The Impact of Channel Estimation Error on Space-Time Block Codes

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Abstract—In this paper we examine the impact of channel estimation error on the performance of 2 and 4 antenna space-time block codes. This is examined specifically in the context of space-time spreading as defined for the cdma2000,standard [1], [2]. The performance is determined by finding an expression for the probability of bit error when using BPSK or QPSK modulation and modeling the channel estimation error as a Gaussian random variable.

I. Introduction

Third generation CDMA cellular systems incorporate a downlink transmission technique called Transmit Diversity (TD). Transmit Diversity has been studied in various forms by several researchers. A common method of transmit diversity is the use of a simple two antenna space-time block code (STBC) [3], [4], [1], [2]. Such codes achieve diversity advantage (without suffering rate reductions for the two antenna case) by relying on orthogonal matrix formulation. For schemes with coherent detection, the degree to which the transmit diversity signals remain orthogonal after receiver processing depends on the quality of the channel estimation. In this paper we examine the sensitivity of coherent two-antenna and four-antenna space-time block codes to channel estimation error and compare it to the sensitivity of ideal 2-way and 4-way transmit diversity.

II. Space Time Spreading (STS)

Diversity is a common method for combatting the fading experienced in wireless channels. Motivated by the desire for simple receivers at the mobile unit, techniques for adding diversity on the downlink without requiring an additional receive antenna, wasting bandwidth, or causing self-interference have been explored. Space-Time Spreading (STS) is one such technique and is an extension of the work by Tarokh and Alamouti [4], [3] applied to CDMA [2], [5], [1].

Consider a CDMA system where Walsh codes are used on the downlink for code division multiplexing. Let's consider a single user, labeled user i. For space-time spreading with two antennas and Walsh codes of length N, we transmit on the first antenna

$$\begin{aligned} x_1(t) &= \left(\sqrt{\frac{P_i}{2}} s_e(t) w_i(t) - \dots \right. \\ & \left. \sqrt{\frac{P_i}{2}} s_o^*(t) \overline{w}_i(t) + \sqrt{P_{p1}} w_{p1} \right) p(t) \quad (1) \end{aligned}$$

and on the second antenna we transmit,

$$\begin{aligned} x_2(t) &= \left(\sqrt{\frac{P_i}{2}} s_e^*(t) \overline{w}_i(t) + \dots \right. \\ & \left. \sqrt{\frac{P_i}{2}} s_o(t) w_i(t) + \sqrt{P_{p2}} w_{p2} \right) p(t) \quad (2) \end{aligned}$$

where s_e and s_o are the even and odd symbols respectively, $p(t)$ is a pseudo-random scrambling code, P_i

is the transmit power for user i, $w_i(t)$ is an extended Walsh code and $\overline{w}_i(t)$ is its compliment where both are length $2N$ and derived from the original Walsh code assigned to user i [1]. The variables P_i and w_{pi} represent the transmit power and Walsh code dedicated to the i^{th} transmit antenna respectively. Note that a separate Walsh code is required for each transmit antenna to support a pilot on each antenna. These Walsh codes are associated with the full rate of transmission and are length N. At the receiver, we uncover and correlate with the two Walsh codes. At the output of the two Walsh correlations we obtain

$$z_1 = \sqrt{\frac{P_1}{2}} h_1 s_1 + \sqrt{\frac{P_2}{2}} h_2 s_2 + n_1 \tag{3}$$

$$z_2 = \sqrt{\frac{P_1}{2}} h_2 s_1^* - \sqrt{\frac{P_2}{2}} h_1 s_2^* + n_2 \tag{4}$$

where h_1 and h_2 are the channels experienced by the signals from antenans 1 and 2 respectively. Obviously this introduces interference terms in the decision statistics. However, if we have estimates of the channel distortions $\hat{h}_1$ and $\hat{h}_2$ from pilot signals 1 and 2, we can obtain a symbol estimate for the even symbols as

$$\begin{aligned} \hat{s}_e &= f\left\{\hat{h}_1^* z_1 + \hat{h}_2 z_2^*\right\} \\ &= f\left\{\hat{h}_1^* \left(\sqrt{\frac{P_1}{2}} h_1 s_1 + \sqrt{\frac{P_2}{2}} h_2 s_2 + n_1\right) + \ldots \right. \\ &\qquad \left. \hat{h}_2 \left(\sqrt{\frac{P_1}{2}} h_2 s_1^* - \sqrt{\frac{P_2}{2}} h_1 s_2^* + n_2\right)^* \right\} \\ &= f\left\{\left(\sqrt{\frac{P_1}{2}} |h_1|^2 + \sqrt{\frac{P_1}{2}} |h_2|^2\right) s_1 + \ldots \right. \\ &\qquad h_1^* n_1 + h_2 n_2^* \} \end{aligned} \tag{5}$$

where we've assumed that the channel estimation is perfect, $\hat{h}_1 = h_1$ and $\hat{h}_2 = h_2$. Similarly, we can estimate the symbol for the odd symbols as

$$\begin{aligned} \hat{s}_o &= f\left\{\hat{h}_2^* z_1 - \hat{h}_1 z_2^*\right\} \\ &= f\left\{\left(\sqrt{\frac{P_2}{2}} |h_1|^2 + \sqrt{\frac{P_2}{2}} |h_2|^2\right) s_2 + \ldots \right. \\ &\qquad h_2^* n_1 - h_1 n_2^* \} \end{aligned} \tag{6}$$

It can be easily shown that this is identical to the decision statistic for two-antenna diversity reception (without the 3dB aperture gain)[6]. Thus, we achieve symbol level diversity gain (i.e., before the decoding) without employing additional antennas at the receiver, without requiring additional bandwidth, and without causing self-interference.

Further, this scheme can be extended to four transmit antennas by using 4 double extended Walsh codes along with a 3/4 rate space-time block code [4], [7]. This will increase the diversity order to 4 with a rate reduction penalty of 3/4.

A. The Impact of Channel Estimation Error

As we have shown, the decision statistics for STS rely on the elimination of cross-terms by a simple linear combination of despreader outputs. The removal of the cross-terms was shown to be complete when channel estimation is perfect (i.e.,when $\hat{h}_i = h_i$). In general this is not true since estimation is never perfect. As a result we expect that STS is more sensitive to channel estimation then straightforward two branch diversity. That is, both STS and two-branch diversity will be sensitive to channel estimation error in the demodulation process when coherent demodulation is used. However, STS will have the added penalty of residual cross-terms. In this section we analyze this performance difference. Returning to the decision statistic for STS (5)

$$\begin{aligned} \hat{s}_e &= f\{z_e\} \\ &= f\left\{\hat{h}_1^* z_1 + \hat{h}_2 z_2^*\right\} \\ &= f\left\{\hat{h}_1^* \left(\sqrt{\frac{P_1}{2}} h_1 s_1 + \sqrt{\frac{P_2}{2}} h_2 s_2 + n_1\right) \ldots \right. \\ &\qquad \left. + \hat{h}_2 \left(\sqrt{\frac{P_1}{2}} h_2 s_1^* - \sqrt{\frac{P_2}{2}} h_1 s_2^* + n_2\right)^* \right\} \\ &= f\left\{(h_1 + e_1)^* \left(\sqrt{\frac{P_1}{2}} h_1 s_1 + \sqrt{\frac{P_2}{2}} h_2 s_2 + n_1\right) + \ldots \right. \\ &\qquad \left. (h_2 + e_2) \left(\sqrt{\frac{P_1}{2}} h_2 s_1^* - \sqrt{\frac{P_2}{2}} h_1 s_2^* + n_2\right)^* \right\} \end{aligned}$$

$$\begin{aligned} &= f\left\{\left(\sqrt{\frac{P_1}{2}}|h_1|^2 + \sqrt{\frac{P_1}{2}}|h_2|^2 + h_1 e_1^* + \dots\right.\right. \\ &\quad h_2^* e_2)\, s_1 + \sqrt{\frac{P_1}{2}} h_2 e_1^* s_1 - \sqrt{\frac{P_1}{2}} h_1^* e_2 s_o + \dots \\ &\quad h_1^* n_1 + h_2 n_2^* + e_1^* n_1 + e_2 n_2^* \} \end{aligned}$$

where we have modeled the channel estimation error as $\hat{h}_i = h_i + e_i$ and e_i is a Gaussian random variable with power inversely related to the channel estimation accuracy $\sigma_e^2 = \frac{1}{SNR_{ch}}$. Now, assuming BPSK we can represent the probability of bit error as

$$P_e = Prob\left\{Re\left\{s_e z_e\right\} < 0\right\} \tag{7}$$

The probability of error P_e can be shown to be equivalent to $Prob\left\{\mathsf{x}^\dagger \mathsf{M}\, \mathsf{x} < 0\right\}$ where

$$\mathsf{x} = \begin{bmatrix} h_1 \\ n_1 \\ e_1 \\ h_2 \\ n_2 \\ e_2 \end{bmatrix} \tag{8}$$

is a Gaussian vector with correlation matrix R

$$\mathsf{R} = \begin{bmatrix} P_1 & 0 & 0 & \sqrt{P_1 P_2}\rho & 0 & 0 \\ 0 & \sigma_n^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 & 0 & 0 \\ \sqrt{P_1 P_2}\rho & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_n^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} \tag{9}$$

where ρ is the correlation between channels h_1 and h_2 and the matrix M is defined as

$$\mathsf{M} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{8}} & 0 & 0 & \frac{1}{\sqrt{8}} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{2} & 0 & \frac{1}{\sqrt{8}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{8}} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{8}} & 0 & 0 & \frac{1}{\sqrt{8}} & \frac{1}{2} & 0 \end{bmatrix} \tag{10}$$

Further, it can be shown that $P_e | \, s_e = P_e | \, s_o$. Note that since we have assumed BPSK, this is both the probability of symbol error and bit error. This result also applies to the probability of bit error for QPSK. Now, in order to determine the probability of error we must determine the probability distribution function of $z = \mathsf{x}^\dagger \mathsf{M}\, \mathsf{x}$. To accomplish this, we first determine the characteristic funtion of z denoted as $\Phi_z(s)$:

$$\begin{aligned} \Phi_z(s) &= \mathrm{E}\left\{e^{-sz}\right\} \\ &= \frac{1}{\pi^3 |\mathsf{R}|} \int e^{-s\mathsf{x}^\dagger \mathsf{M}\, \mathsf{x}} e^{-\mathsf{x}\, \mathsf{R}^{-1} \mathsf{x}} d\mathsf{x} \\ &= \frac{1}{|s\mathsf{M} - \mathsf{R}^{-1}|\, |\mathsf{R}|} \\ &= \frac{1}{|s\mathsf{M} R - \mathsf{I}|} \end{aligned} \tag{11}$$

The probability distribution function of z is then

$$f(z) = \begin{cases} \sum_{i=1}^{n} -\frac{k_i}{\lambda_i} e^{-\frac{z}{\lambda_i}} & z < 0 \\ \sum_{i=n+p+1}^{N} \frac{k_i}{\lambda_i} e^{-\frac{z}{\lambda_i}} & z > 0 \end{cases} \tag{12}$$

where λ_i with $1 \le i \le n$ are the negative eigenvalues of $\mathsf{M}\, \mathsf{R}$, λ_i with $n + p < i \le N$ are the positive eigenvalues of $\mathsf{M}\, \mathsf{R}$, N is the dimension of R and M, and λ_i with $n < i \le n + p$ are the zero eigenvalues. The coefficients k_i are the residues of $\Phi_z(s)$ evaluated at λ_i. For distinct eigenvalues, $k_i = \prod_{k \neq i} \frac{\lambda_i}{\lambda_i - \lambda_k}$. The probability of error is then the integration of $f(z)$ from $-\infty$ to 0

$$\begin{aligned} P_e &= \int_{-\infty}^{0} f(z) dz \\ &= \int_{-\infty}^{0} \sum_{i=1}^{n} -\frac{k_i}{\lambda_i} e^{-\frac{z}{\lambda_i}} dz \\ &= \sum_{i=1}^{n} -\lambda_i \left(-\frac{k_i}{\lambda_i}\right) e^{-\frac{z}{\lambda_i}} \Bigg|_{-\infty}^{0} \\ &= \sum_{i=1}^{n} k_i \end{aligned} \tag{13}$$

In the preceding analysis we have assumed that the eigenvalues of $\mathsf{M}\, \mathsf{R}$ are distinct. However, if they are not distinct the result is similar. That is, the probability of error is the sum of the residues of $\Phi_z(s)$ corresponding to the negative eigenvalues of $\mathsf{M}\, \mathsf{R}$. To test this expression we simulated STS and two branch receive diversity (with one half of the transmit power to remove the aperture gain). In Figure 1 we plot the simulated performance of STS and two branch diversity in Rayleigh fading with a channel estimation SNR of 10dB and independent channels (i.e.,$\rho = 0$). We have plotted both the simulated performance of STS as well as the predicted performance from (13). Also plotted are the theoretical performance of one branch and two branch diversity in Rayleigh fading with perfect channel estimation. Clearly the presence of the cross terms in STS introduces more sensitivity to channel estimation than traditional two branch diversity. Further, we see that the predicted performance from (13) matches the simulated performance well. Figure 2 further shows the relative sensitivity of STS and two branch diversity to channel estimation. Clearly, a channel estimation SNR of 20dB or better provides nearly equialent performance between the two. As a point of reference, in cdma2000,the pilot is typically transmitted at -7dB of full transmit power (i.e., $\frac{E_c}{I_{or}}$=-7dB) and a typical receiver signal-to-noise ratio is $\frac{I_{or}}{I_{oc}} = 6dB$. A simple channel estimation scheme simply creates an average over one power control group which corresponds to 1.67ms or 1536 chips. Thus, the channel estimation SNR is SNR = -7dB + 6dB + $10 \log 1536 \approx$ 30dB. Thus, for cdma2000,channel estimation should not cause problems for STS.

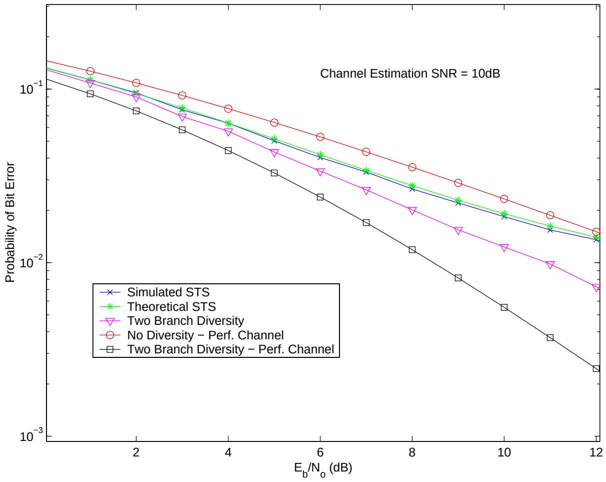

Fig. 1. Effect of Channel Estimation on STS and Two Branch Diversity (Channel Estimation SNR = 10dB)

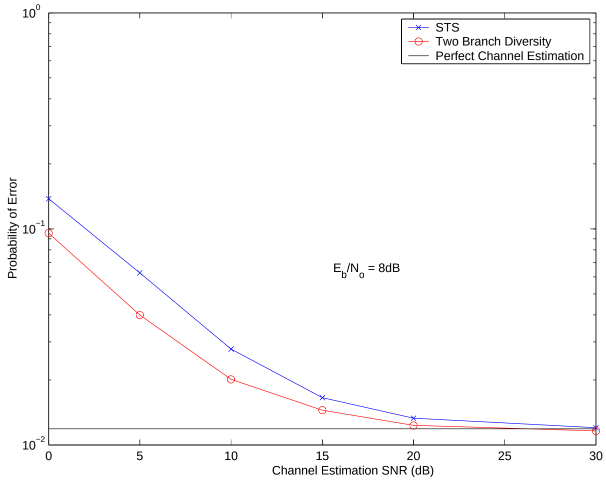

Fig. 2. Effect of Channel Estimation on STS and Two Branch Diversity ($\mathsf{E_b/N_o}$ = 8dB)

We can extend this analysis to four antennas by using four double extended Walsh codes of length $4N$ [7] and the transmission matrix for four antennas from [4]. The analysis is the similar except that there each of the four decision statistics is the linear combination of four outputs leading to 12 total terms including channel, AWGN and estimation error for each antenna. The matrix R is extended simply from (9) to a 12×12 matrix and M becomes

$$\mathsf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 \end{bmatrix} \tag{14}$$

The resulting performance of STS with channel estimation error is plotted in Figures 3 and 4. Again, we plot the performance of four-antenna STS (simulated and theory) as well as four-antenna receive diversity (minus the 6dB aperture gain) with and without channel estimation error. Also plotted for reference is the performance with no diversity and perfect channel estimation. As in the two antenna case, the sensitivity of STS to channel estimation error is greater than simple four-branch diversity due to the existence of cross-terms. Further, with four transmit antennas, less power can be afforded to the pilots per antenna. Thus, either longer estimation times are needed or channel estimation error will be higher than for the two antenna case. Note this analysis can be equally applied to any variation of space-time block codes not just STS, as the decision statistics apply equally well. Additionally, the analytical approach can be extended to an arbitrary number of antennas.

References

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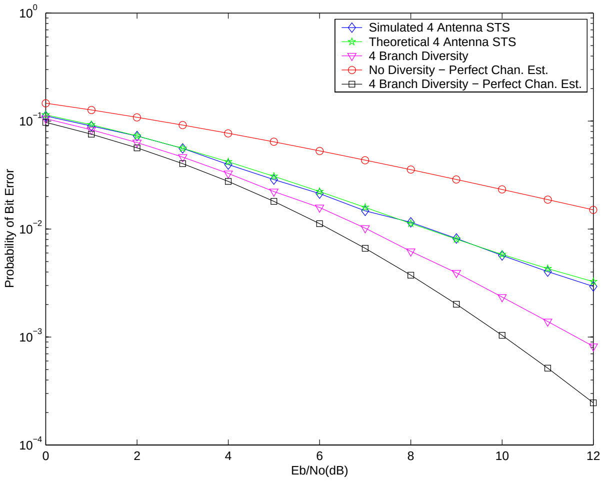

Fig. 3. Effect of Channel Estimation on 4 Antenna STS and Four Branch Diversity (Channel Estimation SNR = 10dB)

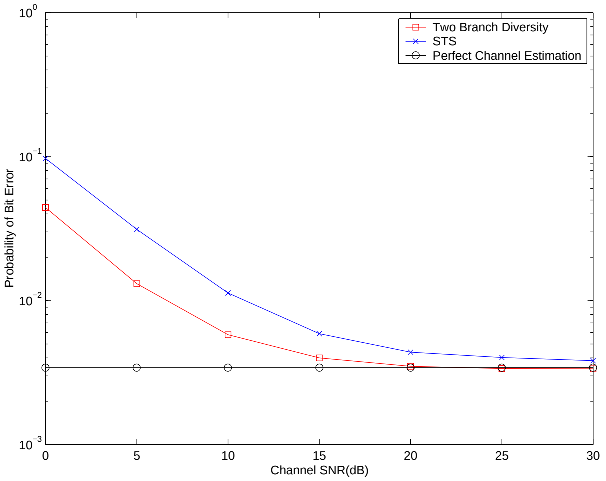

Fig. 4. Effect of Channel Estimation on 4 Antenna STS and Four Branch Diversity ($E_b/N_o = 8$dB)