A Convergence Result for Potential Games

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1 INTRODUCTION

A special class of normal form games, potential games [1], are easy to analyze because they are equivalent, in some sense, to a coordination game in which every player's utility function is replaced by the so called potential function. Hence, pure-strategy Nash Equilibria (N.E.) of the potential game can be found by finding the N.E. of its equivalent coordination game. While finite potential games are by now well understood, little has been written about continuous potential games on compact action spaces. We classify all N.E. as those that are local maxima of the potential function (l.N.E.), and those that are not, which we call elusive N.E. (e.N.E.). l.N.E. are arguably much easier to find than e.N.E., and when the potential function can be considered a social welfare function, l.N.E. have an intuitively appealing interpretation. In contrast, elusive N.E. are such a nuisance that it is beneficial to discard them. This paper shows that for a special class of continuous potential games, which we call Nash-separable potential games, e.N.E. are unstable in the following sense: when arbitrarily small noise is added to a best-response dynamic, play almost surely converges to an arbitrarily small neighborhood of an l.N.E. component. The proof is a generalization of the result in [2], who considered a specific game that happened to have a unique best response for every player and every action profile. Our chief contribution is handling the possibility of multiple best-responses. We will start with some preliminary definitions that will help us to transcend this difficulty.

2 PRELIMINARIES

Consider a normal form game [1] expressed as the following tuple

$$\Gamma = \left\langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N} \right\rangle, \tag{1}$$

where $N = \{1,\ldots,|N|\}$ is the set of players. For a given player $i \in N$, A_i, is the set of actions available, and u_i is its utility function. If $A = \times_{i\in N} A_i$ then $u_i : A \to \mathbb{R}$. Player $i \in N$ prefers $x \in A$ over $x' \in A$ if $u(x) \geq u(x')$. We will call an element, $x \in A$ an *action profile*. We assume that A_i is a compact metric space, which implies that A is a compact metric space when endowed with the product metric. In this paper we will adopt the following notation. $A_{-i} = \times_{j\in N-\{i\}} A_j$, $x_{-i} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{|N|})$. Given $x_0 \in A$, $N_\delta(x_0) = \{x \in A : d(x,x_0) < \delta\}$ is *a neighborhood of* x_0. $N^*_\varepsilon(x) = N_\varepsilon(x) - \{x\}$ is the so called *deleted neighborhood*. The relations $B \subseteq A$ and $B \subset A$ are used to denote that "B is a subset of A", and "B is a strict subset of A", respectively. Given $B \subseteq A$, $\overline{B}$ denotes its closure with respect to A, and $P(B)$ denote its *power set* (i.e. set of all subsets). Given $V : A \to \mathbb{R}$, $V(B) = \{V(x) : x \in B\}$.

A fundamental concept for normal form games is the so called Nash Equilibria (N.E.). An action profile, $x \in A$ is a *Nash Equilibria* if[1] $\forall k \in N$,

$$u_k(x) \geq u_k(x'_k, x_{-k}), \forall x'_k \in A_k \tag{2}$$

[1] With some of abuse of notation, let $V(x'_i, x_{-i})$ denote $V(x_1,\ldots,x_{i-1},x_i{}',x_{i+1},\ldots,x_{|N|})$ where $x_i{}' \in A_i$ and $x_{-i} \in A_{-i}$.

Given an action profile, $x \in A$, a *better response for player* $i \in N$ is any $x_i' \in A_i$ such that $u_i(x_i', x_{-i}) \geq u_i(x)$, and a *best response* for player $i \in N$ is any $x_i' \in \arg\max\{u_i(z_i, x_{-i}) : z_i \in A_i\}$. With this definition in hand, an N.E. is an action profile $x \in A$ such that x_i is a best response for every player $i \in N$. Two games with the same sets N and $\{A_i\}$ are said to be *best response equivalent* if $\forall x \in A$ and every player, the best response of both games coincide. *Better response equivalence* has a corresponding definition. For a game, Γ, a function $V : A \to \mathbb{R}$ is said to be:

1) an exact potential function if $\forall i \in N, \;\; x \in A, x_i' \in A_i$

$$u_i(x) - u_i(x_i', x_{-i}) = V(x) - V(x_i', x_{-i}) \tag{3}$$

2) an ordinal potential function if $\forall i \in N, \;\; x \in A, x_i' \in A_i$

$$u_i(x) \geq u_i(x_i', x_{-i}) \Leftrightarrow V(x) \geq V(x_i', x_{-i}) \,. \tag{4}$$

3) a *best response* (BR) *potential function* if $\forall i \in N, \; x_{-i} \in A_{-i}$,

$$\arg\max_{x_i \in A_i} u_i(x_i, x_{-i}) = \arg\max_{x_i \in A_i} V(x_i, x_{-i}) \tag{5}$$

A game is an exact, ordinal, or BR potential game if, for the game, there exists an exact, ordinal, or BR potential function, respectively. The fact that both exact and ordinal potential games are best-response potential games follows directly from their definitions. In addition, a game is a *transformable ordinal potential game* if there exists an ordinal transformation[2], $f : u(A) \to \mathbb{R}^{|N|}$, such that the game $\langle N, \{A_i\}_{i\in N}, \{\tilde{u}_i\}_{i\in N} \rangle$ with $\tilde{u}_i(x) = f_i(u_i(x))$, is an exact potential game. A game is a *potential game* if it is any of these games. While much is known about exact potential games, little is known about ordinal potential games. The easiest way to show that a game has an ordinal potential is to construct a set of ordinal transformations; however, it is neither known if an ordinal transformation exists for all ordinal potential games nor is it known if there exists a procedure to find such an ordinal transformation.

The importance of potential games hinges on its relation to the following so called coordination game

$$\Gamma' = \left\langle N, \{A_i\}_{i\in N}, \{V\}_{i\in N} \right\rangle \tag{6}$$

where each player's utility function is replaced by the (player-independent) potential function. Exact and ordinal potential games are better response equivalent to this coordination game, while BR potential games are only best-response equivalent. In both cases, when the potential function is also a measure of social welfare, this equivalence says that players can serve the greater good by following their own best interest. In a *round robin best-response dynamic*, players take turns in a round robin fashion choosing a best response until there are no more improving solutions[3]. In such a case, the best response dynamic enters a Nash Equilibrium. Since a potential game is best-response equivalent to its corresponding coordination game any result for the best-response dynamic of a coordination game also applies to a potential game.

Although, on a given player's turn, there may be many possible best-responses to choose, he must choose only one, possibly in a random or deterministic manner. Toward the goal of establishing convergence results independent of the nature of these choices, we sill consider all possible choices. This requires an understanding of set-valued functions, or correspondences [3].

A *correspondence* from metric space A to metric space S is a mapping, $\Phi : A \to P(S)$, where $P(S)$ denotes the power set of S. A correspondence is *compact valued* if the set, $\Phi(x)$ is compact for every $x \in A$. A

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[2] An ordinal transformation is a transformation $f : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$, $f(u) = (f_1(u_1), \ldots, f_{|N|}(u_{|N|}))$, $u \in \mathbb{R}^{|N|}$ such that $\forall i \in N$, f_i is a monotonically increasing function.

[3] For brevity, henceforth we will drop the round-robin distinction.

correspondence is *upper-semi-continuous* (u.s.c.) if, for a given point $x \in A$, for every open neighborhood of $\Phi(x)$, Ψ, there is an open neighborhood of x, U, such that $\Phi(U) \subseteq \Psi$. Upper-semi-continuity is a useful generalization of continuity about which much is known. For instance, the direct sum and composition of compact-valued u.s.c. correspondences is compact-valued u.s.c. Also, the image of a compact set under a compact-valued u.s.c. correspondence is compact, and the pre-image of open sets are open [3].

When considering a best-response dynamic, the set of best-responses for a player, $i \in N$, is the correspondence $D_i^* : A_{-i} \to P(A_i)$,

$$D_i^*(x_{-i}) = \arg\max_{x_i \in A_i} V(x_i, x_{-i}) \tag{7}$$

The best-response iteration for player i, is the correspondence $\Phi_i : A \to P(A)$,

$$\Phi_i(x) = (D_i^*(x_{-i}),\ x_{-i}) \tag{8}$$

It will be convenient to consider the composite best response iteration as the composition of all players' best responses after one round-robin iteration, $\Phi : A \to P(A)$

$$\Phi(x) = \Phi_{|N|}(\cdots \Phi_2(\Phi_1(x))\cdots) \tag{9}$$

Since, for every $i \in N$, V is continuous function on the compact set A_i, $D_i^*(x_{-i}) \neq \varnothing\ \forall i \in N$, $x_{-i} \in A_{-i}$. The well known maximum theorem [3] states that for every $i \in N$, D_i^* is a compact-valued u.s.c. correspondence. Hence, so are $\{\Phi_i\}$ and Φ. Finally, $x \in A$ is an N.E. if and only if it is a fixed point of the round-robin best response iteration, i.e. $x \in \Phi(x)$. For this reason we let Φ_F denote the set of all N.E. For potential games on compact action spaces there is at least one local-maximum N.E. because all points in the set $X_{\max} \triangleq \arg\max\{V(x) : x \in A\}$ are N.E. In the subsequent discussion, it will be convenient to define $V_{\max} = \max\{V(x) : x \in A\}$.

Note, while $V(\Phi_i(x))$ is singleton $\forall i \in N$, $V(\Phi(x))$ is not necessarily because different best responses for a given player can open up different possibilities for subsequent players. Of course, any inequalities involving V over best-response iterations must account for this fact. So, given $x \in X \subseteq A$, by the definition of best response, $x' \in \Phi_i(x)$ only if $V(x') \geq V(x)$, and this inequality holds for both the infimum and supremum. On a similar note, if $V(x') = V(x)$ for every such x, x', then no player can increase its utility on its given turn. That is, $X \subseteq \Phi_F$. These observations give rise to the following propositions.

Proposition 1: Given $X \subseteq A$, $\inf V(X) \leq \inf V(\Phi(X))$, $\sup V(X) \leq \sup V(\Phi(X))$

Proposition 2: Given $X \subseteq A$ with $V(\Phi(X))$ singleton, then $X \subseteq \Phi_F$ if and only $V(\Phi(X)) = V(X)$.

We conclude this section with the definition of the *noisy best response iteration* (NBRI), which concerns this paper's key result. This dynamic is a best-response variation of the iteration considered in [2]. In this case, the best response of each player is perturbed by bounded noise, with bound $\delta > 0$. The motivation behind the noise is that in the process of determining their actions, players may make random mistakes, however small, in a manner whose nature may depend upon the best response itself. For each $\chi \in A$, let $z(\chi)$ be a random variable with arbitrary distribution $p_z(z;\chi,\delta)$. We only require that $p_z(z;\chi,\delta)$ be positive almost everywhere on $\overline{N_\delta(\chi)}$ and zero on $A \setminus \overline{N_\delta(\chi)}$. For instance, this noise may be uniformly distributed on $\overline{N_\delta(\chi)}$. The NBRI for noise bound, δ, (δNBRI) is thus given as follows.

Noisy Best Response Iteration (δNBRI)
Given noise bound $\delta > 0$, and $x[0] \in A$
For each $t \in \mathbb{N}_0$,

1. Choose $\chi[t] \in \Phi(x[t])$
2. $x[t+1] = z(\chi[t])$

Here, $\{\chi[t]\}$ is the sequence of chosen best-responses, $\{x[t]\}$ is the sequence of action profiles actually taken (including the mistakes). The index $t \in \mathbb{N}_0$ indexes one round robin iteration. The choice in step 1 is a convenient representation of all the choices that players make on one round-robin iteration. In the event that there is not a unique best response (Φ is not singleton valued), the NBRI does not specify the mechanism of choice. However, the coming Theorem 3 holds for any sequence of choices.

A continuous function, $V : A \to \mathbb{R}$, on compact A is *Nash separable* if

1. There are no suboptimal local maxima on A,
2. It's maximum is isolated from the image of other fixed points (i.e. $\exists\ \varepsilon_m > 0,$ s.t. $N^*_{\varepsilon_m}(V_{\max}) \cap V(\Phi_F) = \varnothing$).
3. Best response iterations are strictly improving in a neighborhood of the maximum (i.e. $\exists \delta_m > 0$ such that $\forall x \in N^*_{\delta_m}(X_{\max}),\ \forall x' \in \Phi(x),\ V(x') > V(x)$).

A potential game is *Nash separable* if its potential is Nash separable. Not all continuous functions on compact sets are Nash separable as evidenced by the potential function[4] $V : [-1,1]^2 \to \mathbb{R}$, $V(x_1, x_2) = \left| |x_1 + x_2| - \tfrac{1}{2} \right|_+^3 - |x_1 - x_2|^{3/2}$. It is routine but tedious to show that V has a connected component of l.N.E. along the line $x_1 = x_2, -1/4 < x_1 < 1/4$ but e.N.E. at $(x_1, x_2) = \pm(1/4, 1/4)$. In fact, V is continuously differentiable, suggesting that a simple characterization of Nash separability based on differentiability is difficult to achieve. The following proposition is straightforward but tedious to prove.

Proposition 3: Given conditions 1 and 2 in the definition of a Nash separable potential function, condition 3 is equivalent to the following: $\exists \varepsilon_m > 0,$ s.t. $\forall \varepsilon \in (0, \varepsilon_m)$, $\Phi(V^{-1}([V_{\max} - \varepsilon, V_{\max}])) \subset V^{-1}((V_{\max} - \varepsilon, V_{\max}])$

3 Convergence of the Noisy Best Response Iteration (NBRI)

If an elusive N.E. exists it is unstable in the sense that arbitrarily small noise will drive the NBRI away from the equilibria. To see this, note that by definition, eN.E. are not local maxima, and hence, every neighborhood contains points with higher potentials. Moreover, by continuity, every neighborhood contains a measurable region with a higher potential. Hence, noise will drive the NBRI into this finite region with finite probability, and increasing probability on successive trials; finally, once the NBRI enters this region, it is unlikely to escape. In contrast, as the NBRI approaches a l.N.E., it will remain attracted to it as long as the noise is small enough to not over-power the natural best-response dynamic.

The following result is a generalization of Theorem 8 in [2] and says that even if a Nash separable potential game has sub-optimal N.E., arbitrarily small noise will cause the NBRI to almost surely converge to a neighborhood of the global optima.

Theorem 1: Consider a Nash separable potential game with the form of Equation (1) and potential function V. Then, $\forall \varepsilon > 0$, $\exists \delta_0 > 0$, such that $\forall \delta$ with $0 < \delta < \delta_0$ and $\forall x[0] \in A$, the $\delta NBRI$ with iterates $(x[t])$ obeys $\liminf_{t\to\infty} V(x[t]) \underset{a.s.}{\geq} \max V - \varepsilon$.

[4] Here $|a|_+ = a$ if $a \geq 0$, and zero otherwise.

The type of convergence in the Theorem's statement is almost sure (a.s.)[4]. That is, the inequality holds for every realization of the NBRI except for on a set of measure zero.

Proof:

Since V is continuous on a compact set, A, there exist $V_{\min} = \min_{x \in A} V(x)$ and $V_{\max} = \max_{x \in A} V(x)$. By the fact that V is Nash separable, choose $\varepsilon_0 > 0$ such that $N^*_{\varepsilon_0}(V_{\max}) \cap V(\Phi_F) = \varnothing$.

Now given $\varepsilon > 0$ in the Hypothesis of this theorem, without loss of generality, we may consider $\varepsilon < \varepsilon_0$. Let

$$\begin{aligned} W_+ &= V^{-1}([V_{\min}, V_{\max} - \varepsilon]) \\ W_0 &= V^{-1}([V_{\max} - \varepsilon, V_{\max}]) \end{aligned} \tag{10}$$

Since both W_+ and W_0 are defined as the continuous pre-image of a closed set, they are closed. Since both W_+ and W_0 are contained in the compact set A, they are compact. Note that if $W_+ = \varnothing$, this theorem holds trivially; so henceforth, we will consider $W_+ \neq \varnothing$.

We seek a noise bound, $\delta_0 > 0$, small enough to ensure that $x \in W_0$ implies $N_{\delta_0}(x) \subseteq W_0$. Toward this end, let

$$\delta_0 = \min\{d(x, x') : x \in W_+, x' \in \Phi(W_0)\} \tag{11}$$

Note, that this minimum exists since the metric, d, is continuous on the compact set, $W_+ \times \Phi(W_0)$ (since Φ is a compact valued u.s.c. correspondence, $\Phi(W_0)$ is compact). Since, by Proposition 3, $\Phi(W_0) \subset V^{-1}((V_{\max} - \varepsilon, V_{\max}])$ and by definition, $W_+ = V^{-1}([V_{\min}, V_{\max} - \varepsilon])$, we have $\Phi(W_0) \cap W_+ = \varnothing$; hence, δ_0 is the minimum distance between two non-empty, non-overlapping compact sets, and so $\delta_0 > 0$.

Now let $\delta \in (0, \delta_0)$ be given. Note that by choice of δ_0, once the $\delta NBRI$ enters W_0 containing the local maximum, it cannot escape. We now wish to show that outside of W_0, the NBRI increases the potential function by a fixed finite amount with finite probability. Toward this end, define $\beta : \mathrm{A} \to \mathbb{R}$ as follows.

$$\beta(x) = \max\{V(x_0) : x_0 \in \overline{N_\delta(x)}\} \tag{12}$$

Note that this maximum exists because V is continuous on the compact set, $\overline{N_\delta(x)}$. Moreover, by the maximum theorem [3], β is continuous. Hence, $V - \beta$ also attains the following minimum on the compact set, W_+.

$$\gamma = \tfrac{1}{2} \min\{V(x) - \beta(x) : x \in W_+\} \tag{13}$$

We claim that $\gamma > 0$. If $\gamma = 0$, then there is a x_0 such that $V(x_0) = \beta(x_0)$. That is, x_0 is a local maximum; but since $V(x)$ is Nash separable, all local maxima are global maxima and $V(x_0) = V_{\max}$, which contradicts the fact that $x_0 \in W_+$.

Now, given $x \in A$, $\chi \in \Phi(x)$, let

$$P(x, \chi) = \Pr\big((V(z(\chi)) > V(x) + \gamma) \cup (z(\chi) \in W_0)\big). \tag{14}$$

This is a continuous function of x and χ since all primitives are continuous. Let $P : A \to P([0,1])$ be defined as

$$P(x) = P(x, \Phi(x)). \tag{15}$$

This is a compact valued u.s.c. correspondence. We claim that $P(W_+) \subseteq (0,1]$. Given $p \in P(W_+)$, choose $x \in W_+$ and $\chi \in \Phi(x)$ such that $p = P(x,\chi)$. Consider two cases. First, suppose $\chi \in W_+$. By the definition of γ, choose χ' $\chi' \in N_\delta(\chi)$ such that $2\gamma < V(\chi') - V(\chi)$. By the continuity of V, choose $\delta_0 > 0$ such that $V(\chi'') > V(x) + \gamma$, $\forall \chi'' \in N_{\delta_0}(\chi') \cap N_\delta(\chi)$. Since $p_z(z;\chi)$ is non-zero almost everywhere on $N_{\delta_0}(\chi') \cap N_\delta(\chi) \neq \varnothing$, $p \geq \Pr\left(V(z(\chi)) > V(x) + \gamma\right) \geq \Pr\left(z(\chi) \in N_{\delta_0}(\chi') \cap N_\delta(\chi)\right) > 0$. Second, suppose $\chi \in W_0$. By the choice of δ, $N_\delta(\chi) \subseteq W_0$, so $p \geq \Pr\left(z(\chi) \in W_0\right) \geq \Pr\left(z(\chi) \in N_\delta(\chi)\right) > 0$.

Now since, $P(x)$ is compact valued u.s.c., $P(W_+)$ is also compact. So, there exists a $p = \min P(W_+)$ and by the previous claim, $p > 0$.

Thus far, we have shown that

$$\Pr\left((V(z(\chi)) > V(x) + \gamma) \cup z(\chi) \in W_0\right) \geq p, \ \forall x \in A, \chi \in \Phi(x), \tag{16}$$

and now whish to apply this fact to the NBRI. Let $Q = \left\lceil \frac{V_{\max} - \varepsilon - V_{\min}}{\gamma} \right\rceil$, and let

$$V_q = \left\{ \begin{array}{cc} [V_{\max} - \varepsilon, V_{\max}] & q = 0 \\ [V_{\min} + \gamma(Q-q), V_{\max} - \varepsilon) & q = 1 \\ [V_{\min} + \gamma(Q-q), V_{\min} + \gamma(Q-q+1)) & 2 \leq q \leq Q \end{array} \right]. \tag{17}$$

Thus, $\{V_q\}$ partition $V(A)$.

Now let $t \in \mathbb{N}_0$, $x[t] \in A$, and $\chi[t] \in \Phi(x[t])$ be given. Define the state of the NBRI as

$$\sigma[t] \triangleq \{q : V(x[t]) \in V_q\} \tag{18}$$

and let G_t denote the event that $\sigma[t+1] < \sigma[t]$, while E_t denotes the event that $\sigma[t] = 0$. Since G_t is only determined by knowledge of $x[t]$ and $\chi[t]$, we have

$$\Pr\left(G_t \left| \bigcap_{l=1}^{\mu_1} G_{t-l}, \bar{E}_{t-\mu_2}, x[t], \chi[t] \right.\right) = \Pr\left(G_t \left| x[t], \chi[t] \right.\right) \geq p, \ \forall \mu_1, \mu_2 \geq 0 \tag{19}$$

where the last inequality follows from (16). Moreover, since the inequality in (19) holds for arbitrary $x[t]$ and $\chi[t]$, we have

$$\Pr\left(G_t \left| \bigcap_{l=1}^{\mu_1} G_{t-l}, \bar{E}_{t-\mu_2} \right.\right) \geq p, \ \forall \mu_1, \mu_2 \geq 0 \tag{20}$$

In particular, given an $m \geq 0$,

$$\Pr\left(G_{Qm+q} \left| \bigcap_{l=0}^{q-1} G_{Qm+l}, \bar{E}_{Qm} \right.\right) \geq p, \ \forall q, 0 \leq q \leq Q-1. \tag{21}$$

So,

$$\Pr\left(\bigcap_{q=0}^{Q-1} G_{Qm+q}, \left| \bar{E}_{Qm} \right.\right) \geq p^Q. \tag{22}$$

Note, $\bigcap_{q=0}^{Q-1} G_{Qm+q}$ implies $\sigma[Q(m+1)] \le \sigma[Qm] - Q$, which in turn, implies $\sigma[Q(m+1)] = 0 \Rightarrow E_{Qm}$. Thus,

$$\Pr(E_{Q(m+1)} \mid \bar{E}_{Qm}) \ge \Pr\left(\bigcap_{q=0}^{Q-1} G_{Qm+q} \mid \bar{E}_{Qm}\right) \ge p^Q. \tag{23}$$

Also, $E_{Qm} \Rightarrow \sigma[Qm] = 0 \Rightarrow x[Qm] \in W_0 \Rightarrow x[t] \in W_0, \forall t > Qm \Rightarrow E_{Q(m+1)}$. Thus $\Pr(E_{Q(m+1)} \mid E_{Qm}) = 1$. Therefore,

$$\begin{aligned} \Pr(E_{Q(m+1)}) &= \Pr(E_{Q(m+1)} \mid E_{Qm}) \Pr(E_{Qm}) + \Pr(E_{Q(m+1)} \mid \bar{E}_{Qm}) \Pr(\bar{E}_{Qm}) \\ &\ge (1)(1 - \Pr(\bar{E}_{Qm})) + p^Q \Pr(\bar{E}_{Qm}) \end{aligned} \tag{24}$$

So,

$$\Pr(\bar{E}_{Q(m+1)}) \le \Pr(\bar{E}_{Qm})(1 - p^Q) \tag{25}$$

and by induction,

$$\Pr(\bar{E}_{Q(m+1)}) \le \Pr(\bar{E}_0)(1 - p^Q)^m \le (1 - p^Q)^m \to 0 \tag{26}$$

as $m \to \infty$. Thus, $\lim_{t\to\infty} \sigma[t] = 0$ and $\limsup_{t\to\infty} V(x[t]) \ge V_{\max} - \varepsilon$. This concludes the proof.

4 CONCLUSIONS

This paper proves an intuitive convergence result for a certain class of continuous potential game, Nash separable games: when arbitrarily small noise is added to a best-response dynamic, play almost surely converges to an arbitrarily small neighborhood of an l.N.E component. Although our definition of Nash separable games is broad, we eventually hope to obtain a result for a larger class of potential games. On a final note, the method by which noise is added in our Noisy Best Response Dynamic is admittedly artificial. If noise represents player's mistakes, then our NBRI says that the effect of their mistakes only arise after one round-robin iteration. It would be better if noise were added after each player's response.

REFERENCES

[1] D. Monderer, L. Shapley, "Potential Games," *Journal of Games and Economic Behavior*, vol. 14, no. 0044, pp. 124-143, 1996.

[2] P. Anigstein, V. Anantharam, "Ensuring Convergence of the MMSE Iteration for Interference Avoidance to the Global Optimum," *IEEE Trans. on Information Theory*, vol. 49, no. 4, pp. 873-885, April 2003.

[3] R. Sundaram, *A First Course in Optimization Theory*, Cambridge University Press, Cambridge, U.K. © 1996.

[4] H. Stark, J. Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, Prentice Hall, Englewood Cliffs, NJ, © 1994.

[5] M. Voorneveld, *Potential Games and Interactive Decisions with Multiple Criteria*, PhD Dissertation, Tilburg University, September 1999.

[6] Prajit K. Dutta, *Strategies and Games, Theory and Practice*, MIT Press, Cambridge Massachusetts, © 1999.