

Chapter 8: Applications of Weak FIP

*“Two roads diverged in a wood, and I,
I took the one less traveled by,
And that has made all the difference.”*
- R. Frost, The Road Not Taken

In a cognitive radio network, each cognitive radio repeatedly makes choices that impact the evolution of the network state. But unlike Robert Frost, cognitive radios may have the chance to make the same choice again and again if play does not converge. However, in networks that can be modeled as a game with weak FIP, there always exists at least one choice that “makes all the difference” and adaptations will lead to an NE.

In Chapter 4, we identified weak FIP as a critical property for the convergence of cognitive radio networks. Specifically, we asserted that without weak FIP, a cognitive radio network could not be guaranteed to converge to an NE if the radios are making myopic individually rational adaptations under round-robin or random timing. However, for games with weak FIP there always exists some improvement path that leads to an NE, so this repetition must come to an end in a game with weak FIP.

While we have made an extensive discussion of FIP (which implies weak FIP), we have not discussed any specific techniques for identifying when a network has weak FIP without FIP nor have we presented any cognitive radio applications of weak FIP. This chapter addresses these shortcomings and discusses a readily identified game model that can be used to establish that a game has weak FIP (Section 8.1 – Supermodular games), and two cognitive radio applications where weak FIP occurs – ad-hoc power control (Section 8.2) and sensor network formation (Section 8.3). For practical considerations, this chapter does not provide the same deliberate presentation given in the preceding chapters. However, the most widely applicable insights into how to identify when a cognitive radio network has weak FIP are covered – demonstration that the game is a finite supermodular game and identification of an everywhere convergent improvement path..

8.1 Supermodular Games

A normal form game, $\Gamma = \langle N, A, \{u_i\} \rangle$ is termed a *supermodular game* if the action space forms a *lattice* and the utility functions are *supermodular*. A partially ordered set, X , is termed a lattice if for all $a, b \in X$, $a \wedge b \in X$ and $a \vee b \in X$ where $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. A function $f : X \rightarrow \mathbb{R}$ where X is a lattice, is termed supermodular if for all $a, b \in X$, $f(a) + f(b) \leq f(a \wedge b) + f(a \vee b)$.

8.1.1 Model Identification

While the definition may seem complicated, a game can be identified as a supermodular game if all players' utility functions satisfy the relationship given in (8.1) and the action space is compact subset of real space [Milgrom_90]. Such a game is called a *smooth supermodular game*.

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} \geq 0 \forall j \neq i \in N \quad (8.1)$$

8.1.2 Steady-states

As shown in [Topkis_98], the best response function for a supermodular game is a monotonic function of a , i.e., $a^1 \geq a^2 \Rightarrow \hat{B}(a^1) \geq \hat{B}(a^2)$. By Tarski's fixed point theorem given in [Topkis_98], monotonic functions on a compact space (not necessarily convex) have a fixed point. Thus the best response function for a supermodular game has a fixed point, which implies the game must have at least one NE.

By [Topkis_98], all NE for a game form a lattice. While this does not particularly aid in the process of initially identifying NE, from every pair of identified NE, e.g., a^* and b^* , additional NE can be found by evaluating $a^* \wedge b^*$ and $a^* \vee b^*$. In general, NE identification for a supermodular game has to proceed as it did in Chapter 4 – by simultaneously solving the system of best response equations for fixed points.

More usefully, it is possible to establish a condition where a supermodular game has a unique NE by leveraging the Standard Interference Function of [Yates_95] which is an example of a monotonic best response function. This novel result turns out to be very useful for convergence and stability properties of supermodular games.

Theorem 8.1: NE Uniqueness in a Supermodular Game (*)

Given a supermodular game with a real convex compact action space, suppose the best response function satisfies the following conditions:

1. *Uniqueness* - $\{b_i \in A_i : u_i(b_i, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in A_i\}$ is a singleton for all a .
2. *Positivity* - $\hat{B}(a) > 0$
3. *Scalability* - For all $\mathbf{a} > 1$, $\mathbf{a}\hat{B}(a) > \hat{B}(\mathbf{a}a)$.

then the game has a unique NE.

Proof: (Paralleling the proof in [Yates_95] that the Standard Interference Function has a unique fixed point) By Tarski's fixed point theorem, an NE exists for this game. Now suppose that a^* and b^* are both fixed points. By positivity, we know that $a_i^* > 0$ and $b_i^* > 0$ for all $i \in N$. As these are distinct fixed points, there must be some $a_i^* > b_i^*$ (or some $b_i^* > a_i^*$, simply interchange the following comparisons). By scalability, there exists $\mathbf{a} > 1$ such that a) $\mathbf{a}a^* \geq \mathbf{a}b^*$ and b) for some i $b_i^* = \mathbf{a}a_i^*$. Then by the monotonicity of \hat{B} and the scalability property, it must hold that $b_i^* = \hat{B}_i(b^*) \leq \hat{B}_i(\mathbf{a}a^*) < \mathbf{a}\hat{B}_i(a^*) = \mathbf{a}a_i^*$, a contradiction as $b_i^* = \mathbf{a}a_i^*$.

8.1.3 Desirability

In general, little can be said about the desirability or optimality of a supermodular game's NE. However, as we saw for potential games in Chapter 5, an NE in a supermodular game whose utility function satisfies (8.1) can be adjusted by introducing any additive self-motivated function.

8.1.4 Convergence

By [Friedman_01], finite supermodular games have weak FIP, i.e., from any initial action vector, there exists a sequence of selfish adaptations that lead to a NE. Thus the convergence results of Chapter 4 for games with weak FIP apply to supermodular games.

A variation on the simultaneous best response algorithm is presented in [Milgrom_90] wherein the players follow what is termed an *adaptive dynamic process*. In an adaptive dynamic process, all players play a best response to some arbitrary weighting of recent past actions by other players.

Definition 8.1: *Adaptive dynamic process* ([Milgrom_90] (A6))

Formally, a decision rule is defined as an *adaptive dynamic process* if $\forall t^* \in T$ there exists a $t' \in T$ such that $\forall t \geq t'$, $d_i(a^i) \in \bar{U} \left(\left[\inf(P(t^*, t)), \sup(P(t^*, t)) \right] \right)$ where $P(t^*, t)$ denotes the action tuples observed between times t^* and t , $U(a)$ is the list of undominated responses to a for each player, and $\bar{U}(a) = \left[\inf(U(a)), \sup(U(a)) \right]$.

The corollaries to Theorem 8 in [Milgrom_90] show that a smooth supermodular game following an adaptive dynamic process with any timing converges to a region bounded by the Nash equilibrium lattice and that iterative elimination of dominated strategies converges to a region defined by the Nash equilibrium lattice. Note that when the NE is unique, the adaptive dynamic process converges to the NE.

8.1.5 Stability (*)

In general little can be said about the stability of a supermodular game. However, if the best response function satisfies Theorem 8.1, then the unique NE will be asymptotically stable under best response decision rules with any timing with a Lyapunov function given by $L(a) = \|a^* - a\|$ - the same Lyapunov function we defined for the standard interference function. This can be quickly verified by noting that the best response decision rule is monotonic under any timing [Topkis_98] and the uniqueness of the NE implies that every adaptation must bring the network closer to the NE. Interestingly, this implies that the best response function decision rule in a supermodular game constitutes a *pseudo-contraction* – a topic covered in Chapter 3.

8.2 Ad-hoc Power Control¹

As we showed in Chapter 5, when cognitive radios distributed implement power control at a single cluster head where each radio is guided by the utility function given in (8.2), the system forms an ordinal potential game.

$$u_i(\mathbf{p}) = \left| \hat{g}_i - \frac{g_i p_i}{1/K \left(\sum_{k \in N \setminus i} g_k p_k + \mathbf{s} \right)} \right| \quad (8.2)$$

Unfortunately, this result is not so easily extended to ad-hoc networks. Yet, we can still establish broad convergence and stability results for target-SINR power control algorithms in ad-hoc networks.

[Neel_05], considers the analysis of distributed power control in an ad hoc network where each link, j , varies its transmit power in an attempt to achieve a target SINR, γ_j , measured in dB. This scenario can be thought of as analogous to the fixed assignment scenario presented in [Yates_95]. Indeed this analysis can be considered an extension of [Yates_95] to ad-hoc networks with additional consideration given to stability.

8.2.1 Stage Game Model

Based on the preceding discussion, a normal form stage game can be formulated as follows.

- Player Set N – Set of decision making links
- Player Action Set A_j – The real convex, compact set of powers, $[0, p_j^{\max}]$ where p_j^{\max} is the maximum transmit power of cognitive radio j . The action space, $A \subset \mathbb{R}^n$, is given by $A = A_1 \times A_2 \times \dots \times A_n$.
- Utility – An appropriate action based utility function for a target SINR (dB) algorithm is given by (8.3) where \hat{g}_j is the SINR target of cognitive radio j .

¹ This text is taken from [Neel_06].

$$u_j(\mathbf{p}) = - \left(\hat{\mathbf{g}}_j - 10 \log_{10}(g_{jj} p_j) + 10 \log_{10} \left(\frac{\sum_{k \in N \setminus j} g_{kj} p_k + \mathbf{s}_j}{K} \right) \right)^2 \quad (8.3)$$

Here, communications theory provides the necessary connection between action and outcome as SINR. Using the notation we presented in Chapters 3 and 5, in a network, N , of cognitive radios the SINR of the signal transmitted by j and received by its node (radio) of interest measured in dB is given by (8.4) where g_{kj} is the effective fraction of power transmitted by node k that is received at j 's node of interest (receiving end of j 's link) and N_j is the noise at the receiving end of link j .

$$\mathbf{g}_j = 10 \log_{10}(g_{jj} p_j) - 10 \log_{10} \left(\sum_{k \in N \setminus j} g_{kj} p_k + N_j \right) \text{ (dB)} \quad (8.4)$$

8.2.2 Analysis

In [Altman_03] it is claimed that the cellular fixed assignment scenario of [Yates_95] on which this ad-hoc network model is based is supermodular. The following parallels the analysis in [Neel_05] where we showed that this stage game constitutes a supermodular game for an ad-hoc network.

A stage game can be shown to be a smooth supermodular game by applying the second order conditions we presented in Section 8.1. First, notice that the action space forms a complete lattice (compact subset of Euclidean space). Then evaluating the second derivative with respect to p_j and p_k where k is any cognitive radio $k \in N \setminus j$ yields (8.5).²

$$\frac{\partial^2 u_j(p)}{\partial p_j \partial p_k} = \frac{200 g_{kj}}{p_j \left(\sum_{k \in N \setminus j} g_{kj} p_k + N_j \right) \ln(20)} \quad (8.5)$$

As (8.5) is strictly positive, the last conditions for a smooth supermodular game is satisfied. Accordingly, we know that the network

- Has at least one steady state and

² Note that as g_{kj} will not generally equal g_{jk} , this game will not be an exact potential game.

- Converges for synchronous and asynchronous best response algorithms (local optimization).

As (8.5) is not a function of the target SINRs, each radio can have its own target SINR and the game will remain a supermodular game. Further, since the best response algorithm given by (8.6) is a known standard interference function (or equivalently since \hat{B} satisfies Theorem 8.1), we know the following:

- The network has a unique fixed point.
- The network achieves the target SINR vector with the smallest possible power vector (when the SINR vector is feasible).
- A Lyapunov function is given by the distance between the current power vector and the fixed point.

$$\hat{B}_j(\mathbf{p}) = p_j \frac{\hat{\mathbf{g}}_j}{\mathbf{g}_j} p_j^{t_{k+1}} = p_j^k \frac{\bar{\mathbf{g}}_j}{\mathbf{g}_j} \quad (8.6)$$

Finally, for a feasible SINR target vector, the unique steady state for this game can be found by solving the linear system of equations $\mathbf{Z}\bar{\mathbf{p}} = \mathbf{?}$

where $\mathbf{Z} = \begin{bmatrix} h_{1n_1} & -\hat{\mathbf{g}}_1 h_{1n_2} & \cdots & -\hat{\mathbf{g}}_1 h_{1n_n} \\ -\hat{\mathbf{g}}_2 h_{2n_1} & h_{2n_2} & \cdots & -\hat{\mathbf{g}}_2 h_{2n_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\mathbf{g}}_n h_{nn_1} & -\hat{\mathbf{g}}_n h_{nn_2} & \cdots & h_{nn_n} \end{bmatrix}$, $\mathbf{?} = [\hat{\mathbf{g}}_1 N_1 \quad \hat{\mathbf{g}}_2 N_2 \quad \cdots \quad \hat{\mathbf{g}}_n N_n]^T$, and

$$\bar{\mathbf{p}} = [p_1 \quad p_2 \quad \cdots \quad p_n]^T. \text{ Here, } h_{jk} = g_{jk} / K \text{ for } j \neq k \text{ and } h_{jj} = g_{jj}.$$

To determine the desirability of this fixed point, we can consider a network design objective function that seeks the minimum power vector that provides at least the target SINR for all links. For this design objective function, the fixed point power vector is optimal as no smaller power vector achieves the target SINR (assuming the target SINRs are feasible) because otherwise the smaller power vector would also be a fixed point in violation of a property of being a Standard Interference Function.

8.2.3 Validation

Consider the ad hoc network shown in Figure 8.1 where at a particular frequency each terminal is attempting to maintain a target SINR at a cluster head and each cluster head is maintaining a target SINR at the gateway node. The signals employed by the radios have a statistical spreading factor of K .

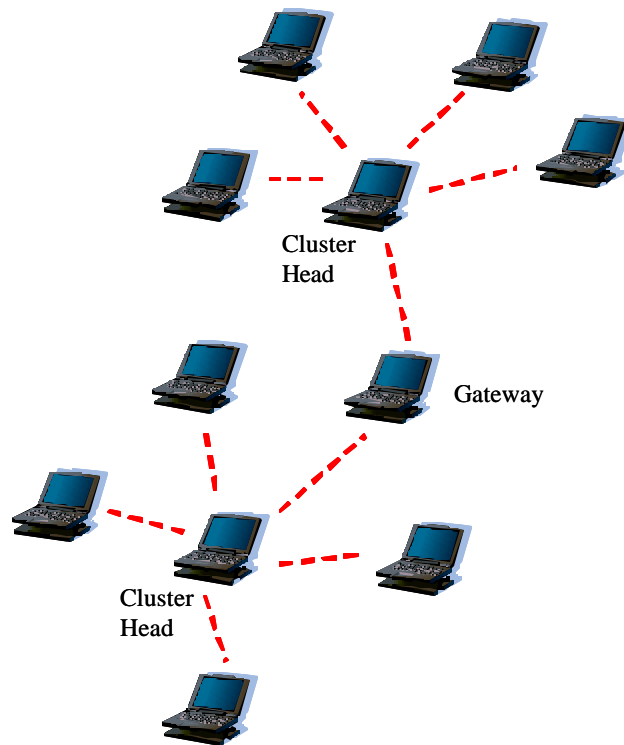


Figure 8.1: Simulation scenario for ad-hoc power control example

Assuming these devices are adjusting their power levels in a locally optimal manner, then the network conforms to the model described in the preceding. Accordingly, we would expect that any initial power vector would converge to a unique power vector and that even when corrupted by noise, the system would remain in a region near this steady state as the network is Lyapunov stable.

A simulation was constructed for deterministic and stochastic simulation scenarios. The simulation results for these scenarios are shown in Figure 8.2 and Figure 8.3, respectively. Note that the locally optimal algorithm rapidly converges to the steady state in both scenarios and that even in the presence of random noise-induced perturbations, the network remains in a region around the deterministic steady-state. However, as in

Chapter 7, adaptations could be further smoothed by introducing a threshold to the decision rule. Further, rather than the best response decision rule employed in this simulation, the radios could also be implementing an averaged best response (a particular example of an adaptive dynamics process).

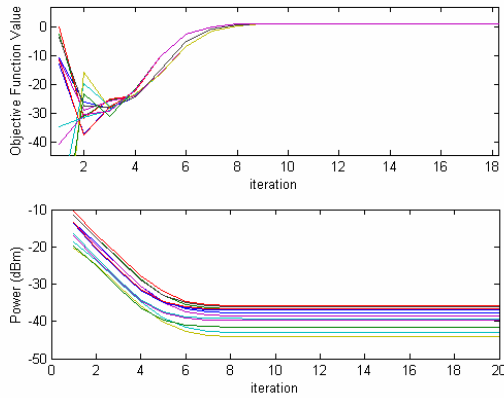


Figure 8.2: Noiseless simulation.

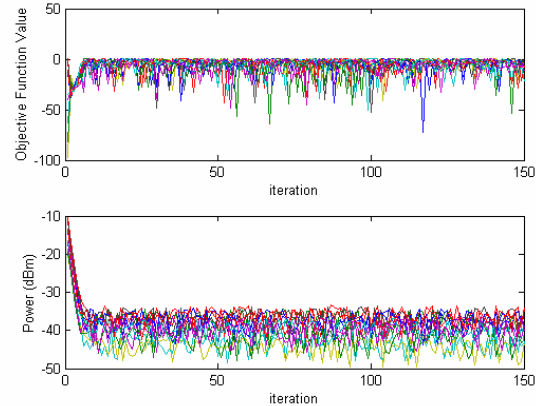


Figure 8.3: Noisy simulation.

8.3 Sensor Network Formation (*)

Consider a network of sensors collecting information and charged with transporting the information back to some data base (the information *sink*). It is frequently more efficient (in terms of cost, battery life, coverage, and covertness) for the radios to indirectly transport their information to the sink. In this example, we study a collection of wireless sensors which are guided by a desire to transport their data back to a common data sink (perhaps connected to the IP cloud) balanced against a desire to minimize power consumption – a term which is assumed to be dominated by transmit power. Because the sensors contain recording devices, if the cost to transmit is too high, the sensors have the option of simply storing the data for later retrieval.

Under the assumptions that forwarding other sensors' data is costless and that only a single sink is present, the following presents a model of this situation and a proof that the network has weak FIP.

8.3.1 Model

We can form a game model for this network as follows. We consider the player set to be the set of sensors (or nodes), N , and assume that each sensor i can form any number of directional links to the other sensors $j \in N \setminus i$ (perhaps via beam forming) and denote a particular link between from i to j by the symbol l_{ij} . Each sensor has an action set, L_i , given by the power set of $\{l_{ik}\} \forall k \in N \setminus i$ (i.e., a sensor can choose to maintain any combination of links to other sensors or no link at all.) Each sensor network expresses its utility as a function of the network g which is defined by the set of links implemented by each player. In general $g \subset g^N$ where, $g^N = \{l_{ij} \mid \forall i, j \in N, i \neq j\}$ is the complete network. For notational convenience, we also define g_i as the set of directed links in g formed by node i , i.e., $g_i = \{l_{ij} \in g\}$, and g_{-i} as the set of all links in g other than g_i , i.e., $g_{-i} = g \setminus g_i$.

Let l^k denote a particular link and for l_{ij} we refer to i as the *originating node* and j as the *terminating node*. We say that $\mathbf{g} = (l^1, l^2, \dots, l^m)$ is a *path* in the network g if the originating vertex for every l^{k+1} is the terminating node for every l^k . We say that i is *connected* to j if there exists a path from i to j and refer to this path as ij . If i is connected to j via a path consisting of a single link, then we say that i is *directly connected* to j and refer to the single link as a *direct connection*. If i is connected to j yet there exists no direct connection between i and j , then we say that i is *indirectly connected* to j . We say that g is a *connected network* if every $i \in N$ is connected to every $j \in N$.

One particularly useful utility function model for sensor network formation is the link connections model. Originally, introduced in [Jackson_96], a slightly modified network valuation function for the link connections model is given by (8.6)

$$u_i(g) = \sum_{j \in N \setminus i} b_{ij} \mathbf{d}_{ij}^{t_{ij}} - \sum_{l_{ij} \in g_i} c(l_{ij}) \quad (8.6)$$

where b_{ij} is the benefit i receives for being connected to j , \mathbf{d}_{ij} is a hop decay factor (perhaps reflecting increasing probability of transmission failure as the length of a connection increases from queue overflow or link failures) between i and j , t_{ij} is the

length of shortest path between i and j , with $t_{ij} = \infty$ when no path exists, and $c(l_{ij})$ is the cost to sensor i of forming link l_{ij} .

This cost parameter could be simplified as $c(l_{ij}) = c_i$ where each player has a cost that is applied across all local links. This situation is encountered when cost is a function of transmit power and the nodes can only select a fixed transmit power for all links. Another appropriate choice of cost parameter is $c(l_{ij}) = \mathbf{a}_i d(l_{ij})^n$ where $d(l_{ij})$ is the Euclidean distance between i and j , n is the path loss exponent, and \mathbf{a}_i is the free space loss factor. This can be used to model situations where a particular received power must be achieved at the terminating end of a link and the environment has a uniform path loss model. In general, however, $c(l_{ij})$ will vary by link in a sensor network due to various obstructions that may make the signal propagation environment non-uniform.

For a sensor network we assume there exists some $k \in N$ (the sink) such that for all $i \in N \setminus k$ $b_{ik} = b$ and $b_{ij} = 0 \forall j \in N \setminus k$, i.e., there is one and only sensor to which every other sensor assigns a benefit of being connected. For this example we assume $\delta=1$ so no information degradation occurs over multiple hops - a reasonable assumption with sufficiently large queues and sufficiently long periods of time for retransmissions.

8.3.2 Steady-states

The study of network formation in game theory (primarily social network) is somewhat unique in the sense that numerous network stability concepts have been introduced. These include *Nash networks*, *pairwise stability*, *link deletion proofness*, and *link addition proofness*. For this example where we are assuming that each sensor can arbitrarily change its links with each adaptation, the concept of the *Nash network* is the most natural equilibrium concept.

Definition 8.2: *Nash Network*

A network, g , is a *Nash network* if for every node $i \in N$, $u_i(g_i, g_{-i}) \geq u_i(g'_i, g_{-i})$ for all $g'_i \in L_i$.

In other words, a Nash network is a network where no node can improve its payoff by unilaterally altering any combination of its links.

Determining if and when a Nash network exists for this model of sensor network formation can be best done by simultaneously establishing a convergence condition performed in the following section. However, we can make some characterizations about the topology of any Nash network because we know that each sensor's best response is either a single link or no link (the utility function expresses no benefit for maintaining a redundant path). Thus in a Nash network, no paths branch (as that requires two outbound links) and no paths lead out from sink (as that would be costly to the sink without any benefit). Thus for this model of sensor network formation, all Nash networks lack cycles – paths such that some sensor i is indirectly connected to itself.

8.3.3 Convergence

To establish convergence of this sensor network to a Nash network we demonstrate that when the radios implement best response decision rules under round-robin timing, the network must progress through a sequence of readily characterized networks concluding in a Nash network. The first network we consider in this sequence is a trimmed network.

Definition 8.3: *Trimmed network* (*)

A network, g , is said to be a trimmed network if all sensors maintain no more than one link.

Now consider any initial distribution of links and what happens after every sensor has had a chance to implement its best response.

Theorem 8.2: Convergence to a Trimmed Network (*)

After a complete round-robin best response, any starting network must be in a trimmed network.

Proof: As there is no benefit to maintaining multiple links, the best response for every sensor is a single link or no link. So after each node has had a chance to play its best response, each node must have a single link or no link.

To further the convergence analysis, we must establish that certain paths are not contained in the network.

Definition 8.4: *Poison path* (*)

A path is said to be a *poison path* if it is a profitable path for some sensor j but is not profitable for some sensor after j in the path.

With a poison path in the network, the network cannot be a Nash network as at least one sensor will have to adapt. Further, other sensors may make adaptations which will have to be later reversed because of the unstable path to the sink. Fortunately, poison paths quickly disappear from any sensor network under round-robin best responses.

Theorem 8.3: *Withering of Poison Paths* (*)

For $\delta=1$, trimmed sensor networks playing a best response do not create new poison paths.

Proof: With a trimmed sensor network and $\delta=1$, creating a new poison path implies that a path exists to the sink such that some sensor has added a link where the cost outweighs the benefit. However, not playing any link is strictly preferable to creating such a link so creating a poison path cannot be a best response so a new poison path cannot be created.

With no poison paths being created, it is valuable to consider the situation where all poison paths are eliminated the network, a condition we term a *pruned network*.

Definition 8.5: *Pruned network* (*)

A network g is a *pruned network* if it contains no poison paths.

Starting from an arbitrary network, we can show that the sensor network rapidly converges to a pruned network.

Theorem 8.4: *Convergence to a Pruned Network* (*)

Starting from any initial network, the finite sensor network with $\delta=1$ is pruned after the first best-response round-robin and remains pruned thereafter.

Proof: After the first round, the network is trimmed and no node maintains a link whose cost outweighs the potential benefit of having a path to the sink (if it did, then it would not be preferable to not playing a link implying the network is not trimmed). As this is a requirement for a poison path in a sensor network, no poison paths can exist and the network is pruned.

While there may not be any poison paths after the first complete round-robin, there may be sensors which do not have a path to the sink either because the sensor has no links or that one of its paths was broken when an unprofitable link was eliminated. Some portions of the network, however, may have positive utility.

Definition 8.6: *Healthy network* (*)

A *healthy network* is a pruned network in which every sensor has positive utility

In general, the sensors in a healthy network constitute a subset of N and the number of sensors in a healthy network is a nondecreasing sequence for round-robin best responses. For this sensor network, membership in the healthy network implies a path to the sink as such a path is required for positive utility.

Theorem 8.5: *Healthy Network Stability* (*)

For $\delta=1$, all sensors in a healthy network remain in a healthy network when playing a round-robin best response (or better response) to a pruned network.

Proof: A sensor falls out of a healthy network if it chooses to disconnect or a sensor ahead of it in its path to the sink disconnects. As disconnecting drops the sensor's utility to 0 (or worse), this is never preferable as membership in the healthy network implies positive utility. Further no downstream sensor will profitably disconnect as this implies the existence of a poison path which contradicts the assumption of a pruned network.

We can make an interesting characterization of the sensor network after one round of best responses to the pruned network.

Theorem 8.6: *Sensor Network Link Characterization* (*)

After the first round of best responses to a pruned network, each sensor in the network is either in a healthy network or has no links.

Proof: By Theorem 8.5, once a sensor is part of a healthy network, its best response is to remain part of the healthy network. If it is not part of the healthy network and joining the healthy network would yield a positive utility, then the best response is to join the network. If a sensor cannot profitably join the healthy network, then utility is maximized when no link is implemented.

For those sensors in the healthy network, we can establish a convergence condition.

Theorem 8.7: *Health Network Monotonicity* (*)

For $\delta=1$, all finite healthy networks converge to a Nash Network when following a round-robin best response.

Proof: Consider the function $f(g) = \sum_{j \in H} u_j(g)$. As all sensors in the healthy network, H ,

have a path to the sink, any better response adaptations (which are assured of preserving H by Theorem 8.5) are taken only if it decreases the sensor's cost. For $\delta=1$ this adaptation causes no change in the utilities of the other sensors in H so $f(g)$ is a nondecreasing

function on a finite action space play must converge. Note that any profitable deviation increases the value of f .

We can then extend this result by incorporating the movement of sources into the healthy-network.

Theorem 8.8: Pruned Network Convergence(*)

For $\delta=1$, all finite pruned sensor networks converge to a Nash Network under a round-robin best response.

Proof: By Theorem 8.6 after its first round-robin best response, pruned sensor networks consist of nodes either in H or disconnected. By Theorem 8.7, nodes in H converge to a Nash network. When profitable, nodes from not in H transition to H (and thus converge). Nodes not in H that can never profitably join H remain disconnected and thus are at their steady-state once disconnected.

Then combining this result with the earlier theorem, we can show that all finite sensor networks with a goal given by (8.6) and $\delta=1$ converges to a Nash Network

Theorem 8.9: Arbitrary Network Convergence (*)

For $\delta=1$ the sensor network scenario converges to a Nash network under a round-robin best response from every starting network.

Proof: By Theorem 8.6, every initial network necessarily converges under a round-robin best response to a pruned network. By Theorem 8.8 a finite pruned network necessarily converges under these conditions.

We can also characterize when a round-robin best response will result in a Nash network with no disconnected sensors.

Theorem 8.10: Guaranteed Paths to Sink (*)

Suppose it is possible to number the sensors such that $c(l_{k,k-1}) < b$ with the sink as sensor 0 and $\delta=1$, then the round-robin best response network will converge to a network where all sensors have paths to the network.

Proof: Assume any initial distribution of links. After a single round of best responses, the network is a pruned network. As we assumed $c(l_{1,0}) < b$, the next best response of sensor 1 must place sensor 1 in the healthy network if it was not already. Likewise if node k is in H then node $k+1$ will join in its next iteration (if not before). By induction, all sensors in finite N must eventually join H implying that every sensor has a path to the sink.

Explicitly returning to the topic of this chapter, Theorem 8.9 supplies the sufficient condition to establish weak FIP.

Theorem 8.11: Sensor Networks and Weak FIP (*)

For $\delta=1$ the sensor network scenario has weak FIP.

Proof: Theorem 8.9 provides the requisite improvement path.

As Theorem 8.9 implies convergence from the empty network (which is a pruned network) and Theorem 8.11 implies that asynchronous timing can be employed with best response decision rules, a simple algorithm for autonomous sensor network formation can be written.

Algorithm 8.1: Sensor Network Formation

- 1) When first deployed, let all sensors (including the sink) in the network broadcast signals at the same power level.
- 2) Because it is at a common level, the initial signal should be sufficient for each sensor to calculate link gains to each of its detected sensors and thus the required transmit power to communicate and to estimate link formation costs. Let each sensor maintain the set N^i which is initialized to the set of sensors for which i 's cost of link formation is less than the benefit of a path to the sink.
- 3) At intervals determined by a random timer, each sensor i pings each sensors in its N^i to request its path to the sink, if one exists.
- 4) The pinging sensor either adds or switches a link to the lowest cost sensor that reports a path to the sink that does not pass through the pinging sensor and drops from N^i all sensors whose costs are greater than or equal to chosen link (thus there is no need to ping the sensor on the other end of the implemented link).
- 5) If $N^i = \emptyset$, the sensor is done as it has found its lowest cost path to the sink. Otherwise each sensor continues again from step 3) until the random timer has triggered a predetermined number of times in which case the sensor terminates this algorithm.

Because the system has weak FIP, rather than choosing the lowest cost link, each sensor could also be randomly choosing links and keeping those that improve its payoff. However, the best response algorithm should converge faster than the random better response algorithm.

8.4 Conclusions

This chapter introduced two different techniques for identifying when a game has weak FIP – showing that the game is a finite supermodular game and showing that there exists convergent improvement paths from all network states.

8.4.1 Analysis Summary

This chapter primarily considered the application of supermodular games to the analysis of cognitive radio networks. It was seen that a supermodular game always has at least one NE and that the NE of a game would form a lattice. However, identifying a supermodular game's NE requires that we solve the system of equations $a^* = \hat{B}(a^*)$ as we did in Chapter 4. By leveraging a relationship between standard interference functions and supermodular games, we established a novel condition on $\hat{B}(a)$ that ensures the uniqueness of an NE in a supermodular game and allows us to introduce a Lyapunov function for best response decision rules. Via adaptive dynamics, we also know that if the radios play best responses to observations over a finite history, a supermodular game will converge.

We then confirmed the claim in [Altman_03] that target SINR power control games are supermodular games by applying the concept of smooth supermodular games and extended this result to ad-hoc networks. As the power control game satisfied the condition to have a unique NE, the game has a Lyapunov function and is also stable. While a single reactive decision rule was simulated, the adaptive dynamics result of [Milgrom_90] informs us that the best responses also could have been based on previous observations as long as a finite history is employed.

We also studied sensor network formation and showed that for $\delta=1$ the network has weak FIP. Unlike previous efforts, this relied on demonstrating that that from every action vector there exists an improvement path that converges to an NE. In general such an approach is not as straight forward as evaluating the second order conditions of supermodular games or potential games as evidenced by the seven theorems that had to

be introduced to show convergence. However, game theory allows us to quickly identify other convergent decision rules based on the existence of weak FIP.

8.4.2 Design Implications

To an extent, supermodular games are closely related to the procedural analysis we performed in Chapter 3, particularly pseudo-contractions and standard interference functions. Partly this is because unlike potential games, not all myopic self-interested decision rules are guaranteed to converge under weak FIP or with a supermodular game and more specific decision rules have to be employed. Because of this limitation, cognitive radio networks which are supermodular games will generally not be appropriate for most ontological radio implementations. Instead, a cognitive radio should implement a specific decision rule such as its best response or a random better response (for finite supermodular games). In either case, this is best performed with a procedural radio though we see again that weak FIP implies that cognitive radios with properly designed random decision rules, e.g. a genetic algorithm cognitive radio, will be suitable.

As a properly best response algorithm will converge faster than a random better response, it is seen again that cognitive radios should incorporate the ability to perform scenario classification so that when a known scenario is encountered the best response algorithm can be employed and when an unknown scenario is encountered the cognitive radio uses a random better response as it converges under the broadest range of conditions.

As we saw in previous chapters, once we establish that a network satisfies a particular game model, it is trivial to develop low complexity convergent cognitive radio algorithms, as analysis of the game model in this document and elsewhere have yields lists of convergent algorithms. As was the case for both the power control and the sensor network formation examples, many of these algorithms are low complexity and well suited for use in rapidly deployed networks.

8.5 References

- [Altman_03] E. Altman and Z. Altman. "S-Modular Games and Power Control in Wireless Networks" *IEEE Transactions on Automatic Control*, Vol. 48, 839-842, May 2003.
- [Friedman_01] J. Friedman, C. Mezzetti, "Learning in Games by Random Sampling," *Journal of Economic Theory* **98**, 55-84, 2001.
- [Jackson_96] Jackson, M.O. and A. Wolinsky "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* vol. 71, pp. 44-74, 1996.
- [Milgrom_90] Milgrom, Paul and John Roberts, "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, Volume 58, Issue 6 (Nov 1990), pp. 1255-1277.
- [Neel_05] J. Neel, R. Menon, A. MacKenzie, J. Reed, "Using Game Theory to Aid the Design of Physical Layer Cognitive Radio Algorithms," *Conference on Economics, Technology and Policy of Unlicensed Spectrum*, May 16-17 2005, Lansing, Michigan.
- [Neel_06] J. Neel, J. Reed, A. MacKenzie, "Analyzing Cognitive Radio Networks" in **Cognitive Radio**, ed. B. Fette, Elsevier Publications, August 11, 2006.
- [Topkis_98] D. Topkis, **Supermodularity and Complementarity**, Princeton University Press, Princeton, New Jersey, 1998.
- [Yates_95] R. Yates, "A Framework for Uplink Power Control in Cellular Radio Systems," *IEEE Journal on Selected Areas in Communications*, Vol. 13, No 7, September 1995, pp. 1341-1347.