

# Convergence of Cognitive Radio Networks<sup>1</sup>

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## Abstract

*In this paper, we examine the conditions and behavior of several common convergence dynamics from game theory and show how they influence the structure of networks of cognitive radios. We then apply these to previously proposed distributed power control algorithms and describe how they impact network complexity.*

## 1. Introduction

Cognitive radio is an enhancement on the traditional software radio concept wherein the radio is aware of its environment and its capabilities, is able to independently alter its physical layer behavior, and is capable of following complex adaptation strategies [1]. Cognitive radio has received significant interest as a technology that could enable improved performance and more efficient spectrum usage.

On May 19, 2003, the FCC convened a workshop to examine the impact that cognitive radio could have on spectrum utilization and to study the practical regulatory issues that cognitive radio would raise [2]. There, Lauren Van Wazer, Special Counsel to the Chief of the FCC's Office of Engineering and Technology, said the FCC desires to "improve access [to radio spectrum] through better use of time, frequency, power, bandwidth, and geographic space" and its belief that cognitive radio held the potential to accomplish this goal [3]. However, the FCC has the following concerns. 1) "What operational parameters are necessary to have the capability for adaptation and flexibility?" 2) "Will system reliability be affected by the incorporation of cognitive radio technologies?" 3)

"Where would their use be particularly beneficial?"

This paper addresses some of the issues associated with the first two questions by examining the network structure, adaptation algorithms, and adaptation criteria such that cognitive radio networks (CRN) exhibit convergent behavior. The establishment that a CRN exhibits convergent behavior is desirable as it permits the analysis and planning needed to predict network reliability. The conditions under which a CRN will exhibit convergent behavior identifies parameters necessary for reliable adaptation.

To establish the necessary conditions for convergent behavior, we utilize existing techniques and models from game theory commonly used to study systems with dynamic decision making. We then introduce conditions and structures for cognitive radio networks to exhibit convergent behavior. We then consider several classes of distributed power control algorithms which are suitable for implementation in a CRN.

## 2. Game theory

Game theory is a set of mathematical tools used to analyze interactive decision makers. The fundamental component of game theory is the notion of a game, expressed in normal form:  $G = \langle N, A, \{u_i\} \rangle$  where  $G$  is a particular game,  $N = \{1, 2, \dots, n\}$  is a finite set of players (decision makers),  $A_i$  is the set of actions available to player  $i$ ,  $A = A_1 \times A_2 \times \dots \times A_n$  is the action space, and  $\{u_i\} = \{u_1, u_2, \dots, u_n\}$  is the set of utility (objective) functions that the players wish to maximize. Each player's objective function,  $u_i$ , is a function of the particular action chosen by player  $i$ ,  $a_i$ , and the particular actions chosen by all of the other players in the game,  $a_{-i}$  and yields a real number. Other games may include additional components, such as the information available to each player and

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communication mechanisms. In a repeated game, players are allowed to observe the actions of the other players, remember past actions, and attempt to predict future actions of players.

**Definition: Nash equilibrium** An action vector  $a$  is said to be a Nash equilibrium (NE) iff

$$u_i(a) \geq u_i(b_i, a_{-i}) \forall i \in N, b_i \in A_i \quad (1)$$

Restated, a NE is an action vector from which no player can profitably unilaterally deviate. NE correspond to the steady-states of the game and are then predicted as the most probable outcomes of the game. Note: demonstrating that an action vector is a NE says nothing about the desirability or optimality of the action vector.

## 2.1. Repeated games

A repeated game is sequence of stages where each stage is the same normal form game. When the game has an infinite number of stages, the game is said to have an infinite horizon game. Based on their knowledge of the game – past actions, future expectations, and current observations - players choose strategies – a choice of actions at each stage.

These strategies can be fixed, contingent on the actions of other players, or adaptive. Further, these strategies can be designed to punish players who deviate from agreed upon behavior. When punishment occurs, players choose their actions to minimize the payoff of the offending player. However, even when the other players are minimizing the payoff a player  $i$ ,  $i$  is still able to achieve some payoff  $v_i$ . Thus there is a limit to the how much a player can be punished.

As estimations of future values of  $u_i$  are uncertain, many repeated games modify the original objective functions by discounting the expected payoffs in future stages by  $\delta$ , where  $0 < \delta < 1$  such that the anticipated value in stage  $k$  to player  $i$  is given by (2).

$$u_{i,k}(a) = \delta^k u_i(a) \quad (2)$$

**Folk theorem** [4] In a repeated game with an infinite horizon and discounting, for every feasible payoff vector  $v > v_i$  for all  $i \in N$ , there exists a  $\underline{\delta} < 1$  such that for all  $\delta \in (\underline{\delta}, 1)$  there is a NE with payoffs  $v$ .

To generalize the Folk theorem, given a discounted infinite horizon repeated game, nearly any behavior can be designed to be the steady-state through the proper choice of punishment strategies and  $\delta$ . Thus convergent behavior of a repeated game can be achieved nearly independent of the objective function.

## 2.2. Myopic games

A myopic game is defined here as a repeated game in which there is no communication between the

players, memory of past events, or speculation of future events. Any adaptation by a player can still be based on knowledge of the current state of the game. As players have no consideration of future payoffs, the Folk theorem does not hold for myopic games and the convergence to steady-state behavior must occur through other means. Two convergence dynamics possible in a myopic game are the *best response dynamic* and the *better response dynamic*. Both dynamics require additional structure in the stage game to ensure convergence.

**Definition: Best response dynamic** [5] At each stage, one player  $i \in N$  is permitted to deviate from  $a_i$  to some randomly selected action  $b_i \in A_i$  iff

$$u_i(b_i, a_{-i}) \geq u_i(c_i, a_{-i}) \forall c_i \neq b_i \in A_i \text{ and } u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}).$$

**Definition: Better response dynamic** [5] At each stage, one player  $i \in N$  is permitted to deviate from  $a_i$  to some randomly selected action  $b_i \in A_i$  iff

$$u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}).$$

## 2.3. S-modular games

An S-modular game restricts  $\{u_j\}$  such that for all  $i \in N$  either (3) or (4) is satisfied.

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} \geq 0 \forall j \neq i \in N \quad (3)$$

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} \leq 0 \forall j \neq i \in N \quad (4)$$

When (3) is satisfied, the game is said to be supermodular; when (4) is satisfied, the game is said to be submodular. Myopic games whose stages are S modular games with a unique NE and follow a best response dynamic converge to the NE when the NE is unique [6].

## 2.4. Potential games

A potential game is a special type of game where  $\{u_j\}$  are such that the change in value seen by a unilaterally deviating player is reflected in a function  $P: A \rightarrow \mathfrak{R}$ . All myopic games where the stages are the same potential game converge to a NE when decisions are updated according to a better response dynamic [5].

**2.4.1 Exact potential games** A game is an exact potential game (EPG) if there exists some function (EPF)  $P: A \rightarrow \mathfrak{R}$  such that (5) is satisfied  $\forall i \in N, \forall a \in A$ .

$$u_i(a_i, a_{-i}) - u_i(b_i, a_{-i}) = P(a_i, a_{-i}) - P(b_i, a_{-i}) \quad (5)$$

In [7], (6) is given as a necessary and sufficient condition for a game to be an exact potential game.

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i} \forall i, j \in N, a \in A \quad (6)$$

**Coordination-dummy games** [7] As shown in (7), coordination-dummy game is a composite of a coordination game with identical interest function  $V$  and a dummy game with dummy function  $D_i$  whose value is solely dependent on the actions of the other players. An EPF for this game can be written as (8).

$$u_i(a) = V(a) + D_i(a_{-i}) \quad (7)$$

$$P(a) = V(a) \quad (8)$$

**Self-motivated games** As shown in (9), the utility functions of a self-motivated game are a function solely of that player's actions. A self-motivated game can be shown to be an EPG by introducing the EPF listed in (10).

$$u_i(a) = h_i(a) \quad (9)$$

$$P(a) = \sum_{i \in N} h_i(a_i) \quad (10)$$

**Bilateral symmetric interaction games** [8] In a bilateral symmetric interaction (BSI) game, every player's objective function can be characterized by (11) where  $w_{ij}: A_i \times A_j \rightarrow \mathfrak{R}$  and  $h_i: A_i \rightarrow \mathfrak{R}$  such that for every  $(a_i, a_j) \in A_i \times A_j$ ,  $w_{ij}(a_i, a_j) = w_{ji}(a_j, a_i)$ . An EPF for a BSI game is given by (12).

$$u_i(a) = \sum_{j \in N \setminus \{i\}} w_{ij}(a_i, a_j) - h_i(a_i) \quad (11)$$

$$P(a) = \sum_{i \in N} \sum_{j=1}^{i-1} w_{ij}(a_i, a_j) - \sum_{i \in N} h_i(a_i) \quad (12)$$

**2.4.2 Ordinal potential games** A game is an ordinal potential game (OPG) if there exists some function (OPF)  $P: A \rightarrow \mathfrak{R}$  such that (13)(5) is satisfied  $\forall i \in N, \forall a \in A$ . Note that all EPG are also OPG.

$$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \Leftrightarrow P(a_i, a_{-i}) > P(b_i, a_{-i}) \quad (13)$$

**2.4.3 Potential game properties** While there is no necessary and sufficient condition for establishing that a game is an OPG, it is possible to indirectly show that a game is an OPG through the application of the following transformation properties (TP).

**Definition: Ordinal transformation** An ordinal transformation (OT) is a one-to-one mapping of  $\{u_i\}$  to  $\{u_i'\}$  so that the ordering of the values of the objective functions remains the same.

**Transformation property 1:** An OT of an OPG is an OPG.

**Proof:** This is shown in [9].

**Lemma 1** If an OT of a game yields an OPG, then the original game must also be an OPG.

**Proof:** This is shown in [9].

**Transformation property 2:** An OT of an EPG is an OPG, but not necessarily an EPG.

**Proof:** This is shown in [9].

**Definition: Composite game** A composite game is a game,  $G$  formed from two other games  $G_1$  and  $G_2$  with a common set of players,  $N$ , a common action space,  $A$ , and utility functions,  $g_1^1(a)$  and  $g_1^2(a)$ .  $G$  has player set  $N$ , action set  $A$ , and utility functions  $\{u_i\}$  given by  $u_i(a) = g_i^1(a) + g_i^2(a)$ .

The concept of composite games is useful in the analysis of games with complex utility functions and allows analysis to proceed on a "divide and conquer" approach.

**Transformation property 3:** A composite game formed from a linear combination of two EPG is itself an EPG.

**Proof:** This is shown in Proposition 2.7 of [10].

**Transformation property 4:** A composite game formed from a combination of two OPG is not necessarily an OPG.

**Proof:** This is shown in the game of Figure 5.4 of [10].

**Transformation property 5:** A composite game formed from a combination of an EPG and an OPG is not necessarily an OPG.

**Proof:** This is shown in [9].

## 3. Cognitive radio networks

A cognitive radio network (CRN) is characterized by radios which can sense its environment, are knowledgeable of their capabilities, able to alter their physical layer behavior, and follow adaptation algorithms to determine their behavior.

### 3.1. Cognitive radio networks as a game

A CRN can be modeled as a game as follows: The cognitive radios in the network form the game's set of decision makers. The set of physical layer parameters which a radio is permitted to alter forms the player's action set. From these action sets, the action space is formed. Preference relations over the action space are formed by an exhaustive evaluation of the adaptation algorithms. Objective functions are then formed by mapping the preference relations to the real number line so that preferable action vectors are larger than less preferable action vectors.

### 3.2. Repeated games

When the modeled CRN can be shown to be a discounted repeated game, convergence requires the following constraints. 1) Some mechanism must exist for broadcasting the desired operating vector, the discount factor, and the punishment strategy. 2) Full

knowledge of the environment so the radios can differentiate deviant behavior from fades and jammers external to the CRN. 3) Knowledge of the action chosen by each radio at each stage.

These constraints imply significant resources in terms of both memory and processing power within the radios will be required. Additionally, significant overhead in the network will be incurred to facilitate the broadcasting of the necessary information. Note, however, no constraints are placed on the structure of the objective function.

### 3.3. S-modular games

When the modeled CRN can be shown to be an S-modular game, convergence according to a best-response dynamic requires that the following conditions be satisfied. 1) The adaptation algorithms must incorporate perfect knowledge of the objective function. 2) The network must have a unique steady-state. 3) Some method must exist for measuring current performance and for sensing relevant environmental parameters. Depending on the particular adaptation algorithm, it might be necessary to know the number and type of other radios in the network.

### 3.4. Potential games

When the modeled CRN can be shown to be a potential game, convergence requires that the radios merely need to be able to measure their performance. For power control and adaptive interference avoidance applications, [11] describes sufficient conditions for modeling networks as a potential game.

### 3.5. Comments

The complexity of CRNs is strongly influenced by the convergence mechanism that it adopts. The kind of convergence mechanisms available to a CRN can be determined from its associated game theoretic model. In order of network complexity it is seen that OPG < S-modular < Repeated. Thus it will be beneficial for CRNs to utilize algorithms that can be modeled as an OPG, when possible.

## 4. Distributed power control

In this section we examine several proposed distributed power control algorithms within the context of CRNs. To formalize the following discussion, we introduce the notation in Table 1. Based on these conventions, a power control game  $G$  can be formulated as  $G = \langle N, \mathbf{P}, \{u_j\} \rangle$ .

**Table 1 Power control game notation**

Symbol	Meaning
$N$	The set of decision making (cognitive) radios in the network
$i, j$	Two different cognitive radios, $i, j \in N$
$P_j$	The set of power levels available to radio $j$ . This is presumed to be a segment of the real number line $\mathcal{R}$
$p_j$	A power level chosen by $j$ from $P_j$
$\mathbf{P}$	The power space ( $\mathcal{R}^n$ ) formed from the Cartesian product of all $P_j$ . $\mathbf{P} = P_1 \times P_2 \times \dots \times P_n$
$\mathbf{p}$	A power profile (vector) from $\mathbf{P}$ formed as $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$
$u_j(\mathbf{p})$	The utility that $j$ receives from $\mathbf{p}$ . This is the specific objective function that $j$ is looking to maximize.

### 4.1. Repeated power games

[12] considers a discounted repeated power control game implemented on a packet based network wherein the original objective function for each radio  $j$  is the modified function of throughput given by (14)

$$u_j(\mathbf{p}) = \frac{R}{p_j} (1 - 2BER(\mathbf{p}))^L \quad (14)$$

where  $R$  is the transmission rate,  $L$  is the packet length, and  $BER$  is the bit error rate which is a function of the SINR seen by  $j$ .

As the modeled game has an infinite horizon, a CRN implemented in this manner will exhibit convergent behavior if the conditions of Section 3.2 are satisfied.

### 4.2. S-modular games

[6] examines the application of super-modular games to distributed power control. Thus the objective functions for this game are characterized by (3) and (4). Herein each game follows a General Updating Algorithm (GUA) which is actually a best response dynamic. Thus if these network games have a unique NE, behavior converges to the NE from any initial  $\mathbf{p}$ .

Thus for a CRN which satisfies this characterization, the conditions of Section 3.3 must hold for convergent behavior.

[6] asserts that the classes of power control algorithms considered in by Yates in [13] are S-modular games. Yates examines power control in the context of the uplink of a cellular system under five scenarios: 1) Fixed assignment where each mobile is assigned to a particular base station, 2) Minimum power assignment where each mobile is assigned to the base station where its SINR is maximized, 3) Macro diversity where all base stations combine the signals of the mobiles, 4) Limited diversity where a

subset of the base stations combine the signals of the mobiles, and 5) Multiple connection reception where the target SINR must be maintained at a number of base stations.

In each scenario, each mobile,  $j$ , tries to achieve a target SINR  $\mathbf{g}$  as measured by a function,  $I(\mathbf{p})$ .  $I(\mathbf{p})$  is the standard effective interference function which has the following properties: 1) *Positivity*, e.g.,  $I(\mathbf{p}) > 0$ , 2) *Monotonicity*, e.g., If  $\mathbf{p} \geq \mathbf{p}^*$ , then  $I(\mathbf{p}) \geq I(\mathbf{p}^*)$ , 3) *Scalability*, e.g., for all  $\mathbf{a} > 1$ ,  $\mathbf{a}I(\mathbf{p}) > I(\mathbf{a}\mathbf{p})$  where the convention that  $\mathbf{p} > \mathbf{p}^*$  means that  $p_i > p_i^* \forall i \in N$

It should be noted that single cell target SINR games are also ordinal potential games which can be shown as follows.

**Theorem 1:** All target SINR games are OPG.

**Proof:** Consider a modified game where the action sets are the received power levels at the base station. A received power level for mobile  $j$ ,  $r_j$ , is the product of its path loss to base station  $k$ ,  $h_{j,k}$ , and its transmitted power level. Then the objective functions of a target SINR game can be expressed as in (15)

$$u_j(\mathbf{p}) = 1 - \left[ -\mathbf{g}_j + \left( r_j - \left( \sum_{i \in N \setminus j} r_i \right) - \mathbf{s}_k \right) \right]^2 \quad (15)$$

where  $\mathbf{s}_k$  is the noise power at  $k$ . Expanding (15) yields (16).

$$\begin{aligned} u_j(\mathbf{p}) &= 1 - \mathbf{g}_j^2 - \mathbf{s}_k^2 - 2\mathbf{g}_j\mathbf{s}_k \\ &- \left( \sum_{i \in N \setminus j} r_i \right)^2 - 2\mathbf{g}_j \left( \sum_{i \in N \setminus j} r_i \right) - 2\mathbf{s}_k \left( \sum_{i \in N \setminus j} r_i \right) \\ &- r_j^2 + 2\mathbf{g}_j r_j + 2\mathbf{s}_k r_j \\ &+ 2r_j \left( \sum_{i \in N \setminus j} r_i \right) \end{aligned} \quad (16)$$

Now notice that the first two lines of (16) are dummy games; line three of (16) is a self-motivated game – both EPG. The final line is also an EPG as it is a BSI game. Thus by applying TP 3, it is seen that the target SINR game is an EPG. As other forms of target SINR games are ordinal transformations of (16), by TP 1 all target SINR games are OPG. So in this instance, the convergence conditions can be relaxed to the ones outlined in Section 3.4.

**Theorem 2:** All target throughput power control games are OPG.

**Proof:** A target throughput power control game is just an OT of a target SINR game. As target SINR is an OPG, by TP 1, the target SINR game is also an OPG.

**Lemma:** All throughput maximization power control games are OPG.

**Proof:** This is simply a target throughput game with an infinite target throughput.

### 4.3. Linear individual pricing

[14] considers a variation of a fixed assignment scenario wherein only a single cell exists and the mobile devices are trying to maximize a function of their throughput. Note that the only NE of games whose utility functions are pure throughput function is the power vector where all mobiles transmit at maximum power. To avoid this problem [14] introduces the modified throughput function (17)

$$u_j(\mathbf{p}) = \frac{Ef(\mathbf{p})}{p_j} - tRp_j \quad (17)$$

where  $R$  is the transmission rate,  $t$  is a constant,  $E$  is the battery life, and  $f$  is the throughput function.

By the techniques of this paper, this can neither be shown to be an OPG nor not an OPG. The left most term is generally only an OPG whereas the pricing function is an EPG as it is a self-motivated game. By TP5, the full game cannot be guaranteed to be an OPG implying that it does not have to converge under a better response dynamic.

[14] uses a best response dynamic in its experiments which converge to a NE. Though not stated in [14], [15] shows that (17) is indeed a supermodular game. Also note that without the cost function, (14) is a particular instantiation of (17). So a CRN implementing the repeated games of [12], [14], or [15] are supermodular games and will exhibit convergent behavior if the properties of Section 3.3 are satisfied. However, it should be noted that when left as a repeated game, there is greater flexibility in dynamically selecting the steady-state.

### 4.4. Nonlinear group pricing

[16] considers another single cell distributed power control game with pricing characterized by (18) where  $\lambda$  is a constant. Note the base station subscript has been dropped in this case as all mobiles are communicating with the same base station.

$$u_j(\mathbf{p}) = R_j f(\mathbf{p}) - \frac{I h_j p_j}{\sum_{i \in N} h_i p_i} \quad (18)$$

The pricing function is to reflect that the damage caused by  $p_j$  to the other devices in the network is a function of the relative difference in powers rather than absolute power level. Note that (18) is just a composite game of throughput maximization and a cost function. As we have shown that throughput maximization is an OPG, (18) can only be guaranteed to be an OPG only if, though not necessarily if, the pricing function has an EPF which can only be true if (6) is satisfied. Differentiating the price function twice yields:

$$\frac{\partial^2 C_i}{\partial p_i \partial p_j} = \frac{\left(\sum_{k \in N} h_k p_k\right)^2 I h_i h_j - 2h_j \left(\sum_{k \in N} h_k p_k\right) g_i(\mathbf{p})}{\left(\sum_{k \in N} h_k p_k\right)^4}$$

$$\frac{\partial^2 C_j}{\partial p_j \partial p_i} = \frac{\left(\sum_{k \in N} h_k p_k\right)^2 I h_i h_j - 2h_i \left(\sum_{k \in N} h_k p_k\right) g_j(\mathbf{p})}{\left(\sum_{k \in N} h_k p_k\right)^4}$$

where  $g_i(\mathbf{p}) = I h_i \sum_{k \in N} h_k p_k - I h_i^2 p_i$ . Further

evaluation leads to the result that this price function has an EPF iff  $h_j p_j = h_i p_i$ . Note that this is only satisfied when the received powers are identical, which will generally not be true. Thus the cost function does not have an EPF and the nonlinear group priced game is not an OPG. Also note that (18) cannot be guaranteed to be a supermodular game either as properly chosen power vectors  $\mathbf{p}$  and  $\mathbf{p}^*$  will yield different signs for the second partial derivative of the cost functions.

Therefore neither the better response dynamic nor the best response dynamic will assuredly converge, and a more complex convergence dynamic is required. As the repeated game dynamic is guaranteed for convergence it can still be used. Thus this modification of the pricing function significantly complicated the network.

## 5. Conclusions

We examined three convergence dynamics: coordinated behavior, best-response, and better response. We saw that these dynamics are supported by discounted repeated games, S-modular games, and potential games, respectively. We introduced a game theoretic model suitable for analyzing the behavior of a CRN. We saw then saw how the ability of a CRN to support these dynamics is influenced by the cognitive radios' objective functions. We then saw how the objective functions impact the complexity of the CRN by examining several different distributed power control algorithms as summarized in Table 2. Finally, we showed that relatively minor changes to a CRN's objective functions, such as a modification to the pricing function, can prevent the application of better-response based adaptation algorithm.

**Table 2 Surveyed algorithms' CRN complexity**

Algorithm	Game Model	Complexity
Repeated Game	Repeated	High
S-modular	S-modular	Low
Target SINR	OPG	Minimal
Target Throughput	OPG	Minimal
Linear Pricing	S-Modular	Low
Nonlinear Group Pricing	Repeated	High

Cognitive radios promise to be an exciting new technology, and game theory will be a valuable tool for understanding the behavior of CRNs. However, when analyzing proposed algorithms for CRNs, the convergence process and the steady-state behavior should be considered.

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