

Game Theoretic Models of Network Formation

An overview

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Outline

1. Game Theory: A tool box
2. Purely individualistic models: Nash networks
3. Link based models: Pairwise stability
4. Models with consent: Trust
5. Some conclusions

Game Theory: Purpose

- **Game Theory** is a part of (applied) mathematics that describes and studies interactive decision problems.
- In an **interactive decision problem** the decisions made by each decision maker affect the outcomes and, thus, the resulting situation for all decision makers involved.
- Game theory evolved from recreational mathematics (Borel, Zermelo) and the mathematical economics of general equilibrium analysis (von Neumann, Wald, Menger). The foundations were provided by von Neumann (1928), von Neumann and Morgenstern (1944), and Nash (1950, 1951 and 1953).

Game Theory: Purpose

- Game theory consists of a set of mathematical tools that
 1. describe and
 2. analyzethe interactive decision situations at hand.
- Interactive decision situations are called **games**.
- The decision makers within a “game” are called **players**.

Game Theory: Branches

- Non-cooperative Game Theory
 - Extensive form games
 - Strategic or normal form games
- Cooperative Game Theory
 - Transferable utilities (think “money”)
 - Non-transferable utilities

Strategic form games

1. **Players:** A precise description of the decision makers in the game.

$$i \in N = \{1, \dots, n\}.$$

2. **Action sets:** Each player is assigned a set of decisions or actions. Each player selects *exactly one* action from his action set.

$$a_i \in A_i \text{ and } a \in A = A_1 \times \dots \times A_n.$$

3. **Payoff functions:** Each player is assigned a payoff function describing his evaluation of the resulting outcomes in the game.

$$\pi_i: A \rightarrow \mathbb{R}.$$

Strategic form games

A **game** is now fully represented in strategic form as

$$\Gamma = \langle N, (A_i, \pi_i)_{i \in N} \rangle$$

Strategically stable action tuples $a \in A$ within a game are called “equilibria”.

Nash equilibrium

$\hat{a} \in A$ is a **Nash equilibrium** of the strategic game Γ if for every player $i \in N$, \hat{a}_i maximizes $\pi_i(\cdot, \hat{a}_{-i})$ over A_i .

Or, for every $i \in N$:

$$\pi_i(\hat{a}) \geq \pi_i(b_i, \hat{a}_{-i})$$

for every $b_i \in A_i$.

Network formation principles

- **Links** are in principle undirected and formed between two players in N . A link between players i and j is therefore the binary set $\{i, j\}$.
- A **network** is given by $g \subset g_N = \{ij \mid i \neq j\}$, where ij stands for $\{i, j\}$.
- Let $\mathbb{G} = \{g \mid g \subset g_N\}$ be the class of all (undirected) networks.
- **Payoffs to network formation** accrue to the players within the formed network, represented by $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{G} \rightarrow \mathbb{R}^N$.

Networks: Basics

- A network $g \in \mathbb{G}$ is **Pareto optimal** if there is no other network $h \in \mathbb{G}$ such that $\varphi_i(h) \geq \varphi_i(g)$ for all players i and $\varphi_j(h) > \varphi_j(g)$ for at least one player $j \in N$.
- A network $g \in \mathbb{G}$ is **Efficient** if it maximizes the total payoff given by $\Phi: \mathbb{G} \rightarrow \mathbb{R}$ with

$$\Phi(h) = \sum_{j \in N} \varphi_j(h) \text{ for all } h \in \mathbb{G}.$$

Efficiency implies Pareto optimality.

Nash networks

Bala and Goyal (2000) base network formation on purely individualistic considerations:

The **Bala-Goyal model** is a game Γ_{bg} such that

- $A_i = N \setminus \{i\}$ describes the players that i wants to link with;
- If $a \in A$, then the network formed is given by

$$g_\alpha(a) = \{ij \in g_N \mid i \neq j \text{ and } j \in a_i\}.$$

- Now the game-theoretic payoff for player i is given by $\pi_i(a) = \varphi_i(g_\alpha(a))$.

Nash networks

The Nash equilibria in the Bala-Goyal model Γ_{bg} are known as “Nash networks”.

Bala and Goyal (2000) and follow-up publications investigated the Nash networks that resulted for different payoff functions. Familiar architectures that could be supported are stars, wheels, and empty and complete networks.

Nash networks

There are problems with the Bala-Goyal framework:

- Consent is not required for link formation. For social networks, this is unacceptable.
- Applications are very limited. Links have to be viewed as confirmations of existing relationships.
⇒ Applications in engineering?
- The payoff functions investigated thus far are rather unrealistic.
- The set of Nash networks can be rather large for certain payoff structures. On the other hand, Nash networks certainly do not have to be Pareto optimal.

Link-based models

- Instead of looking at individualistic decisions to make and break links, we can also look at link-based considerations. Here the links $ij \in g_N$ are the primary unit of investigation and decision making.
- A link is **broken** when an individual player participating in that link prefers to have it deleted.
- A link is **created or added** if *both* participating players prefer to have it created within the existing network.

Auxiliary notation

Let $g \in \mathbb{G}$ be some network. Then we define:

- The set of direct neighbors of player i in g is given by

$$N^d(i, g) = \{j \in N \mid ij \in g\}.$$

- $g + ij = g \cup \{ij\}$ is the network resulting from adding the link ij to g .
- $g - ij = g \setminus \{ij\}$ is the network resulting from deleting the link ij from g .

Link deletion proofness

- A network $g \in \mathbb{G}$ is **link deletion proof** if for every player $i \in N$ and every $j \in N^d(i, g)$ it holds that $\varphi_i(g - ij) \leq \varphi_i(g)$.
- A network $g \in \mathbb{G}$ is **strong link deletion proof** if for every player $i \in N$ and every $M \subset N^d(i, g)$ it holds that $\varphi_i(g \setminus h_M) \leq \varphi_i(g)$, where

$$h_M = \{ij \in g \mid j \in M\} \subset g.$$

Pairwise stability

- A network $g \in \mathbb{G}$ is **pairwise stable** if g is link deletion proof and, moreover, for all players $i, j \in N$: $\varphi_i(g + ij) > \varphi_i(g)$ implies $\varphi_j(g + ij) < \varphi_j(g)$.
- A network $g \in \mathbb{G}$ is **strongly pairwise stable** if g is strong link deletion proof and, moreover, for all players $i, j \in N$: $\varphi_i(g + ij) > \varphi_i(g)$ implies $\varphi_j(g + ij) < \varphi_j(g)$.

Strong stability

- Network $g' \in \mathbb{G}$ is **obtainable** from network $g \in \mathbb{G}$ through coalition $S \subset N$ if
 - $ij \in g'$ and $ij \notin g$ implies that $\{i, j\} \subset S$, and
 - $ij \in g$ and $ij \notin g'$ implies that $\{i, j\} \cap S \neq \emptyset$.
- A network $g \in \mathbb{G}$ is **strongly stable** if for any coalition $S \subset N$ and any network g' that is obtainable from g through S it holds that for every $i \in S$ with $\varphi_i(g') > \varphi_i(g)$ there exists a player $j \in S$ such that $\varphi_j(g') < \varphi_j(g)$.

Some relational insights

- Deletion proofness properties are considered very weak; they just represent minimal requirements.
- Pairwise stability is the corner stone of the work starting with Jackson and Wolinsky (1996). Models have been developed by Jackson and Watts (2001, 2002) to identify these networks.
- Strong stability implies efficiency and, thus, Pareto optimality. Strongly stable networks only rarely exist, but these networks have many appealing properties.

Comparing properties: An example

Consider the following three player example:

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	Stability
$g_0 = \emptyset$	2	2	4	D_s
$g_1 = \{12\}$	6	4	2	P_s
$g_2 = \{13\}$	2	2	2	
$g_3 = \{23\}$	0	3	0	
$g_4 = \{12, 13\}$	1	1	1	
$g_5 = \{12, 23\}$	2	6	2	
$g_6 = \{13, 23\}$	5	5	3	P
$g_7 = g_N$	7	4	4	S

Consent and Trust: Principles

We return to game theoretic foundations of network formation based on individualistic principles. However, this time we introduce the necessity of *consent* in link formation.

Consent requires that players develop a form of *trust* in order to create non-trivial networks. Without such trust, networks would not form and can therefore not be explained.

Analysis supports the previous claim.

Consent and Trust: Formalities

The **standard (consent) model** is a game Γ_s such that

- $A_i = N \setminus \{i\}$ describes the players that i intends to link with;
- If $a \in A$, then the network formed is given by

$$g_\beta(a) = \{ij \in g_N \mid i \in a_j \text{ and } j \in a_i\}.$$

- The game-theoretic payoff for player i is given by $\pi_i(a) = \varphi_i(g_\beta(a))$.

A network $g \in G$ is **individually stable** if it can be supported by a Nash equilibrium in Γ_s .

Individual stability

- A network $g \in \mathbb{G}$ is individually stable if and only if g is strong link deletion proof for the network payoff function φ .
- The empty network is always individually stable.
- If φ is link monotonic in the sense that $\varphi_i(g) < \varphi_i(g + ij)$ for all networks g and players $i \in N$ with $ij \notin g$ where $j \neq i$, then *every* network is individually stable.

Modelling trust

Let $a \in A$ be an arbitrary action tuple. For every player $i \in N$ we define his **myopic belief system** $a^{i*} \in A$ by

1. for every $j \neq i$ we let

- $i \notin a_j^{i*}$ if $\varphi_j(g_\beta(a) + ij) < \varphi_j(g_\beta(a))$ and
- $i \in a_j^{i*}$ if $\varphi_j(g_\beta(a) + ij) \geq \varphi_j(g_\beta(a))$,

2. and for all $k \in N$ with $k \neq i, j$ we define $k \in a_j^{i*}$ if and only if $k \in a_j$.

Modelling trust

- An action tuple $\hat{a} \in A$ is a **Network Trust Equilibrium in Γ_s** if for every player $i \in N$: $\hat{a}_i \in A_i$ is a best response to $\hat{a}_{-i}^{i*} \in A_{-i}$ for the payoff function φ_i .
- Hence, an NTE is similar to the Nash equilibrium definition, except that each player bases his actions on his belief system rather than the actual actions selected by the other players.
- A network $g \in G$ is **monadically stable in Γ_s** if it can be supported by an NTE in Γ_s .

Monadic stability

- Every monadically stable network in Γ_s is strongly pairwise stable.
- There is no guarantee that monadically stable networks exist.
- Monadic stability is not related in any fashion to strong stability.

Example: Nonexistence

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	Stability
$g_0 = \emptyset$	2	2	2	I
$g_1 = \{12\}$	3	3	0	I
$g_2 = \{13\}$	5	2	3	I
$g_3 = \{23\}$	2	3	5	P_s
$g_4 = \{12, 13\}$	4	1	3	
$g_5 = \{12, 23\}$	1	4	3	
$g_6 = \{13, 23\}$	1	3	6	
$g_7 = g_N$	3	0	0	

Example: Strong stability

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	Stability
$g_0 = \emptyset$	0	0	0	
$g_1 = \{12\}$	0	0	0	
$g_2 = \{13\}$	0	0	0	
$g_3 = \{23\}$	1	1	1	S , not M
$g_4 = \{12, 13\}$	0	0	0	M , not S
$g_5 = \{12, 23\}$	0	2	0	
$g_6 = \{13, 23\}$	0	0	2	
$g_7 = g_N$	2	0	0	M , not S

Concluding remarks

- Network formation requires a balance between “cooperation” and “competition”. Too much of either makes the analysis less valuable to understand social networks.
- Engineering applications can be based on Nash networks or the standard model preferably if it is a potential game.

Concluding remarks

- What dynamic processes support the formation of certain stable networks? Many game theoretic processes seem to have less relevancy in network environments.
- Link to physics literature on statistical mechanics of social networks: Can game theory contribute to our understanding of the formation of scale free and small world networks? What are the defining hypotheses underlying such models? (Possibly, “social space”.)