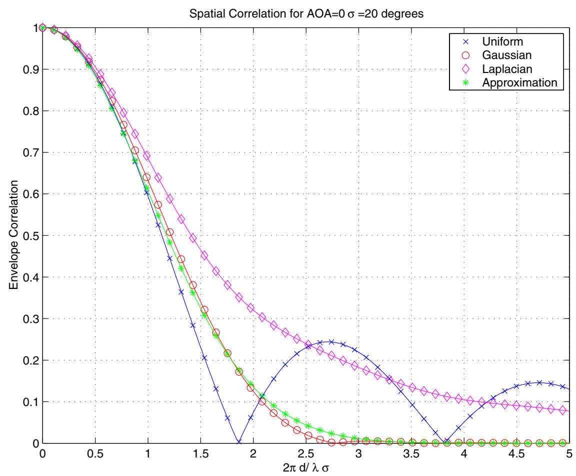

Fig. 2. Spatial Correlation versus $2\pi \frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions and Proposed Approximation ($\sigma = 20°$, $\phi = 0$)

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Appendix A: Derivation of Generalized Equation for Gaussian Distribution

Let us assume a Gaussian distribution for angular energy such that the angular distribution function can be represented as

$$p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(\theta-\phi)^2}{2\sigma^2}\right\} \qquad (8)$$

where σ is the standard deviation of the distribution in radians and ϕ is the central angle of arrival in radians.

Then we know that the spatial correlation can be determined as [4]

$$\rho(d) = \int_{-\pi}^{\pi} \exp\left\{j2\pi\frac{d}{\lambda}\sin(\theta)\right\} p_\Theta(\theta)d\theta \qquad (9)$$

Now, substituting (2) into (9) and making a change of variables we get

$$\begin{aligned}\rho(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}\sin(\sigma z+\phi)\right\}\exp\left\{-\frac{z^2}{2}\right\}dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}(\sin(\sigma z)\cos(\phi)+\ldots\right. \\ &\quad \left.\cos(\sigma z)\sin(\phi))\right\}\exp\left\{-\frac{z^2}{2}\right\}dz\end{aligned}$$

Now, assuming that σz is small over the range where $\exp\left(-\frac{z^2}{2}\right)$ is significant, we can approximate the above with

$$\begin{aligned}\rho(d) &\approx \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}(\sigma z\cos(\phi)+\sin(\phi))\right\}\times \\ &\quad \exp\left\{-\frac{z^2}{2}\right\}dz \\ &\approx \frac{\exp\left\{j\frac{2\pi d}{\lambda}\sin(\phi)\right\}}{\sqrt{2\pi}}\times \\ &\quad \int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}(\sigma z\cos(\phi))\right\}\exp\left\{-\frac{z^2}{2}\right\}dz\end{aligned}$$

Evaluating the integral [7] then gives

$$\boxed{\rho(d) \approx \exp\left\{j\tfrac{2\pi d}{\lambda}\sin(\phi)\right\}\exp\left\{-\frac{\left(\frac{2\pi d}{\lambda}\sigma\cos(\phi)\right)^2}{2}\right\}}$$

Appendix B: Derivation of Generalized Equation for Uniform Distribution

Let us assume that the angular energy is distributed according to

$$p_\Theta(\theta) = \frac{1}{2\Delta} \qquad \phi-\Delta \le \theta \le \phi+\Delta \qquad (10)$$

$$\begin{aligned}\rho(d) &= \frac{1}{2\Delta}\int_{\phi-\Delta}^{\phi+\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(\theta)\right)d\theta \\ &= \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(z+\phi)\right)dz \\ &= \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}(\sin(z)\cos(\phi)+\cos(z)\sin(\phi))\right)dz\end{aligned}$$

Now for small Δ, we can approximate $\sin(z) \approx z$ and $\cos(z) \approx 1$ which gives

$$\rho(d) = \frac{1}{2\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(\phi)\right)\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}\cos(\phi)z\right)dz$$

Evaluating the integral [7] then gives

$$\boxed{\rho(d) = \exp\left(j\tfrac{2\pi d}{\lambda}\sin(\phi)\right)\operatorname{sinc}\left(\tfrac{2\pi d}{\lambda}\cos(\phi)\Delta\right)}$$

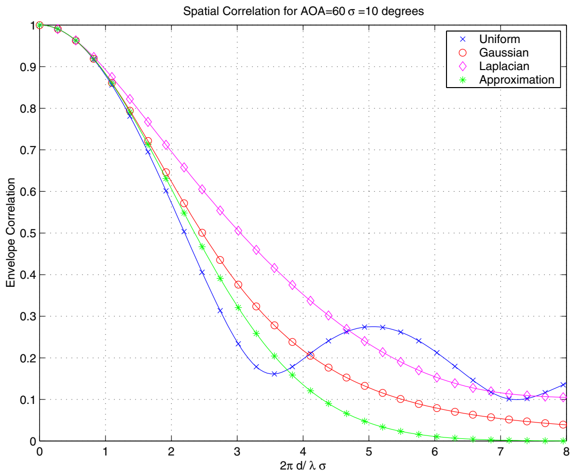

Fig. 3. Spatial Correlation versus $2\pi\frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions and Proposed Approximation ($\sigma = 10°$, $\phi = 60$)

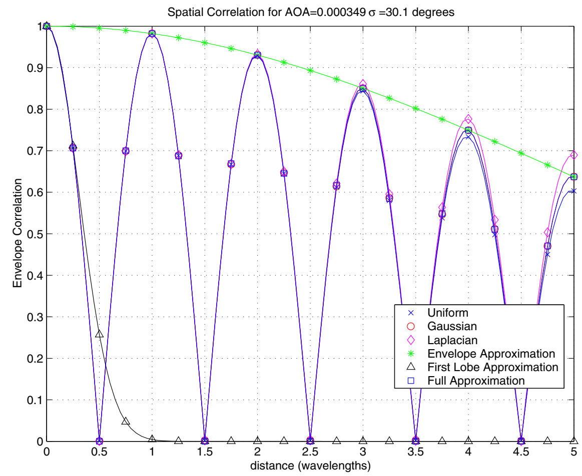

Fig. 5. Spatial Correlation versus $2\pi\frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions as well as Predicted Correlation Using Various Approximations ($\sigma_a = \sigma_b = 2°$, $\phi_a = -30°$,$\phi_b = 20°$.)

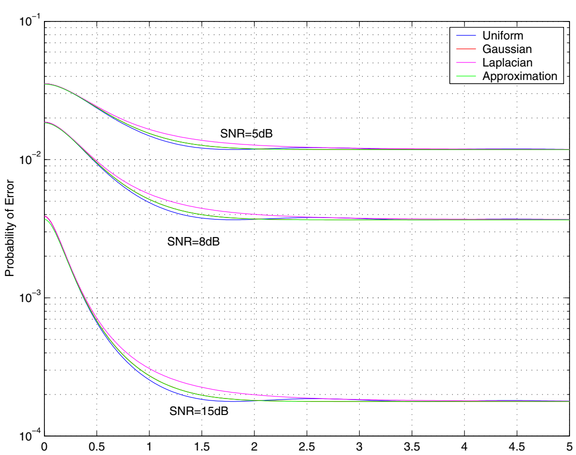

Fig. 4. Probability of Bit Error versus $2\pi\frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions as well as Predicted Performance Using Proposed Approximation ($\sigma = 2°$, $\phi = 0$, SNR=5dB,8dB,15dBm. Within each SNR range curves are Laplacian, Guassian, Approximation, and Uniform from top to bottom)

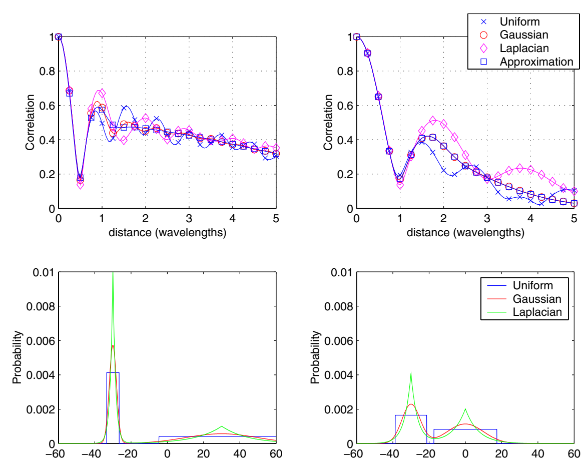

Fig. 6. Spatial Correlation versus $2\pi\frac{d}{\lambda}\sigma$ for Uniform, Gaussian, and Laplacian Distributions and Corresponding Distributions ($\sigma_a = \sigma_b = 2°$, $\phi_a = -30°$,$\phi_b = 30°$.)