Generalized Equations for Spatial Correlation for Low to Moderate Angle Spread

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Abstract In this paper we derive generalized formulas for three types of angular energy distributions: a Gaussian angle distribution, the angular energy distribution arising from a Gaussian spatial distribution, and a uniform angular distribution. These generalized equations are parameterized by $d\sigma$ where d is the distance between antennas and σ is the standard deviation of the angular energy distribution and approximate the true correlation with the approximation being very good for angle spreads below approximately 25^o.

1 Introduction

Previous work on antenna correlation has relied on numerical integration or infinite series to evaluate the spatial correlation between two points using the angular energy distribution [1, 2]. As a result, separate curves must be generated for each distribution parameter of interest (e.g., each variance of a Gaussian distribution). However, it was recently suggested by Chizhik and Gans [3] that when the energy arriving at a linear array has a distribution in angle which is Gaussian, the spatial correlation function can be parameterized by $\frac{2\pi d}{\lambda}\sigma$ where d is the distance, λ is the carrier wavelength, and σ is the standard deviation of the angular distribution. In other words, we can create a generalized spatial correlation curve which would be useful for practical values of σ. In this paper we show that indeed for low values of σ ($< 25^o$) we can derive a generalized equation which approximates the spatial correlation for any central angle-of-arrival for a Gaussian angular energy distribution, a uniform angular energy distribution, and a Gaussian spatial distribution.

2 Spatial Correlation

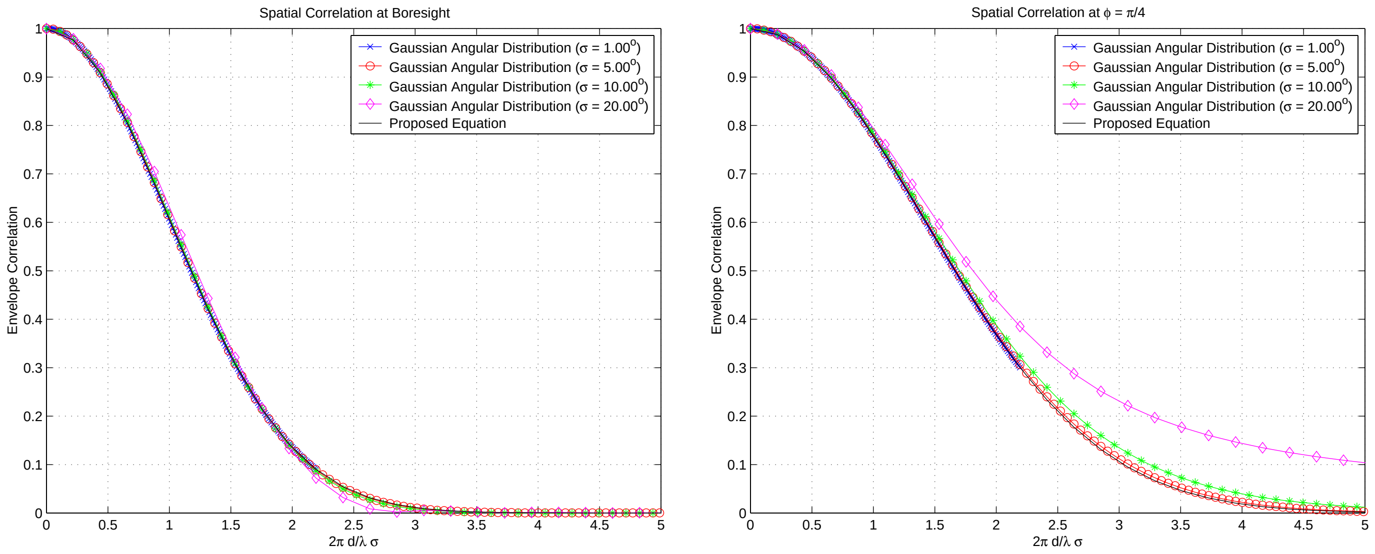
Consider a plane wave signal arriving at an array from angle θ with respect to the normal bisecting two points of interest separated by d meters. The signals seen at the two points can be represented as $s_1(t) = m(t)$ and $s_2(t) = m(t)e^{-j2\pi d/\lambda \sin(\theta)}$. If the power of the message signal $m(t)$ is unity, then $\mathrm{E}\left\{s_1(t)s_2^*(t)\right\} = e^{j2\pi d/\lambda \sin(\theta)}$. Thus, if a signal of interest arriving at an array can be described by the summation of plane waves arriving from angles with distribution $p_\Theta(\theta)$, then we know that the spatial correlation between two points a distance d apart can be determined as [4]

$$\rho(d) = \int_{-\pi}^{\pi} \exp\left\{j2\pi\frac{d}{\lambda}\sin(\theta)\right\} p_\Theta(\theta)d\theta \tag{1}$$

where θ is defined relative to the normal.

First, let us assume a Gaussian distribution for angular energy which is common for spatial channel modeling. Thus, the angular distribution function can be represented as

$$p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(\theta-\phi)^2}{2\sigma^2}\right\} \tag{2}$$



(a) (b)

Figure 1: Exact and Approximated Spatial Correlation Values for Gaussian Angular Energy Distribution with Various Values of σ for (a) $\phi = 0$[bore-sight] and (b) $\phi = \pi/4$

where σ is the standard deviation of the distribution in radians assumed to be small enough that there is not significant energy beyond $\pm\pi$ and ϕ is the central angle of arrival in radians. It is shown in Appendix A that in the case of $p_\Theta(\theta)$ defined by (2) the correlation can be approximated by

$$\rho(d) \approx \exp\left\{j\frac{2\pi d}{\lambda}\sin(\phi)\right\}\exp\left\{-\frac{\left(\frac{2\pi d}{\lambda}\cdot\sigma\cos(\phi)\right)^2}{2}\right\} \tag{3}$$

where we now see that we can parameterize the spatial correlation by $\frac{2\pi d}{\lambda}\sigma\cdot\cos(\phi)$.

In Figure 1 (a) we plot the spatial correlation versus $\frac{2\pi d\sigma}{\lambda}$ using the integral in (1) as well (3) for $\phi = 0$ and several values of σ. We can see that for σ up to 20^o, (3) is a very good fit. Figure 1 (b) presents the same results for $\phi = \frac{\pi}{4}$. As ϕ increases the fit is not as good for moderate values of σ. Figure 5 (a) plots (3) for four values of ϕ. Using this single plot, we can determine the correlation for almost any scenario, provided that the angular spread is Gaussian with a standard deviation less than about $20^o - 25^o$.

3 Antenna Separation

Using the above equation (3), we can easily show that the required distance to ensure a correlation $|\rho|$ can be approximated by

$$\frac{d}{\lambda} = \frac{\sqrt{-2\ln(|\rho|)}}{2\pi\sigma\cos(\phi)} \tag{4}$$

The above equation is plotted for $|\rho| = 0.5$ for four values of ϕ in Figure (2). From this plot we see that the classic rule-of-thumb of 10λ spacing holds for $1 \leq \sigma \leq 2$. It can be shown that for $\phi = 0$ $d = \frac{11}{\sigma}$ where sigma is in degrees provides a good approximation for required antenna separation to achieve correlation values below 0.5.

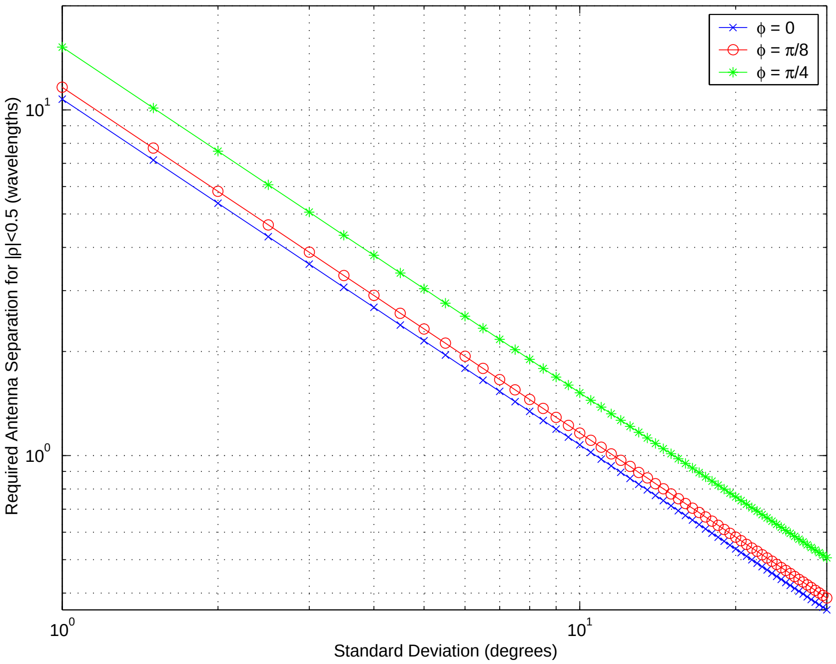


Figure 2: Minimum Required Separation Distance for Various Values of ϕ to Insure $\rho < 0.5$)

4 Gaussian Spatial Distribution

A second model for a spatial channel which is commonly used is a Gaussian spatial distribution. A Gaussian spatial distribution models the scatterers surrounding the mobile using a bi-variate Gaussian distribution in space [5]. In other words, the scatterers have position $[x, y]$ with probability

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sigma_s^2}\exp\left(-\frac{(x-x_o)^2+(y-y_o)^2}{2\sigma_s^2}\right) \tag{5}$$

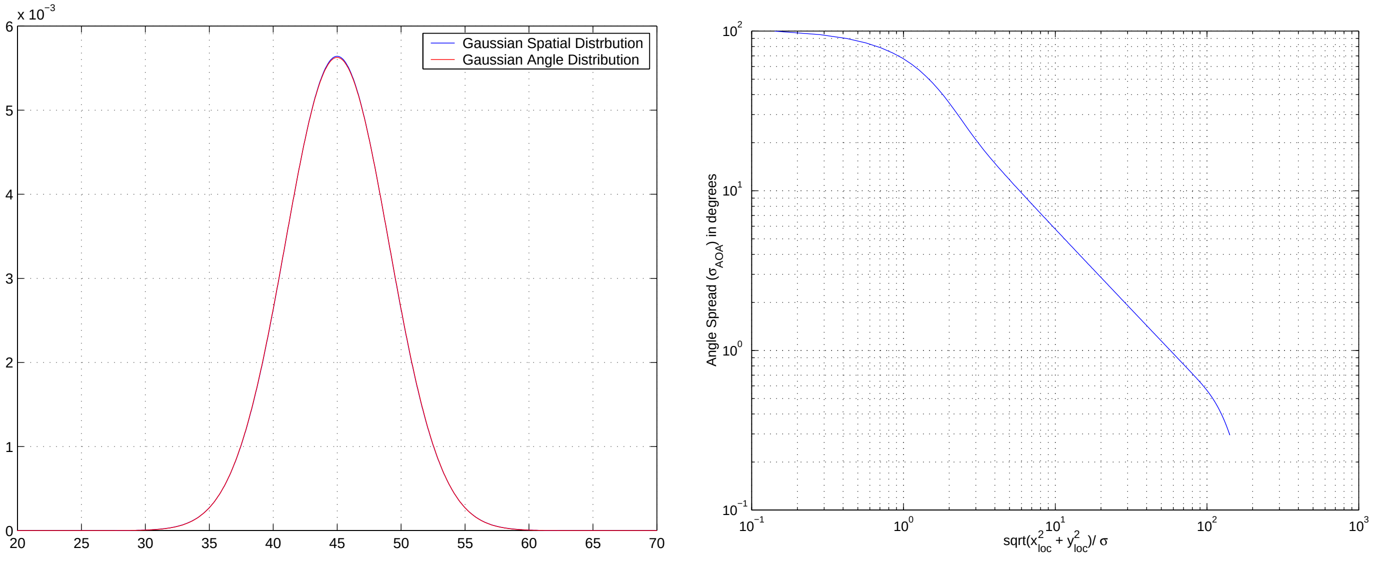
where σ_s is the standard deviation in both the x and y directions and $[x_o, y_o]$ is the center of the distribution. We wish to find the distribution of the angle-of-arrival, i.e.,the angular energy distribution in order to determine the spatial correlation. To do so we make the substitutions $x = r\sin(\theta)$, $y = r\cos(\theta)$, and $dxdy = rdrd\theta$ into (5). Making this substitution and integrating over r results in

$$\begin{aligned} p_\Theta(\theta) &= \frac{1}{2\pi}\exp\left(-\frac{x_o^2+y_o^2}{2\sigma_s^2}\right) + \frac{1}{\sqrt{2\pi}\sigma_s}\left(x_o\sin(\theta)+y_o\cos(\theta)\right)\cdot \\ &\quad \exp\left(-\frac{x_o^2+y_o^2-(x_o\sin(\theta)+y_o\cos(\theta))^2}{2\sigma_s^2}\right)\mathrm{Q}\left(-\frac{x_o\sin(\theta)+y_o\cos(\theta)}{\sigma_s}\right) \end{aligned} \tag{6}$$

Further if we define $R = \sqrt{x_o^2+y_o^2}$ and $\phi = \operatorname{atan}(x_o/y_o)$ we can show that

$$p_\Theta(\theta) = \frac{1}{2\pi}\exp\left(-\frac{R^2}{2\sigma_s^2}\right) + \frac{(R\cos(\theta-\phi))}{\sqrt{2\pi}\sigma_s}\cdot\exp\left(-\frac{R^2\sin^2(\theta-\phi)}{2\sigma_s^2}\right)\mathrm{Q}\left(-\frac{R\cos(\theta-\phi)}{\sigma_s}\right) \tag{7}$$

Thus for small values of $\theta - \phi$ this is nearly identical to a Gaussian distribution and thus we can use a Gaussian distribution to model the angular energy distribution. Figure 3 (a) plots (6) with $[x_o = 300, y_o = 300]$ and $\sigma_s = 30$ as well as a Gaussian angular energy distribution with $\phi = 45$ and



(a) (b)

Figure 3: (a) Comparison of Angular Energy Distribution Arising from a Gaussian Spatial Distribution $[x_o = y_o = 300,\ \sigma_s = 30]$ with a Gaussian Angular Energy Distribution $[\sigma = 4^o, \phi = 45^o]$ and (b) Translation Between Gaussian Spatial Distribution Parameters (x_o, y_o, σ_s) and the Corresponding Gaussian Angular Distribution Variance

$\sigma = 4^o$ (0.07 radians). Thus, we can see that we can approximate the angular energy distribution for a Gaussian spatial distribution with a Gaussian angular energy distribution and thus use (3) to approximate the correlation. To make the translation we can use the plot in Figure 3 (b) to convert the spatial parameters to a standard deviation to use in (1). For example, if we assume a a Gaussian spatial distribution of scatterers with $[x_o, y_o] = [300, 300]$ and $\sigma_s = 30$, we find that $\frac{\sqrt{x_o^2+y_o^2}}{\sigma_s} \approx 14$. Using Figure 3 (b), we find that this corresponds to a standard deviation of approximately 4^o. Alternatively, we can see from (7) that for large R/σ_s and small $\theta - \phi$, we can approximate (7) using a Gaussian with $\sigma = \frac{\sigma_s}{R}$. This corresponds to the linear region of Figure 3 (b). Thus, we can use $\sigma = \frac{180}{\pi} \frac{30}{\sqrt{2}*300} \approx 4^o$ in equation (3) to approximate spatial correlation.

5 Uniform Distribution

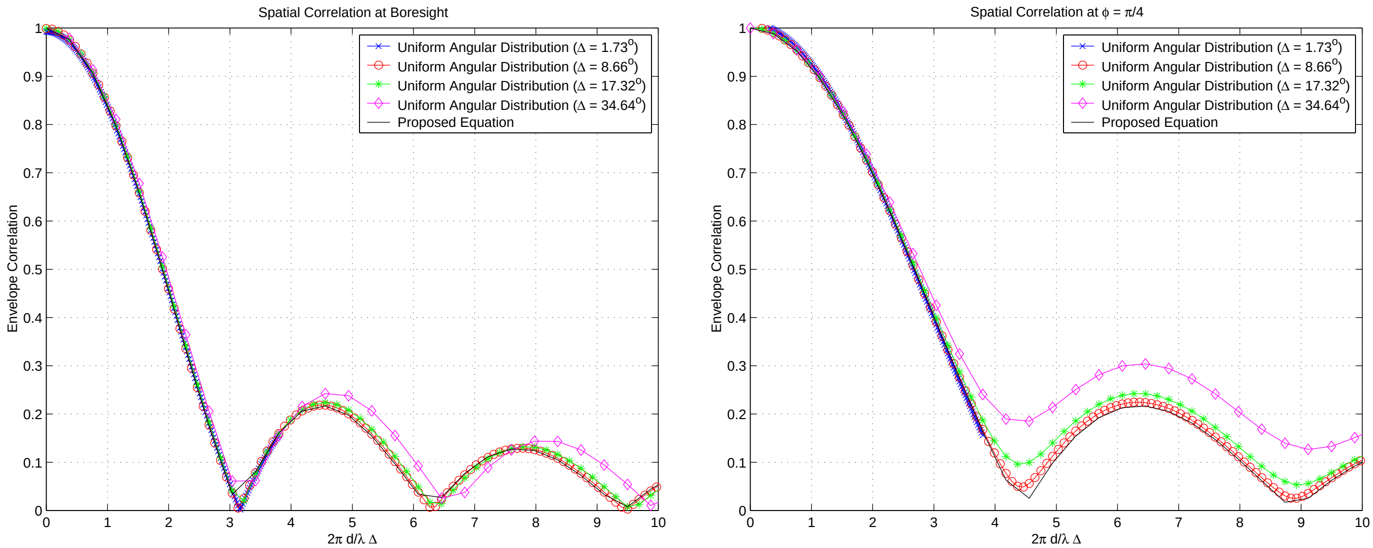
Another common assumption for angular energy distribution is a uniform distribution [2]. A uniform distribution of angular energy is defined as

$$p_\Theta(\theta) = \frac{1}{2\Delta} \qquad \phi - \Delta \leq \theta \leq \phi + \Delta \tag{8}$$

where 2Δ is the range of angles about a central angle-of-arrival ϕ. It is shown in Appendix B that the spatial correlation in this case can be approximated by

$$\rho(d) \approx \exp\left(j\frac{2\pi d}{\lambda}\sin(\phi)\right) \operatorname{sinc}\left(\frac{2\pi d}{\lambda}\cos(\phi)\,\Delta\right) \tag{9}$$

In Figure 4 the spatial correlation is plotted using a numerical integration of (1) along with the approximation in (9) for $\phi = 0$ and $\phi = \pi/4$. Again, for low to moderate values of Δ we find that (9) is a very good approximation. Again, the approximation is less accurate for larger values of ϕ.



(a) (b)

Figure 4: Exact and Approximated Spatial Correlation Values for Uniform Angular Energy Distribution with Various Values of Δ for (a) $\phi = 0$[bore-sight] and (b) $\phi = \pi/4$

If we substitute $\sigma = \frac{\Delta}{\sqrt{3}}$ (the standard deviation of a uniform distribution) we can compare the Gaussian and uniform distributions for several values of ϕ as shown in Figure 5 (a). As expected, the Gaussian distribution decreases more slowly in the main lobe, but lacks the secondary correlation peaks. Otherwise the approximate correlation functions are similar. Additionally, it can be shown that as σ gets large, the Gaussian function approaches a uniform distribution due to a 2π ambiguity and the correlation function will develop secondary peaks.

As a final note, we compare the results presented here with results given in [6]. In [6] a generalized correlation function was derived for general angular energy distributions. Specifically, it is shown that

$$\rho(d) \approx exp\left\{-23\Lambda^2\left(\frac{d}{\lambda}\right)^2\right\} \tag{10}$$

where Λ is a measure of angular spread defined as

$$\Lambda = \sqrt{1 - \frac{|F_1|^2}{|F_0|^2}} \tag{11}$$

and F_n is the nth complex Fourier coefficient

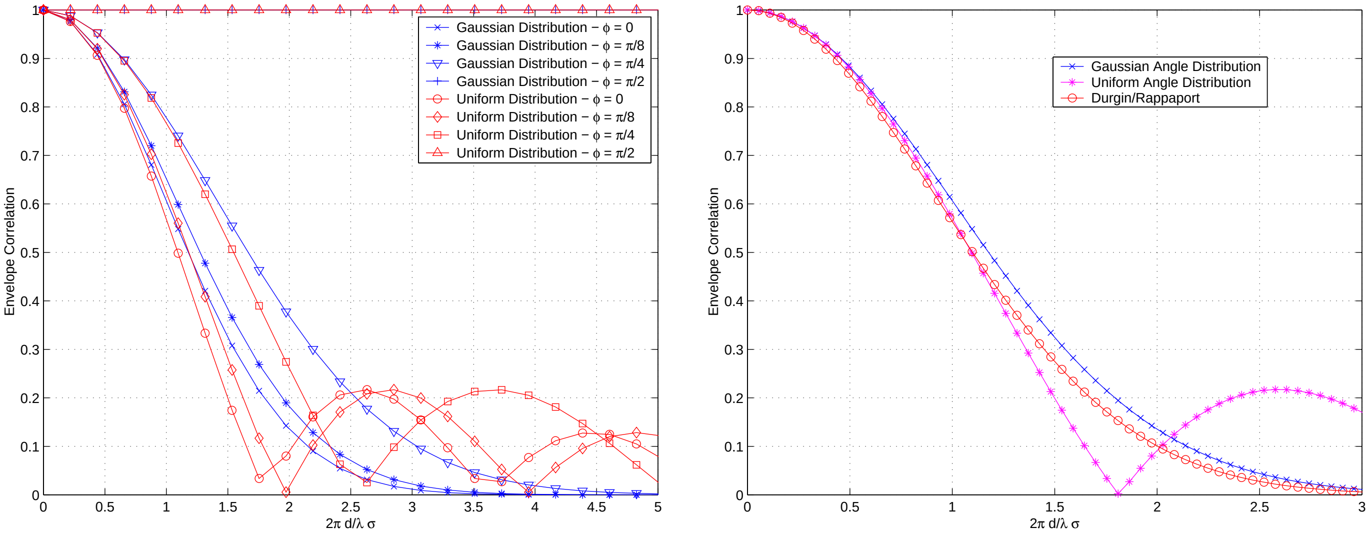
$$F_n = \int_0^{2\pi} p_\Theta(\theta) e^{jn\theta} d\theta \tag{12}$$

Using these definitions we can show that the correlation function for a Gaussian distribution can be approximated as

$$\rho(d) \approx \exp\left(-23\left(1 - e^{-\sigma^2}\right)\left(\frac{d}{\lambda}\right)^2\right) \tag{13}$$

In fact if the approximation $1 - e^{-\sigma^2} \approx \sigma^2$ then we arrive at

$$\rho(d) \approx \exp\left(-23\left(\frac{\sigma d}{\lambda}\right)^2\right) \tag{14}$$



(a) (b)

Figure 5: Comparison of Gaussian and Uniform Distributions (a) and Comparison of the Equations Developed in this Paper and those in [6]

which is very nearly our approximation when we realize that $2\pi^2 = 19.7$. Figure 5 plots this approximation of the correlation function versus $\frac{d}{\lambda}2\pi\sigma$ along with (3) and (9). As can be seen, there is very good agreement down to correlation values of 0.5. The approximation of [6] is slightly optimistic for a Gaussian distribution and would be more so for values of $\phi > 0$. This makes sense since ϕ is not directly involved in (13).

6 Conclusions

In this paper we have derived generalized (i.e.,for multiple values of σ) correlation functions for three distributions of angular energy. The generalized equations allow the correlation to be found for any practical standard deviation and distance. We have shown that the approximations are good for standard deviations of about 25^o or less.

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Appendix A: Derivation of Generalized Equation for Gaussian Distribution

First, let us assume a Gaussian distribution for angular energy such that the angular distribution function can be represented as

$$p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(\theta-\phi)^2}{2\sigma^2}\right\} \tag{15}$$

where σ is the standard deviation of the distribution in radians and ϕ is the central angle of arrival in radians.

Then we know that the spatial correlation can be determined as [4]

$$\rho(d) = \int_{-\pi}^{\pi} \exp\left\{j2\pi\frac{d}{\lambda}\sin(\theta)\right\} p_\Theta(\theta)d\theta \tag{16}$$

Now, substituting (2) into (16) and making a change of variables we get

$$\begin{aligned}\rho(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}\sin(\sigma z+\phi)\right\}\exp\left\{-\frac{z^2}{2}\right\}dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}\left(\sin(\sigma z)\cos(\phi)+\cos(\sigma z)\sin(\phi)\right)\right\}\exp\left\{-\frac{z^2}{2}\right\}dz\end{aligned} \tag{17}$$

Now, assuming that σz is small over the range where $\exp\left(-\frac{z^2}{2}\right)$ is significant, we can approximate the above with

$$\begin{aligned}\rho(d) &\approx \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}\left(\sigma z\cos(\phi)+\sin(\phi)\right)\right\}\exp\left\{-\frac{z^2}{2}\right\}dz \\ &\approx \frac{\exp\left\{j\frac{2\pi d}{\lambda}\sin(\phi)\right\}}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\exp\left\{j2\pi\frac{d}{\lambda}\left(\sigma z\cos(\phi)\right)\right\}\exp\left\{-\frac{z^2}{2}\right\}dz\end{aligned} \tag{18}$$

Evaluating the integral [7] then gives

$$\boxed{\rho(d) \approx \exp\left\{j\tfrac{2\pi d}{\lambda}\sin(\phi)\right\}\exp\left\{-\frac{\left(\frac{2\pi d}{\lambda}\sigma\cos(\phi)\right)^2}{2}\right\}}$$

Q.E.D

Appendix B: Derivation of Generalized Equation for Uniform Distribution

In this appendix we wish to show that the spatial correlation function for a uniform angular energy distribution can be approximated according to (9). First we assume that the angular energy is distributed according to

$$p_\Theta(\theta) = \frac{1}{2\Delta} \qquad \phi - \Delta \leq \theta \leq \phi + \Delta \tag{19}$$

$$\begin{aligned}
\rho(d) &= \frac{1}{2\Delta}\int_{\phi-\Delta}^{\phi+\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(\theta)\right)d\theta \\
&= \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(z+\phi)\right)dz \\
&= \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}(\sin(z)\cos(\phi)+\cos(z)\sin(\phi))\right)dz
\end{aligned} \tag{20}$$

Now for small Δ, we can approximate $\sin(z) \approx z$ and $\cos(z) \approx 1$ which gives

$$\rho(d) = \frac{1}{2\Delta}\exp\left(j\frac{2\pi d}{\lambda}\sin(\phi)\right)\int_{-\Delta}^{\Delta}\exp\left(j\frac{2\pi d}{\lambda}\cos(\phi)\,z\right)dz$$

Evaluating the integral [7] then gives

$$\boxed{\rho(d) = \exp\left(j\tfrac{2\pi d}{\lambda}\sin(\phi)\right)\operatorname{sinc}\left(\tfrac{2\pi d}{\lambda}\cos(\phi)\,\Delta\right)}$$

Q.E.D