Cooperative Game Theory and Networks

Middlemen, Critical Links, and Strong Pairwise Stability

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Cooperative games

- Let N be a population of players.
  Players are the nodes in a network.
- A **cooperative game** on N is a function $v\colon 2^N \to \mathbb{R}$.
  Notation: $v \in \mathbb{U}$.
  A cooperative game assigns to every coalition of players a collective value, “payoff”, or benefit.
- An **allocation rule** is a function $Y : \mathbb{U} \to \mathbb{R}^N$, which assigns to every player $i \in N$ and game v an allocated payoff $Y_i(v)$, such that

$$\sum_{i \in N} Y_i(v) = v(N).$$

Networks

- **Links** are in principle undirected and formed between two players in N. A link between players i and j is therefore the binary set $\{i, j\}$.
- A **network** is given by $g \subset g_N = \{ij \mid i \neq j\}$, where ij stands for $\{i, j\}$.
- $\mathbb{G} = \{g \mid g \subset g_N\}$ is the class of all (undirected) networks.

We do not rule out that these relationships have spillover effects on the productive relations between other players. This is captured by the formal description of such network benefits.

Network components

- The network $g' \subset g$ is a **component** of g if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$.
  In other words, a component is simply a maximally connected subnetwork of g.
- We denote the class of network components of the network g by $C(g)$.

Disconnected nodes

The set of players that are not connected in the network g are collected in the set of (fully) **disconnected players** in g denoted by

$$N_0(g) = N \setminus N(g) = \{i \in N \mid N_i(g) = \varnothing\}.$$

Furthermore, we define

$$\Gamma(g) = \{N(h) \mid h \in C(g)\} \cup \{\, \{i\} \mid i \in N_0(g)\}$$

as the partitioning of the player set N based on the component structure of the network g.

Collective benefits for networks

A **(collective) network value function** is given by $v\colon \mathbb{G}^N \to \mathbb{R}$ such that $v(\varnothing) = 0$.

A network value function v assigns a total benefit $v(g) \in \mathbb{R}$ to the network $g \in \mathbb{G}^N$. The space of all network value functions v such that $v(\varnothing) = 0$ is denoted by $\mathbb{V}^N$.

It is clear that $\mathbb{V}^N$ is a $\left(2^{\frac{1}{2}n(n-1)} - 1\right)$-dimensional Euclidean vector space.

Value properties

Let $v \in \mathbb{V}^N$ be some network value function. We consider two fundamental properties of such a network value function:

1. The network value function v is **component additive** if $v(g) = \sum_{h \in C(g)} v(h)$. Component additivity immediately implies that disconnected players $i \in N_0(g)$ generate no value.
2. The network value function v is **anonymous** if $v(g^\pi) = v(g)$ for all permutations π and networks g. Anonymity implies that benefits $v(g)$ depend on the topology of the network g only.

Allocation of values

We extend the definition of an allocation rule from the standard cooperative game theoretic setting to the setting with networks.

An **allocation rule** is a function $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ such that for every pair $(g, v) \in \mathbb{G}^N \times \mathbb{V}^N$ it holds that

$$\sum_{i \in N} Y_i(g, v) = v(g).$$

Egalitarian allocations

Let $v \in \mathbb{V}^N$ be component additive. The **component egalitarian allocation rule** is defined by

$$Y_i^{ce}(g, v) = \frac{v(h_i)}{n(h_i)} \tag{1}$$

where $h_i \in C(g)$ such that $i \in N(h_i)$ and $h_i = \varnothing$ if there is no $h \in C(g)$ such that $i \in N(h)$.

Under this rule, the value generated by a component is split equally among the members of that component. The component-wise egalitarian allocation rule satisfies component balance.

Link stability 1

A network $g \in \mathbb{G}^N$ is **link deletion proof** (LDP) if for every player $i \in N$ and every neighbor $j \in N_i(g)$, it holds that $Y_i(g - ij, v) \leqslant Y_i(g, v)$.

Link deletion proofness requires that each individual player has no incentive to sever an existing link with one of his neighbors.

Link stability 2

A network $g \in \mathbb{G}^N$ is **strong link deletion proof** (SLDP) if for every player $i \in N$ and every set of neighbors $M \subset N_i(g)$, it holds that $Y_i(g \setminus h_M, v) \leqslant Y_i(g, v)$ where $h_M = \{ij \in g \mid j \in M\} \subset g$.

Strong link deletion proofness requires that each player has no incentive to sever links with one or more of his neighbors. Obviously, SLDP implies LDP.

Link Stability 3

A network $g \in \mathbb{G}^N$ is **link addition proof** if for all players $i, j \in N$, it holds that $Y_i(g + ij, v) > Y_i(g, v)$ implies $Y_j(g + ij, v) < Y_j(g, v)$.

Link addition proofness states that there are no incentives to form additional links. This is founded on a process of mutual consent in link formation.

Pairwise stability

- A network $g \in \mathbb{G}^N$ is **pairwise stable** if it is link deletion proof and link addition proof.
- A network $g \in \mathbb{G}^N$ is **strongly pairwise stable** if it is strong link deletion proof and link addition proof.

Strong pairwise stability is a natural link-based stability concept. Since links require mutual consent, it considers the addition of one link at a time. However, link deletion is unilateral and, hence, it allows for a single player to delete multiple links at the same time.

Bad company 1

Consider a three player situation with $N = \{1, 2, 3\}$. For simplification of notation we denote the potential links in this situation as follows: $a = 12$, $b = 13$, and $c = 23$. Hence, $\mathbb{G}^N = \{\varnothing, a, b, c, ab, ac, bc, abc\}$.

Let $\alpha > 0$.

Bad company 2

We consider an allocation rule $\overline{Y} : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}$ which for every $v \in \mathbb{V}^N$ is defined by

$$\overline{Y}(\varnothing, v) = (0, 0, 0),$$

$$\overline{Y}(a, v) = \left(\frac{v(a)}{2}, \frac{v(a)}{2}, 0\right)$$

$$\overline{Y}(b, v) = \left(\frac{v(b)}{2}, 0, \frac{v(b)}{2}\right),$$

$$\overline{Y}(c, v) = \left(0, \frac{v(c)}{2}, \frac{v(c)}{2}\right)$$

Bad company 3

$$\overline{Y}(ab, v) = (v(ab), 0, 0),$$

$$\overline{Y}(ac, v) =$$
$$\left(-\alpha v(abc), v(ac) - \tfrac{1}{2}(1-\alpha)v(abc), \tfrac{1}{2}(1+\alpha)v(abc)\right)$$

$$\overline{Y}(bc, v) =$$
$$\left(-\alpha v(abc), \tfrac{1}{2}(1+\alpha)v(abc), v(bc) - \tfrac{1}{2}(1-\alpha)v(abc)\right)$$

$$\overline{Y}(abc, v) =$$
$$\left(-\alpha v(abc), \tfrac{1}{2}(1+\alpha)v(abc), \tfrac{1}{2}(1+\alpha)v(abc)\right)$$

Note that $\overline{Y}$ is component balanced.

Bad company 4

Claim: *If* $v \in \mathbb{V}^N$ *such that* $v(g) > 0$ *for every* $g \neq \varnothing$, *then the network* $g^\star = abc$ *is link deletion proof, but not strong link deletion proof, with respect to the allocation rule* $\overline{Y}$.

The claim states that without the possibility of a player to remove multiple of his links simultaneously, he might get stuck with “bad company”. Indeed, here player 1 would like to remove his links with player 2 as well as player 3, but using LDP he can only remove at most one of these two links. Under SLDP player 1 is able to remove both links and improve his situation.

Boundedness of payoffs

Proposition

Let v be a component additive network value function and Y be a component balanced allocation rule. Then there exists some $V > 0$ such that $0 \leqslant Y_i(g, v) \leqslant V$ for every strong link deletion proof network $g \in \mathbb{G}^N$ and every player $i \in N$.

Critical links

A link $ij \in g \in \mathbb{G}^N$ is **critical** in the network g if $\#\Gamma(g) < \#\Gamma(g - ij)$.

Let $h \in C(g)$ denote a component that contains a critical link in the network $g \in \mathbb{G}^N$ and let $h_1 \subset h$ and $h_2 \subset h$ denote components obtained from h by severing that critical link. (Note that it may be the case that $h_1 = \varnothing$ or $h_2 = \varnothing$.)

Critical link monotonicity

The pair (g, v) satisfies **critical link monotonicity** if for any critical link $ij \in h$ with $h \in C(g)$ and the two associated components h_1 and h_2 of $h - ij$, we have that

$$v(h) \geqslant v(h_1) + v(h_2) \;\Rightarrow\; \frac{v(h)}{n(h)} \geqslant \max\left[\frac{v(h_1)}{n(h_1)}, \frac{v(h_2)}{n(h_2)}\right]$$

Jackson-Wolinsky theorem

If the network $g \in \mathbb{G}^N$ is maximal relative to a component additive $v \in \mathbb{V}^N$, then g is pairwise stable with respect to the component egalitarian allocation rule Y^{ce} relative to v if and only if (g, v) satisfies critical link monotonicity.

Middleman positions

A player $i \in N$ has a **middleman position** in the network $g \in \mathbb{G}^N$ if there exists some set of links $h^* \subset L_i(g)$ under the control of player i in g such that there are at least two distinct players $j_1, j_2 \in N \setminus \{i\}$ who are connected in g and who are not connected in $g \setminus h^*$.

A player with a middleman position in a network g is denoted as a **middleman** in g.

The set of middlemen in the network g is denoted by $M(g) \subset N$.

Remark on middlemen

Let $n \geqslant 3$ and let $g \in \mathbb{G}^N$ be some network with $\# C(g) = 1$. Now, $i \in M(g)$ if and only if player $i \in N$ controls a critical link set $h^* \subset L_i(g)$ such that exactly one of the following properties holds:

- $\# C(g \setminus h^*) > \# C(g) = 1$;
- $\# C(g \setminus h^*) = 1$ and there is some player $j \in N \setminus N_0(g)$ such that $j \in N_0(g \setminus h^*)$, and
- $\# C(g \setminus h^*) = 0$ and $N_0(g \setminus h^*) = N$.

Middleman security

A pair $(g, v) \in \mathbb{G}^N \times \mathbb{V}^N$ is **middleman secure** if for every component $h \in C(g)$, every middleman $i \in M(h)$, and every critical link set $h^* \subset L_i(h)$ for middleman i we have that

$$v(h) \geqslant \sum_{i=1}^{m} v(h_i) \;\Rightarrow\; \frac{v(h)}{n(h)} \geqslant \frac{v(\widehat{h})}{n(\widehat{h})},$$

where $C(h \setminus h^*) = \{h_1, h_2, \ldots, h_m\}$ and $\widehat{h} \in C(h \setminus h^*)$ such that $i \in N(\widehat{h})$.

Implication

Proposition

Let $v \in \mathbb{V}_{+}^{N}$ be nonnegative in the sense that $v(g) \geqslant 0$ for all $g \in \mathbb{G}^{N}$. If (g, v) satisfies middleman security, then (g, v) satisfies critical link monotonicity as well.

Main result

If the network $g \in \mathbb{G}^N$ is maximal relative to a component additive $v \in \mathbb{V}^N$, then g is strong link deletion proof with respect to the component egalitarian allocation rule Y^{ce} if and only if (g, v) is middleman secure.

Corollary

If $g \in \mathbb{G}^N$ is maximal relative to a nonnegative and component additive $v \in \mathbb{V}_+^N$, then g is strongly pairwise stable for the component egalitarian allocation rule Y^{ce} if and only if (g, v) is middleman secure.

Middleman-free networks

A network $g \in \mathbb{G}^N$ is called **middleman-free** if $M(g) = \varnothing$. Hence, in a middleman-free network there are no middleman positions.

Proposition
A network $g \in \mathbb{G}^N$ is middleman-free if and only if for every network value function $v \in \mathbb{V}^N$ the pair (g, v) is middleman secure.

Middleman-free theorem

If $g \in \mathbb{G}^N$ is middleman-free as well as maximal relative to a nonnegative and component additive $v \in \mathbb{V}_+^N$, then g is strongly pairwise stable for the component-wise egalitarian allocation rule Y^{ce}.

Concluding questions

- Is the collective benefit model combined with the component egalitarian allocation rule useful in the context of ad-hoc wireless networks?
- Given that, what is the role of critical links and middleman positions in such wireless networks?
- Finally, can the main results be applied to the analysis of these wireless networks?